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Approximation Order and Asymptotic Approximation for Generalized Meyer-König and Zeller Operators [1]

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In this study, we define a sequence of generalized linear positive operators L_n, which includes the operators M_n constructed by W. Meyer-König and K. Zeller. Then we compute the difference $|L_n(f;x) - f(x)|$ with the help of asymptotic inequalities.

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1. Introduction

In this study, similarly to the generalization that we have made in [3] for operators defined in [5], we generalize the Meyer-König and Zeller operators defined in [12]. Then we prove some approximation properties of the operators mentioned above.

2. Sequence of generalized Meyer-König and Zeller operators

Let A be a real number in the interval $(0,1)$. Assume that a sequence of functions of $\{\varphi_n\}$ satisfies the following conditions:

1^0 Every function of sequence $\{\varphi_n\}$ is analytic on a domain D containing the disk $B = \{z \in \mathbf{C} : |z| \leq A\}$;

2^0 $\varphi_n^{(0)}(0) = \varphi_n(0) > 0$;

3^0 $\varphi_n^{(k)}(0) = \frac{d^k}{dx^k}\varphi_n(x)|_{x=0} \geq 0$, $k = 1, 2, ...$;

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4^0 $\varphi_n^{(k)}(0) = \gamma_n(k+n)(1+\ell_{n,k})\varphi_n^{(k-1)}(0)$, $k = 1, 2, ...,$ where, $\ell_{n,k}$ and γ_n are sequences of numbers satisfying the conditions

$$\ell_{n,k} = O\left(\frac{1}{n}\right), \ell_{n,k} \geq 0, \gamma_n = 1 + O\left(\frac{1}{n}\right) \text{ and } \gamma_n \geq 1 \quad (n, k = 1, 2, ...).$$

Consider the sequence of linear positive operators

$$(1) \qquad L_n(f; x) = \frac{1}{\varphi_n(x)} \sum_{k=0}^{\infty} f\left(\frac{k}{k+n}\right) \varphi_n^{(k)}(0) \frac{x^k}{k!},$$

where $f \in C[0, A]$.

R e m a r k. We shall show that the operator $L_n(f; x)$ contains, as a particular case, the following operators.

1. Let $\varphi_n(x) = (1-x)^{-n-1}$. Simple calculations show that in this case $\gamma_n = 1$, $\ell_{n,k} = 0$ and the operator $L_n(f; x)$ has the form

$$M_n(f; x) = (1-x)^{n+1} \sum_{k=0}^{\infty} f\left(\frac{k}{k+n}\right) \binom{n+k}{k} x^k.$$

Note that the operators M_n have been defined by W. Meyer-König and K. Zeller [12].

2. Let $\alpha_n\beta_n \geq 1$, $\lim_{n\to\infty} \alpha_n\beta_n = 1$, and

$$\varphi_n(x) = \frac{\alpha_n + \beta_n x}{(\alpha_n - \beta_n x)^{n+2}}.$$

Then, $A < \dfrac{\alpha_n}{\beta_n}$ and the condition to gives

$$\gamma_n = \alpha_n\beta_n, \quad \ell_{n,k} = \frac{2}{2k+n-1}.$$

In this case, the operator $L_n(f; x)$ has the following form

$$G_n(f; x) = \frac{(\alpha_n - \beta_n x)^{n+2}}{(n+1)\alpha_n^{n+1}(\alpha_n + \beta_n x)} \sum_{k=0}^{\infty} f\left(\frac{k}{k+n}\right) \left(\frac{\beta_n}{\alpha_n}\right)^k (n+2k+1) \binom{n+k}{k} x^k.$$

Now we can prove the following theorem.

Theorem 2.1. *The sequence of linear positive operators defined by (1) with the conditions $1^0 - 4^0$ converges uniformly to the function $f \in C[0, A]$ in $[0, A]$.*

Proof. It is enough to prove the conditions of Korovkin theorem which is

$$L_n(t^k;x) \longrightarrow x^k, \ \ k = 0,1,2 \ .$$

uniformly in $[0, A]$.

First, from

$$\frac{1}{\varphi_n(x)}\sum_{k=0}^{\infty}\varphi_n^{(k)}(0)\frac{x^k}{k!} = 1, \tag{2}$$

we have $L_n(1;x) = 1$.

