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Some Properties on a Parallel Method for Factorization of a Polynomials [1]

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In this paper we introduce the concept of bisymmetric function at first and then give some properties of a parallel iteration for factorization of a polynomial into quadratic factors, which was presented in [9]. The results can be applied to consideration of the global convergence of the iteration processes.

Subject AMS Classification: 65H05

Key Words: parallel iteration, factorization of a polynomial, convergence of iteration processes

1. Introduction

The Bairstow method is a well-known method for determining a real quadratic factor of a polynomial with real coefficients $F(x) = \sum_{i=0}^{n} a_i x^{N-i}$. In computations we have to find all factors of the $F(x) = P(x) + KQ(x)$, where $P(x), Q(x)$ take the form $\prod_{i=1}^{n}(x - r_i)$, or $\prod_{i=1}^{n}(x^2 - v_{i1}x - v_{i2})$) and K, r_i, v_{i1}, v_{i2} are real numbers. Zheng [9] gives a useful detailed review about parallel iterations for finding all factors of a polynomial simultaneously. We denote by $\mathbf{R}^n$ the real n - dimensional space. Let $\mathbf{P}$ be a set of polynomials with real coefficients, $\mathbf{P}^n = \{f \in \mathbf{P}|$ the degree of f is not greater than n $\}$, $\mathbf{F} = \{f/g |\, f, g \in \mathbf{P}\}$. For $\mathbf{u} = (u_1, u_2)^T \in \mathbf{R}^2$, we write $Q(\mathbf{u}) = Q(\mathbf{u}, x) = x^2 - u_1 x - u_2$ and denote by $L(f) = L(f; \mathbf{u}, c; x) = l_1(f; \mathbf{u}, c)(x - c) + l_2(f; \mathbf{u}, c)$ the linear interpolation polynomial for $f \in \mathbf{F}$ with nodes α_1, α_2 are the roots of $Q(\mathbf{u}, x)$ and $c \in \mathbf{R}^1$ is a

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number independent of f. It is clear that finding $L(f)$ is equivalent to find $\mathbf{l}(f) = \mathbf{l}(f;\mathbf{u},c) = (l_1(f;\mathbf{u},c), l_2(f;\mathbf{u},c))^T$. We see that $\mathbf{l}(fg;\mathbf{u},c) = A(f;\mathbf{u},c)\mathbf{l}(f;\mathbf{u},c)$, where

$$A(f;\mathbf{u},c) = \begin{pmatrix} (u_1 - 2c)l_1(f;\mathbf{u},c) + l_2(f;\mathbf{u},c) & l_1(f;\mathbf{u},c) \\ (u_2 + u_1 c - c^2)l_1(f;\mathbf{u},c) & l_2(f;\mathbf{u},c) \end{pmatrix}.$$

and $\det A(f;\mathbf{u},c) = f(\alpha_1)f(\alpha_2)$. It is clear that

$$\mathbf{l}(f;\mathbf{u},0) = \mathbf{l}(f;\mathbf{u},c) - c(0, l_1(f;\mathbf{u},c))^T.$$

Therefore, we always suppose $c = 0$ in the following and write

$$L(f;\mathbf{u};x) = L(f;\mathbf{u},c;x),\ \mathbf{l}(f;\mathbf{u}) = \mathbf{l}(f;\mathbf{u},0),\ A(f;\mathbf{u}) = A(f;\mathbf{u},0).$$

