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A Generalization of the K Transform on Spaces of Generalized Functions

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In this paper, we study a generalization of the K-transform. For the kernel-function $t^{-\gamma}K_{\rho,\nu}(t)$ we show that it is a solution of two differential equations of fractional order. A new real inversion formula is given and a study is realized on some spaces of generalized functions, $\mathcal{F}_{p,\mu}$ and $\mathcal{F}'_{p,\mu}$, by employing the adjoint method.

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1. Introduction

The integral transform defined by

$$ {}_\gamma K_\nu^{(\rho)} f(t) = \int_0^\infty (t\tau)^{-\gamma} K_{\rho,\nu}(t\tau) f(\tau) d\tau \tag{1.1}$$

was introduced by J. Rodríguez [11]. Here $K_{\rho,\nu}(z)$ denotes the function

$$ K_{\rho,\nu}(z) = 2^{-\nu-1} z^\nu \eta\left[\rho, \nu+1; \left(\frac{z^2}{4}\right)^\rho\right], \tag{1.2}$$

where $\eta[\rho,\beta,z]$ is the function

$$ \eta[\rho,\beta,z] = \int_0^\infty \tau^{-\beta} e^{-\tau - z\tau^{-\rho}} d\tau, \tag{1.3}$$

with $\rho > 0$ and $|\arg z| < \pi/2$. This function has been studied in [5] and [6] and it generalizes the modified Bessel function of the third kind:

$$ K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty \tau^{-\nu-1} e^{-\tau - \frac{z^2}{4\tau}} d\tau, \quad (\mathrm{Re}\, z^2 > 0).$$

The asymptotic behaviour of the function $t^{-\gamma}K_{\rho,\nu}(t)$ is obtained from [5]: for $t \to 0^+$,

$$t^{-\gamma}K_{\rho,\nu}(t) = \begin{cases} \frac{2^{\nu-1}}{\rho}\Gamma(\nu/\rho)t^{-\gamma-\nu} & if \quad \mathrm{Re}\,\nu > 0 \\ \frac{2^{\nu-1}}{\rho}\Gamma(\nu/\rho)t^{-\gamma-\nu} + 2^{-\nu-1}\Gamma(-\nu)t^{\nu-\gamma} & if \quad \mathrm{Re}\,\nu = 0, \nu \neq 0 \\ -t^{-\gamma}\ln\frac{t}{2} & if \quad \nu = 0 \\ 2^{-\nu-1}\Gamma(-\nu)t^{\nu-\gamma} & if \quad \mathrm{Re}\,\nu < 0; \end{cases}$$

(1.4)

and for $t \to \infty$,

$$t^{-\gamma}K_{\rho,\nu}(t) \sim 2^{-\nu-1}\lambda_1 t^{\nu-\gamma-\frac{\rho}{\rho+1}(2\nu+1)} e^{-\lambda_2 t^{\frac{2\rho}{\rho+1}}}, \tag{1.5}$$

where $\lambda_1 = \left(\frac{2\pi}{\rho+1}\right)^{1/2} 2^{\frac{\rho}{\rho+1}(2\nu+1)} \rho^{\frac{2\nu+1}{2(\rho+1)}}$ and $\lambda_2 = (1+1/\rho)2^{\frac{2\rho}{\rho+1}}\rho^{\frac{1}{\rho+1}}$.

The ${}_\gamma K_\nu^{(\rho)}$-transform includes as a particular case for $\rho = 1, \gamma = -1/2$, the K- transform [14],[15] and for $\rho = n \in \mathbf{N}, \gamma = -1/2$ a variant of the K-transform [10].

The paper is organized as follows. Section 2 is devoted to a new real inversion formula for the ${}_\gamma K_\nu^{(\rho)}$- transformation. In Section 3 we obtain that the function $t^{-\gamma}K_{\rho,\nu}(t)$ is a solution of two differential equations of fractional order:

$$\begin{aligned} 2^{-1+2\rho}t^{\nu-\gamma-2\rho+1}Dt^{2\rho-2\nu}\mathcal{D}^{\rho}_{2,w}\,t^{\nu+\gamma}y(t) + \rho\, y(t) &= 0 \\ 2^{-2+4\rho}t^{\nu-\gamma-2\rho+1}Dt^{1-2\rho}Dt^{4\rho-2\nu}\mathcal{D}^{2\rho}_{2,w}\,t^{\nu+\gamma}y(t) - \rho^2 y(t) &= 0, \end{aligned} \tag{1.6}$$

where $\mathcal{D}^{\alpha}_{2,w} = (-1)^n D_2^n I^{n-\alpha}_{2,w}$ ($n = 1 + [\mathrm{Re}\,\alpha], \alpha \in \mathbf{C}$), $D_m = \frac{d}{dt^m} = m^{-1}t^{1-m}D$ and

$$I^{\alpha}_{m,w} f(t) = \frac{m}{\Gamma(\alpha)}\int_t^{\infty}(\xi^m - t^m)^{\alpha-1}\xi^{m-1}f(\xi)d\xi, \quad (\mathrm{Re}\,\alpha > 0,\ m > 0) \tag{1.7}$$

is the Erdélyi-Kober operator of fractional integration (see e.g. [4], [12]).

The Mellin transform is obtained in Section 4 for the spaces $\mathcal{F}_{p,\mu}$ [7]. Moreover, we define the ${}_\gamma K_\nu^{(\rho)} f$-transform on spaces $\mathcal{F}'_{p,\mu}$, using the adjoint method.

Finally, in Section 5 we investigate compositions of the ${}_\gamma K_\nu^{(\rho)} f$-transform with some differential operators and fractional calculus operators on spaces $\mathcal{F}_{p,\mu}$ and $\mathcal{F}'_{p,\mu}$.