Secondly, since

$$L_n(t;x) = \frac{1}{\varphi_n(x)}\sum_{k=1}^{\infty}\frac{k}{k+n}\varphi_n^{(k)}(0)\frac{x^k}{k!},$$

we get by 4^0,

$$L_n(t;x) = \frac{x\gamma_n}{\varphi_n(x)}\sum_{k=1}^{\infty}(1+\ell_{n,k})\,\varphi_n^{(k-1)}(0)\frac{x^{k-1}}{(k-1)!} \ .$$

Since $\ell_{n,k} = O\left(\frac{1}{n}\right)$, there exists a positive constant d such that $\ell_{n,k} \leq \frac{d}{n}$ for any k. Therefore, by using (2) we have

$$L_n(t;x) \leq x\gamma_n\left(1+\frac{d}{n}\right).$$

Thus,

$$L_n(t;x) - x \leq (\gamma_n - 1)x + \frac{dx\gamma_n}{n} \ . \tag{3}$$

On the other hand, we can also write

$$L_n(t;x) - x = x(\gamma_n - 1) + \frac{1}{\varphi_n(x)}\sum_{k=0}^{\infty}\ell_{n,k+1}\varphi_n^{(k)}(0)\frac{x^k}{k!} \tag{4}$$

and the conditions $\ell_{n,k+1} \geq 0$, $\gamma_n \geq 1$ (see 4^0) give

$$L_n(t;x) - x \geq 0 \ . \tag{5}$$

From (3) and (5) we can write

$$0 \leq L_n(t;x) - x \leq (\gamma_n - 1)x + \frac{dx\gamma_n}{n}. \tag{6}$$

Hence

$$\lim_{n\to\infty} L_n(t;x) = x$$

uniformly in $[0, A]$ and the second condition of Korovkin theorem is satisfied.

Finally, using 4^0 in $L_n(t^2;x)$, after simplifications, we obtain

$$L_n(t^2;x) = x^2\frac{1}{\varphi_n(x)}\sum_{k=2}^{\infty}\gamma_n\,(1+\ell_{n,k})\,\gamma_n\,(1+\ell_{n,k-1})\,\varphi_n^{(k-2)}(0)\frac{x^{k-2}}{(k-2)!}$$

$$+\frac{1}{\varphi_n(x)}\sum_{k=1}^{\infty}\frac{1}{k+n}\gamma_n\,(1+\ell_{n,k})\,\varphi_n^{(k-1)}(0)\frac{x^{k-1}}{(k-1)!}$$

$$\leq x^2\gamma_n^2\left(1+\frac{d}{n}\right)^2\frac{1}{\varphi_n(x)}\sum_{k=2}^{\infty}\varphi_n^{(k-2)}(0)\frac{x^{k-2}}{(k-2)!}$$

$$+\frac{x\gamma_n}{n}\left(1+\frac{d}{n}\right)\frac{1}{\varphi_n(x)}\sum_{k=1}^{\infty}\varphi_n^{(k-1)}(0)\frac{x^{k-1}}{(k-1)!}.$$

Using (2), we have

$$L_n(t^2;x) \leq x^2\gamma_n^2\left(1+\frac{d}{n}\right)^2+\frac{x\gamma_n}{n}\left(1+\frac{d}{n}\right)$$

and so

$$L_n(t^2;x)-x^2 \leq x^2(\gamma_n^2-1)+\frac{2dx^2\gamma_n^2+x\gamma_n}{n}+\frac{x^2\gamma_n^2d^2+x\gamma_nd}{n^2}. \tag{7}$$

On the other hand, it is clear that

$$L_n(t^2;x)-x^2 = x^2(\gamma_n^2-1)+x^2\gamma_n^2\frac{1}{\varphi_n(x)}\sum_{k=0}^{\infty}(\ell_{n,k+2}\ell_{n,k+1}+\ell_{n,k+2}+\ell_{n,k+1})\,\varphi_n^{(k)}(0)\frac{x^k}{k!}$$

$$+\gamma_nx\frac{1}{\varphi_n(x)}\sum_{k=0}^{\infty}\frac{1}{k+1+n}(1+\ell_{n,k+1})\,\varphi_n^{(k)}(0)\frac{x^k}{k!}. \tag{8}$$

Therefore,

$$0\leq L_n(t^2;x)-x^2 \leq x^2(\gamma_n^2-1)+\frac{2dx^2\gamma_n^2+x\gamma_n}{n}+\frac{x^2\gamma_n^2d^2+x\gamma_nd}{n^2}. \tag{9}$$

and

$$\lim_{n\to\infty} L_n(t^2;x) = x^2$$

uniformly in $[0, A]$.

The proof is completed. ■

Note that for Meyer-König and Zeller operators the inequality (9), was proved by Müller [13, p.61].

3. The order of approximation

In this section, we compute the approximation order of $L_n(f;x)$ to $f(x)$ with the help of asymptotic inequalities. For these computations we use Mamedov's theorem.

We use the following notations:

$D_I^{(2)}$: The space of all functions f such that f'' exists on the interval I.

$B_I^{(2)}$: The space of the functions such that f'' is bounded on the interval I.