Suppose $\mathbf{p}_i = (p_{i1}, p_{i2})^T \in \mathbf{R}^2$, $Q(\mathbf{p}_i) = Q(\mathbf{p}_i, x) = x^2 - p_{i1}x - p_{i2}$ is the i-th factor of $F(x) = Q(\mathbf{p}_i, x)F_i(x)$. If $\mathbf{u}_i = (u_{i1}, u_{i2})^T \in \mathbf{R}^2$ is an approximation of $\mathbf{p}_i$ and α_{i1}, α_{i2} are the roots of $Q(\mathbf{u}_i, x) = x^2 - u_{i1}x - u_{i2}$, $F_i(\alpha_{i1})F_i(\alpha_{i2}) \neq 0$, then we obtain $\mathbf{l}(Q(\mathbf{p}_i);\mathbf{u}_i) = \mathbf{u}_i - \mathbf{p}_i$. But $Q(\mathbf{p}_i) = F(x)/F_i(x)$, so

$$\mathbf{p}_i = \mathbf{u}_i - \mathbf{l}(F/F_i;\mathbf{u}_i).$$

For example, if $F(x)$ is given by $F(x) = \sum_{i=0}^{n} a_i x^{N-i}$, comparing the coefficients, we see that for any $\mathbf{u} = (u_1, u_2)^T \in \mathbf{R}^2$,

$$F(x) = (x^2 - u_1 x - u_2)G(x) + b_{N-1}(x - u_1) + b_N,$$

where $G(x) = \sum_{j=0}^{N-2} b_j x^{N-2-j}$ and b_j, $j = 0, 1, \ldots, N$ are determined by

$$b_j = a_j + u_1 b_{j-1} + u_2 b_{j-2},\ b_0 = a_0.$$

If $\mathbf{u}$ is a good approximation of $\mathbf{p}_i$, then b_N, b_{N-1} are close to 0, and we may approximate $F_i(x)$ by $G(x)$. This is just the Bairstow iteration. In the following we suppose that $F \in \mathbf{R}^{2n}$. Then there are $\mathbf{p}_i = (p_{i1}, p_{i2})^T \in \mathbf{R}^2$, $i = 1, \ldots, n$ such that

$$F(x) = a_0 \prod_{j=1}^{n} Q(\mathbf{p}_j, x) = a_0 \prod_{j=1}^{n} (x^2 - p_{j1}x - p_{j2})$$

$$= Q(\mathbf{p}_i, x)F_i(x) = Q(\mathbf{p}_i, x)a_0 \prod_{j\neq i}^{n} Q(\mathbf{p}_j, x).$$

Except when otherwise stated, we always suppose that parameter q is a natural number, and that subscripts i, j, k are evaluated $1, 2, \ldots, n$ in order and we denote by $m = 0, 1, \ldots$ the numbers of iteration steps and by $\mu = 1, \ldots, q$ the numbers of substeps from m-th to $(m+1)$-th step. Let $\mathbf{u_j}^{\left(m+\frac{\mu-1}{q}\right)}$ be the $\left(m+\frac{\mu-1}{q}\right)$-th approximation of $\mathbf{p_j}$. To obtain the $\left(m+\frac{\mu}{q}\right)$-th of $\mathbf{p_j}$, we approximate $F_i(x)$ by

$$G_i^{\left(m+\frac{\mu-1}{q}\right)}(x) = a_0 \prod_{j \neq i} Q(\mathbf{u_j}^{\left(m+\frac{\mu-1}{q}\right)}, x).$$

Therefore we have *Parallel Iteration* $P(q)$ (see Zheng [9]):

$$\mathbf{u_i}^{\left(m+\frac{\mu}{q}\right)} = \mathbf{u_i}^{(m)} - A(G_i^{\left(m+\frac{\mu-1}{q}\right)}; \mathbf{u_i}^{(m)})^{-1} \mathbf{l}(F; \mathbf{u_i}^{(m)}),$$

$$m = 0, 1, \ldots, \ \mu = 1, \ldots, q.$$

Let

(1) $$p(t) = \sum_{\nu=0}^{2n} a_\nu t^{2n-\nu}, \ a_0 = 1$$

be a monic polynomial of degree $2n$ with real coefficients. Then $p(t)$ can be factorized as

(2) $$p(t) = \prod_{j=1}^{n} (t^2 - p_j t - q_j),$$

where $p_j, q_j, j = 1, 2, \ldots, n$ are real. Weierstrass-Durand-Dochev-Kerner's method is a well-known parallel iteration for finding all zeros of the polynomial (1) (see [7], [1], [2], [8]). Dvorcuk [3] and Zheng [9] presented a parallel iteration and a family of parallel iteration methods $P(q)$ with parameter $q = 1, 2, \ldots$, respectively, for factorization into quadratic factors of (1). Both Dvorcuk's and Zheng's method can perform the calculation in real arithmetic only. Kjurkchiev [5] and [6] gave some properties for Weierstrass-Durand-Dochev-Kerner's and Dvorcuk's method, respectively.

2. Main result

Suppose that $(u_i, v_i)^T$ and $(u_i^+, v_i^+)^T$ are the k-th and $k+1$ -th approximation of (p_i, q_i), $i = 1, \ldots n$, respectively, produced by parallel iteration $P(1)$ in [9]. In this paper we introduce the concept of bisymmetric function at first and then some properties for the parallel iteration $P(1)$.

Definition. Let $u = (u_1, v_1, \ldots, u_n, v_n)^T \in R^{2n}$. We denote bisymmetric functions by

$$(3)\qquad \varphi_{l,m} = \sum u_{j_1} u_{j_2} \ldots u_{j_l} v_{k_1} v_{k_2} \ldots v_{k_m},\ \ l, m \geq 0,\ 1 \leq l+m \leq n,$$

where the summation is taken for $1 \leq j_1 < j_2 < \cdots < j_l \leq n$, $1 \leq k_1 < k_2 < \cdots < k_m \leq n$, $\{j_1, j_2, \cdots, j_l\} \cap \{k_1, k_2, \cdots, k_m\} = \phi$, i.e., every term in $\varphi_{l,m}(u)$ is a product of l u_j s and m v_k s and their subscripts are different each other. Naturally,

$$(4)\qquad \begin{aligned} &\varphi_{0,0} = 1, \\ &\varphi_{l,0} = \sum u_{j_1} u_{j_2} \ldots u_{j_l},\ \ 1 \leq l \leq n, \\ &\varphi_{0,m} = \sum v_{k_1} v_{k_2} \ldots v_{k_m},\ \ 1 \leq m \leq n, \\ &\varphi_{l,m} = 0,\ l < 0\ \ or\ m < 0\ \ or\ l+m > n. \end{aligned}$$

Theorem 1. *Let* $u = (u_1, v_1, \ldots, u_n, v_n)^T \in R^{2n}$, *and*

$$\prod_{j=1}^{n} (t^2 - u_j t - v_j) = \sum_{\nu=0}^{2n} b_\nu(u) t^{2n-\nu}.$$

The coefficient $b_\nu(u)$ *can be expressed as*

$$(5)\qquad b_\nu(u) = \sum_{m=0}^{[\frac{\nu}{2}]} (-1)^{\nu+m} \varphi_{\nu-2m,m}(u),\ \ \nu = 0, 1, 2, \ldots, 2n.$$

Theorem 2. *Suppose that*

$$u = (u_1, v_1, \ldots, u_n, v_n)^T \in R^{2n}$$
$$u^+ = (u_1^+, v_1^+, \ldots, u_n^+, v_n^+)^T \in R^{2n}$$

are the k*-th and* $k+1$*-th approximation of* $p = (p_1, q_1, p_2, q_2, \ldots, p_n, q_n)^T \in R^{2n}$, *respectively, produced by parallel iteration* $P(1)$. *We denote*

$$D_i u = (u_1, v_1, \ldots, u_{i-1}, v_{i-1}, u_{i+1}, v_{i+1}, \ldots, u_n, v_n)^T \in R^{2n-2},\ \ i = 1, \ldots, n.$$