Theorem 3.2. *Let $L_n(f;x)$ be a sequence of generalized linear positive operators defined by (1). Then for every $f \in D_{[0,A]}^{(2)}$, and for sufficiently large n the following asymptotic inequality holds*

$$(10)\quad |L_n(f;x) - f(x)| \leq \frac{x\gamma_n(2d\,|f'(x)| + (2d\gamma_n x - 2dx + 1)\,|f''(x)|}{2n} + O\left(\frac{1}{n}\right).$$

Proof. Due to the proof of Theorem 2.1, we obtain

$$L_n(1;x) = 1,$$

$$L_n(t;x) \leq x + \frac{dx\gamma_n}{n} + O\left(\frac{1}{n}\right),$$

$$L_n(t^2;x) \leq x^2 + \frac{2d\gamma_n^2 x^2 + \gamma_n x}{n} + O\left(\frac{1}{n}\right).$$

Moreover, setting $\tau_n^{[4]}(x) = L_n((t-x)^4;x)$, we get

$$0 \leq \tau_n^{[4]}(x) \leq x^4(\gamma_n - 1)^4 + \frac{4x^4\gamma_n d}{n}(\gamma_n - 1)^3 + \frac{6\gamma_n x^3}{n}(\gamma_n - 1)^2 + O\left(\frac{1}{n^2}\right).$$

Since $(\gamma_n - 1) = O\left(\frac{1}{n}\right)$,

$$\lim_{n \to infty} n\tau_n^{[4]}(x) = 0$$

holds uniformly on $[0, A]$.

This case corresponds to the equalities in Mamedov's theorem (see [13], p. 49) below:

$$\psi_1(x) = dx\gamma_n$$

$$\psi_2(x) = 2d\gamma_n^2x^2 + \gamma_n x$$

$$\varphi_n(x) = n.$$

Because of Mamedov's theorem, for $f(x) \in D^{(2)}_{[0,A]}$ as $n \to \infty$, the following asymptotic inequality is satisfied

$$|L_n(f;x) - f(x)| \leq \frac{2|f'(x)|\,|dx\gamma_n| + |f''(x)|\,|\gamma_n x(2d\gamma_n x - 2xd + 1)|}{2n} + O\left(\frac{1}{n}\right). \tag{11}$$

Using $dx\gamma_n \geq 0$ and $2d\gamma_n x - 2xd + 1 \geq 0$ in (11), we obtain (10).

Corollary of Theorem 3.2. *Let $f \in B^{(2)}_{[0,A]}$, then under the conditions of Theorem 3.2 for sufficiently large n,*

$$|L_n(f;x) - f(x)| \leq \frac{MA\gamma_n(2d + 2Ad(\gamma_n - 1) + 1)}{2n} + O\left(\frac{1}{n}\right),$$

where $M = \max\{M_1, M_2\}$, $M_1 = \max_{x\in[0,A]} |f'(x)|$, and $M_2 = \max_{x\in[0,A]} |f''(x)|$.

4. A generalization of r-th order of the sequence $\{L_n\}$ defined by (1)

By $C^{(r)}[0,A]$, we denote the set of the functions f having the continuous r-th derivative $f^{(r)}$ ($f^{(0)}(x) = f(x)$) on the segment $[0,A]$, $(0 < A < 1)$.

We consider a following generalization of the sequence of linear positive operators defined by (1)

$$L_n^{[r]}(f;x) = \frac{1}{\varphi_n(x)}\sum_{k=0}^{\infty}\sum_{i=0}^{r} f^{(i)}\left(\frac{k}{k+n}\right)\frac{\left(x - \frac{k}{k+n}\right)^i}{i!}\varphi_n^{(k)}(0)\frac{x^k}{k!}. \tag{12}$$

We call operators (12) the r-th order of the sequence (1). Note that this definition for linear positive operators was given in [7].

We can prove the following proposition;

Theorem 4.1. *Let $L_n^{[r]}(f;x)$ be a sequence of operators defined by (12). If $f^{(r)} \in Lip_M(\alpha)$ $(0 < \alpha \leq 1)$, then*

$$\left|L_n^{[r]}(f;x) - f(x)\right| \leq \frac{\alpha M}{r+\alpha}\frac{B(\alpha,r)}{(r-1)!}L_n\left(|x-t|^{r+\alpha};x\right), \tag{13}$$

where $B(\alpha,r)$ is a beta function and L_n is the sequence of linear positive operators defined in (1).