Then u^+ *satisfies the system of linear equations*

$$(6)\qquad \begin{aligned} &\sum_{i=1}^{n} u_i^+ \sum_{m=0}^{[\frac{\nu-1}{2}]} (-1)^m \varphi_{\nu-2m-1,m}(D_i u) + \sum_{i=1}^{n} v_i^+ \sum_{m=1}^{[\frac{\nu}{2}]} (-1)^m \varphi_{\nu-2m,m-1}(D_i u) \\ &= \sum_{m=0}^{[\frac{\nu}{2}]} (\nu - m - 1) \varphi_{\nu-2m,m}(u) + (-1)^\nu a_\nu,\ \ \nu = 1, 2, \ldots, 2n, \end{aligned}$$

i.e.,

$$\sum_{i=1}^{n} u_i^{+} = -a_1,$$

$$\sum_{i=1}^{n} u_i^{+} \sum_{j\neq i} u_j - \sum_{i=1}^{n} v_i^{+} = \sum_{1\le j_1<j_2\le n} u_{j_1}u_{j_2} + a_2,$$

$$\sum_{i=1}^{n} u_i^{+}\Big(\sum_{\substack{1\le j_1<j_2\le n\\ j_1,j_2\neq i}} u_{j_1}u_{j_2} - \sum_{j\neq i}^{n} v_j\Big) - \sum_{i=1}^{n} v_i^{+}\sum_{j\neq i}^{n} u_j$$

$$= 2\sum_{1\le j_1<j_2<j_3\le n} u_{j_1}u_{j_2}u_{j_3} - \sum_{j\neq k}^{n} u_j v_k - a_3,$$

$$\sum_{i=1}^{n} u_i^{+}\Big(\sum_{\substack{1\le j_1<j_2<j_3\le n\\ j_1,j_2,j_3\neq i}} u_{j_1}u_{j_2}u_{j_3} - \sum_{\substack{1\le j,k\le n\\ j\neq k;\ j,k\neq i}} u_j v_k\Big)$$

$$+\sum_{i=1}^{n} v_i^{+}\Big(-\sum_{\substack{1\le j_1<j_2\le n\\ j_1,j_2\neq i}} u_{j_1}u_{j_2} + \sum_{k\neq i}^{n} v_k\Big)$$

$$= 3\sum_{1\le j_1<j_2<j_3<j_4\le n} u_{j_1}u_{j_2}u_{j_3}u_{j_4} - 2\sum_{1\le j,k\le n;\ j\neq k} u_j v_k + \sum_{1\le k_1<k_2\le n} v_{k_1}v_{k_2} + a_4,$$

$$\cdots$$

$$\sum_{i=1}^{n} v_i^{+}\prod_{k\neq i}^{n} q_i = (n-1)\prod_{j=1}^{n} q_j + (-1)^n a_{2n}.$$

Theorem 3. *Suppose that the initial approximation*

$$u = (u_1, v_1, \ldots, u_n, v_n)^T \in R^{2n}$$

of $p = (p_1, q_1, p_2, q_2, \ldots, p_n, q_n)^T \in R^{2n}$ *satisfies the conditions:*

1)

$$\sum_{m=0}^{[\frac{\nu}{2}]} (-1)^{m+1}(m+1-\nu)\varphi_{\nu-2m,m}(u) + (-1)^{\nu} a_{\nu} = 0,\ \ \nu = 1, 2, \ldots, 2n; \tag{7}$$

2) *The zeros of* $t^2 - u_i t - v_i$ *are not the zeros of* $t^2 - u_j t - v_j$, $i, j = 1, 2, \ldots, n;\ i \neq j$.

Then the parallel iteration $P(1)$ is not defined.

To prove the theorems we need some lemmas.

Lemma 1. *Let*

$$u = (u_1, v_1, \ldots, u_n, v_n)^T \in R^{2n}$$

$$D_i u = (u_1, v_1, \ldots, u_{i-1}, v_{i-1}, \ldots, u_{i+1}, v_{i+1}, \ldots, u_n, v_n)^T \in R^{2n-2},\ i = 1, 2, \ldots, n$$

and

$$\prod_{j=1}^{n}(t^2 - u_j t - v_j) = \sum_{\nu=0}^{2n} b_\nu(u) t^{2n-\nu}, \tag{8}$$

$$\prod_{j\neq i}^{n}(t^2 - u_j t - v_j) = \sum_{\nu=0}^{2n-2} b_\nu(D_i u) t^{2n-\nu-2},\quad i = 1, 2, \ldots, n. \tag{9}$$

Then

$$\begin{gathered} b_\nu(u) = b_\nu(D_i u) - u_i b_{\nu-1}(D_i u) - v_i b_{\nu-2}(D_i u), \\ i = 1, 2, \ldots, n;\ \nu = 1, 2, \ldots, 2n, \end{gathered} \tag{10}$$

where $b_\nu(D_i u) = 0$ if $\nu < 0$ or $\nu \geq 2n - 1$.

Proof. We see that from (8) and (9)

$$\begin{aligned} \sum_{\nu=0}^{n} b_\nu(u) t^{2n-\nu} &= (t^2 - u_i t - v_i) \sum_{\nu=0}^{2n-2} b_\nu(D_i u) t^{2n-\nu-2} \\ &= \sum_{\nu=0}^{2n-2} b_\nu(D_i u) t^{2n-\nu} - \sum_{\nu=0}^{2n-2} u_i b_\nu(D_i u) t^{2n-\nu-1} - \sum_{\nu=0}^{2n-2} v_i b_\nu(D_i u) t^{2n-\nu-2} \\ &= \sum_{\nu=0}^{2n-2} b_\nu(D_i u) t^{2n-\nu} - \sum_{\nu=1}^{2n-1} u_i b_{\nu-1}(D_i u) t^{2n-\nu} - \sum_{\nu=2}^{2n} v_i b_{\nu-2}(D_i u) t^{2n-\nu} \\ &= \sum_{\nu=0}^{2n} [b_\nu(D_i u) - u_i b_{\nu-1}(D_i u) - v_i b_{\nu-2}(D_i u)] t^{2n-\nu}. \end{aligned}$$

Comparing the coefficients of $t^{2n-\nu}$ of both sides above, we obtain (10). The lemma is proved. ■

Lemma 2. *Let $l \geq 0$, $m \geq 0$. Then*

$$\sum_{i=1}^{n} \varphi_{l,m}(D_i u) = (n - l - m)\varphi_{l,m}(u). \tag{11}$$

Proof. It is clear that (11) is valid when $l+m \geq n$ because both sides of (11) vanish. We now suppose $l+m < n$. From Definition, $\varphi_{l,m}(u)$ is the sum of all terms with form $u_{j_1}u_{j_2}\ldots u_{j_l}v_{k_1}v_{k_2}\ldots v_{k_m}$ and their subscripts are different each other and in $\{1,2,\ldots,n\}$. The number of the terms of $\varphi_{l,m}(u)$ is

$$N(\varphi_{l,m}(u)) = C_n^l.C_{n-l}^m = \frac{n!}{l!m!(n-l-m)!}.$$

Similarly, the form of the terms in $\varphi_{l,m}(D_i u)$ is the same to that in $\varphi_{l,m}(u)$ but their subscripts are in $\{1,2,\ldots,i-1,i+1,\ldots,n\}$ and the number of the terms is

$$N(\varphi_{l,m}(D_i u)) = C_{n-1}^l.C_{n-l-1}^m = \frac{(n-1)!}{l!m!(n-l-m-1)!} = \frac{(n-l-m)}{n}N(\varphi_{l,m}(u)).$$

Therefore, by the symmetry, the form of the terms in $\sum_{i=1}^{n}\varphi_{l,m}(D_i u)$ are the same to that in $\varphi_{l,m}(u)$ and their subscripts are also chosen from $\{1,2,\ldots,n\}$ and its number of the terms is $nN(\varphi_{l,m}(D_i u)) = (n-l-m)N(\varphi_{l,m}(u))$. This complete the proof of the lemma. ∎