Proof. We can write

$$f(x) - L_n^{[r]}(f;x) = \frac{1}{\varphi_n(x)}\sum_{k=0}^{\infty}\left[f(x) - \sum_{i=0}^{r} f^{(i)}\left(\frac{k}{k+n}\right)\frac{\left(x-\frac{k}{k+n}\right)^i}{i!}\right]\varphi_n^{(k)}(0)\frac{x^k}{k!}. \tag{14}$$

Then from the Taylor's formula we get

$$f(x) - \sum_{i=0}^{r} f^{(i)}\left(\frac{k}{k+n}\right)\frac{\left(x-\frac{k}{k+n}\right)^i}{i!} =$$

$$= \frac{\left(x-\frac{k}{k+n}\right)^r}{(r-1)!}\int_0^1 (1-t)^{r-1}\left[f^{(r)}\left(\frac{k}{k+n} + t\left(x - \frac{k}{k+n}\right)\right) - f^{(r)}\left(\frac{k}{k+n}\right)\right]dt. \tag{15}$$

Since $f^{(r)} \in Lip_M(\alpha)$, we can write

$$\left|f^{(r)}\left(\frac{k}{k+n} + t\left(x - \frac{k}{k+n}\right)\right) - f^{(r)}\left(\frac{k}{k+n}\right)\right| \le M\, t^{\alpha}\left|x - \frac{k}{k+n}\right|^{\alpha}. \tag{16}$$

Also,

$$\int_0^1 (1-t)^{r-1}t^{\alpha}dt = B(1+\alpha, r) = \frac{\alpha}{\alpha+r}B(\alpha, r), \tag{17}$$

where $B(\alpha, r)$ is a beta function.

Using (16) and (17) in (15), we obtain

$$\left|f(x) - \sum_{i=0}^{r} f^{(i)}\left(\frac{k}{k+n}\right)\frac{\left(x-\frac{k}{k+n}\right)^i}{i!}\right| \le \frac{\alpha M}{r+\alpha}\frac{B(\alpha,r)}{(r-1)!}\left|x - \frac{k}{k+n}\right|^{r+\alpha}.$$

Substituting of this inequality in (14), we have (13).

5. A differential equation

In this section, we consider differential equation like the one given in Theorem 5.1 for the generalized Meyer-König and Zeller operators, that seems to be fundamental for the investigation of many kinds of linear positive operators.

We refer to some papers, in which equations analogous to this in Theorem 5.1 are given: May [11], Volkov [14], Alkemade [1].

Theorem 5.1. *Let*

$$g(t) = \frac{t}{1-t} \qquad (t \in [0, A],\ A < 1). \tag{18}$$

For each $x \in [0, A]$ and $f \in C[0, A]$, $L_n(f; x)$ as defined in (1), satisfies the differential equation

$$x\frac{d}{dx}L_n(f;x) = -\gamma_n(1+n)(1+\ell_{n,1})xL_n(f;x) + nL_n(fg;x). \tag{19}$$

Remark. Note that (19) is not a differential equation for $L_n(f;x)$ but rather a functional differential equation.

Proof. Since $f \in C[0, A]$, the power series on the right-hand side of (1) converges on $[0, A]$. Hence it is allowed to differentiate this series term by term in $[0, A]$. Thus,

$$\frac{d}{dx}L_n(f;x) = \frac{-\varphi_n'(x)}{\varphi_n^2(x)}\sum_{k=0}^{\infty} f\left(\frac{k}{k+n}\right)\varphi_n^{(k)}(0)\frac{x^k}{k!}$$
$$+\frac{1}{\varphi_n(x)}\sum_{k=1}^{\infty} f\left(\frac{k}{k+n}\right)\varphi_n^{(k)}(0)k\frac{x^{k-1}}{k!}.$$

Multiplying this equation by x and using $g\left(\frac{k}{k+n}\right) = \frac{k}{n}$ and 4^0, it follows that

$$x\frac{d}{dx}L_n(f;x) = \frac{-\gamma_n(1+n)(1+\ell_{n,1})x}{\varphi_n(x)}\sum_{k=0}^{\infty} f\left(\frac{k}{k+n}\right)\varphi_n^{(k)}(0)\frac{x^k}{k!}$$
$$+\frac{n}{\varphi_n(x)}\sum_{k=0}^{\infty} f\left(\frac{k}{k+n}\right)g\left(\frac{k}{k+n}\right)\varphi_n^{(k)}(0)\frac{x^k}{k!}.$$

Using (1) in this equation, we prove the theorem. ∎

Proposition 5.1. *$L_n(t;x)$ is a solution of the differential equation*

$$xy'(x) + [n + \gamma_n(1+n)(1+\ell_{n,1})x]\,y(x) = \gamma_n(1+n)(1+\ell_{n,1})x \quad (x \in [0, A]) \tag{20}$$

which satisfies the condition $y(0) = 0$.

Proof. Setting in (19) $f = 1 - t$ it follows that

$$x\frac{d}{dx}L_n(1-t;x) = -\gamma_n(1+n)(1+\ell_{n,1})xL_n(1-t;x) + nL_n(t;x). \tag{21}$$

Using the linearity of L_n and $L_n(1;x) = 1$ in (21), we obtain (20). ■

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