Lemma 3. ([9]). *Under the conditions of Theorem 2 it holds*

$$p(t) = \sum_{i=1}^{n}[(u_i - u_i^+)t + (v_i - v_i^+)]\prod_{j\neq i}^{n}(t^2 - u_j t - v_j) + \prod_{j=1}^{n}(t^2 - u_j t - v_j).$$

3. Proof of the theorems

Proof of Theorem 1. Clearly, it holds (5) for $\nu = 0$ or $n = 1$. Suppose that (5) is true for $n-1$, i.e.,

$$\prod_{j=1}^{n-1}(t^2 - u_j t - v_j) = \sum_{\nu=0}^{2n-2} b_\nu(D_n u)t^{2n-\nu-2}$$

and

$$b_\nu(D_n u) = \sum_{m=0}^{[\frac{\nu}{2}]}(-1)^{\nu+m}\varphi_{\nu-2m,m}(D_n u), \quad \nu = 0,1,\ldots,2n-2. \tag{12}$$

Evidently, $\varphi_{\nu-2m,m}(D_i u) = 0$ for $\nu < 0$ or $\nu \geq 2n-1$ because of $(\nu-2m)+m = \nu - m \geq 2n-1-(n-1) = n > n-1$ and (12) is valid in these cases. From (10) and (12) we have for $\nu = 1,2,\ldots,2n$

$$
\begin{aligned}
b_\nu(u) &= \sum_{m=0}^{[\frac{\nu}{2}]}(-1)^{\nu+m}\varphi_{\nu-2m,m}(D_n u) + \sum_{m=0}^{[\frac{\nu-1}{2}]}(-1)^{\nu+m}u_n\varphi_{\nu-2m-1,m}(D_n u) \\
&+ \sum_{m=0}^{[\frac{\nu-2}{2}]}(-1)^{\nu+m+1}v_n\varphi_{\nu-2m-2,m}(D_n u).
\end{aligned}
\tag{13}
$$

The third summation above is:

$$
\sum_{m=1}^{[\frac{\nu}{2}]}(-1)^{\nu+m}v_n\varphi_{\nu-2m,m-1}(D_n u) = \sum_{m=0}^{[\frac{\nu}{2}]}(-1)^{\nu+m}v_n\varphi_{\nu-2m,m-1}(D_n u).
$$

It is clear that the superscript of the second summation of (13)

$$
\left[\frac{\nu-1}{2}\right] = \begin{cases} \frac{\nu-1}{2} = [\frac{\nu}{2}], & \nu \text{ odd}, \\ \frac{\nu}{2}-1, & \nu \text{ even}. \end{cases}
$$

When ν is even and $m = \frac{\nu}{2}$, $\varphi_{\nu-2m-1,m}(D_i u) = \varphi_{-1,m}(D_i u) = 0$. So the second summation of (13) can be written as $\sum_{m=0}^{[\frac{\nu}{2}]}$ for any ν. Therefore,

$$
\begin{aligned}
b_\nu(u) &= \sum_{m=0}^{[\frac{\nu}{2}]}(-1)^{\nu+m}[\varphi_{\nu-2m,m}(D_n u) + u_n\varphi_{\nu-2m-1,m}(D_n u) + v_n\varphi_{\nu-2m,m-1}(D_n u)] \\
&= \sum_{m=0}^{[\frac{\nu}{2}]}(-1)^{\nu+m}\varphi_{\nu-2m,m}(u).
\end{aligned}
$$

Thus, Theorem 1 is proved. ■

P r o o f o f T h e o r e m 2. We have from (1) and Lemma 3

$$
\begin{aligned}
\sum_{\nu=0}^{2n}a_\nu t^{2n-\nu} &= \sum_{i=1}^{n}[(u_i-u_i^+)t + (v_i-v_i^+)]\sum_{\nu=0}^{2n-2}b_\nu(D_i u)t^{2n-\nu-2} + \sum_{\nu=0}^{2n}b_\nu(u)t^{2n-\nu} \\
&= \sum_{i=1}^{n}(u_i-u_i^+)\sum_{\nu=1}^{2n-1}b_{\nu-1}(D_i u)t^{2n-\nu} + \sum_{i=1}^{n}(v_i-v_i^+)\sum_{\nu=2}^{2n}b_{\nu-2}(D_i u)t^{2n-\nu} \\
&+ \sum_{\nu=0}^{2n}b_\nu(u)t^{2n-\nu} \\
&= \sum_{\nu=0}^{2n}t^{2n-\nu}[\sum_{i=1}^{n}b_{\nu-1}(D_i u)(u_i-u_i^+) + \sum_{i=1}^{n}b_{\nu-2}(D_i u)(v_i-v_i^+) + b_\nu(u)].
\end{aligned}
$$

Comparing the coefficients of $t^{2n-\nu}$ of two sides of above equation we obtain from (10),

$$
\begin{aligned}
&\sum_{i=1}^{n} b_{\nu-1}(D_i u) u_i^{+} + \sum_{i=1}^{n} b_{\nu-2}(D_i u) v_i^{+} \\
&= \sum_{i=1}^{n} [b_{\nu-1}(D_i u) u_i + b_{\nu-2}(D_i u) v_i] + b_\nu(u) - a_\nu \\
&= \sum_{i=1}^{n} [b_\nu(D_i u) - b_\nu(u)] + b_\nu(u) - a_\nu \\
&= \sum_{i=1}^{n} b_\nu(D_i u) - (n-1) b_\nu(u) - a_\nu, \quad \nu = 1, 2, \ldots, 2n.
\end{aligned} \tag{14}
$$

We have by Theorem 1 and Lemma 2

$$
\begin{aligned}
&\sum_{i=1}^{n} b_{\nu-1}(D_i u) u_i^{+} + \sum_{i=1}^{n} b_{\nu-2}(D_i u) v_i^{+} \\
&= \sum_{i=1}^{n} u_i^{+} \sum_{m=0}^{[\frac{\nu-1}{2}]} (-1)^{\nu+m+1} \varphi_{\nu-2m-1,m}(D_i u) \\
&+ \sum_{i=1}^{n} v_i^{+} \sum_{m=0}^{[\frac{\nu-2}{2}]} (-1)^{\nu+m+2} \varphi_{\nu-2m-2,m}(D_i u) \\
&= \sum_{i=1}^{n} u_i^{+} \sum_{m=0}^{[\frac{\nu-1}{2}]} (-1)^{\nu+m+1} \varphi_{\nu-2m-1,m}(D_i u) \\
&+ \sum_{i=1}^{n} v_i^{+} \sum_{m=1}^{[\frac{\nu}{2}]} (-1)^{\nu+m+1} \varphi_{\nu-2m,m-1}(D_i u),
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
&\sum_{i=1}^{n} b_\nu(D_i u) - (n-1) b_\nu(u) - a_\nu \\
&= \sum_{m=0}^{[\frac{\nu}{2}]} \sum_{i=1}^{n} (-1)^{\nu+m} \varphi_{\nu-2m,m}(D_i u) - (n-1) \sum_{m=0}^{[\frac{\nu}{2}]} \varphi_{\nu-2m,m}(u) - a_\nu \\
&= \sum_{m=0}^{[\frac{\nu}{2}]} (-1)^{\nu+m} (m+1-\nu) \varphi_{\nu-2m,m}(u) - a_\nu.
\end{aligned} \tag{16}
$$

Then we obtain (6) from (14)-(16). The proof of Theorem 2 is completed. ■

P r o o f o f T h e o r e m 3. Under the conditions of Theorem 3, the system of linear equations (6) can be written as

$$
Au^{+} = 0.
$$

It can be proved that $\det(A) \neq 0$ in this case but we omit the details. Therefore, $u^+ = 0$. However, the condition 2) of theorem is necessary for parallel iteration $P(1)$. Theorem 3 is proved. ■

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