For other related results on the subject one can see [2], [3].

2. A real inversion formula for ${}_{\gamma}K_{\nu}^{(\rho)}$-transformation

Nasim [9], in his studies on the convolution transforms, proved an inversion formula for the Meijer transformation. The method employs differential operators of infinite order. Here we use a similar procedure to obtain an inversion formula for the ${}_{\gamma}K_{\nu}^{(\rho)}$-transform and as special cases, inversion theorems for Laplace and Meijer transformations [1].

Theorem 2.1 *Let $f \in L_2(\mathbf{R}^+)$ such that $F(s) = \mathcal{M}\{f\}(s) \in L_1\left(\frac{1}{2} - i\infty, +\frac{1}{2}i\infty\right)$ and*

$$\int_0^1 t^{-\nu-\gamma}\,|f(t)|\,dt < \infty.$$

Given $G(y) = {}_{\gamma}K_{\nu}^{(\rho)}\{f\}(y)$ and $\delta = x\dfrac{d}{dx}$, define

$$H(x) = \rho \int_0^\infty (yx)^{\gamma+1} I_{\rho,\nu}(xy) G(y)dy,$$

where

$$I_{\rho,\nu}(z) = \left(\frac{z}{2}\right)^{\nu} \phi\left(\frac{1}{\rho}, \frac{\nu+1}{\rho}; -\frac{t^2}{4}\right), \qquad \phi(\rho,\beta;z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\rho n+\beta)}.$$

Then,

$$\frac{2}{\pi}\sin\left(\frac{\pi}{2}(\nu+\gamma+1-\delta)\right) H(x) = f(x)$$

for almost all $x > 0$, provided $-1 < \operatorname{Re}\nu < \frac{1}{2} - \operatorname{Re}\gamma$.

Proof. Note that the integral defining $G(y) = {}_{\gamma}K_{\nu}^{(\rho)}\{f\}(y)$ is absolutely convergent due to the hypotheses and the behaviour of $K_{\rho,\nu}$. Further, $H(x)$ can be rewritten as follows:

$$\begin{aligned} H(x) &= \rho \int_0^\infty (yx)^{\gamma+1} I_{\rho,\nu}(xy) G(y)dy \\ &= \rho \int_0^\infty (yx)^{\gamma+1} I_{\rho,\nu}(xy) \int_0^\infty (yt)^{-\gamma} K_{\rho,\nu}(yt) f(t)dtdy \\ &= \rho \int_0^\infty f(t)dt \int_0^\infty (yt)^{-\gamma} K_{\rho,\nu}(yt)(yx)^{\gamma+1} I_{\rho,\nu}(xy)dy. \end{aligned}$$

The interchange in the order of integration is justified by the absolute convergence of the corresponding double integral.

Now, by virtue of [11, p. 310, (2.5)], if $\operatorname{Re}\nu > -1$, by making a simple change of variable one has,

$$H(x) = \int_0^\infty t^{-1} f(t) k(x/t) dt,$$

where $k(u) = \dfrac{u^{\nu+\gamma+1}}{1+u^2} \in L^2(\mathbf{R}^+)$ if $\operatorname{Re}\nu + \operatorname{Re}\gamma < \frac{1}{2}$.

Hence, according to Parseval equality for Mellin transform (see [13, p. 60]), we obtain

$$H(x) = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} F(s) K(s)\, x^{-s} ds,$$

with $K(s) = \mathcal{M}\{k\}(s) = \frac{\pi}{2}\operatorname{cosec}\frac{\pi}{2}(s+\nu+\gamma+1)$ [1, p. 309].

Moreover, if $(K(\delta))^{-1}$ is understood as the differential operator of infinite order

$$(K(\delta))^{-1} = \frac{2}{\pi} \sin\frac{\pi}{2}(\nu+\gamma+1-\delta) = \lim_{l\to\infty} (\nu+\gamma+1-\delta) \prod_{k=1}^{l} \left(1 - \frac{(\nu+\gamma+1-\delta)^2}{4k^2}\right),$$

then, since $F(\frac{1}{2}+it) \in L_1(\mathbf{R})$, by applying the dominated convergence theorem we can conclude

$$(K(\delta))^{-1} H(x) = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} F(s)\, x^{-s} ds = f(x)$$

for almost all $x > 0$.

3. Solution of some differential equations

In this section, we show that the kernel $t^{-\gamma} K_{\rho,\nu}(t)$ of integral transformation (1.1) is a solution of the differential equations (1.6). For this, we begin with the following lemmas which can be found in [11].

Lemma 3.1 ([11, p.307, Prop. 1]) *Let* $\alpha \in \mathbf{C}$ $(\operatorname{Re}\alpha > 0)$ *and* $at^2 > 0$, *then*

(3.1) $$I^{\alpha}_{2,w}(e^{-at^2}) = a^{-\alpha} e^{-at^2}.$$

Lemma 3.2 ([11, p.307, Prop. 3]) *Define* $N_{\rho,\nu}(t) = t^{\nu} K_{\rho,\nu}(t)$. *Let* $\alpha \in \mathbf{C}$ $(\operatorname{Re}\alpha > 0)$, $\nu \in \mathbf{C}$, $n = 1 + [\operatorname{Re}\alpha]$ *and* $\rho > 0$. *Then*

(3.2) $$I^{n-\alpha}_{2,w} N_{\rho,\nu}(t) = 2^{n-\alpha} N_{\rho,\nu+n-\alpha}(t).$$

Lemma 3.3 ([11, p.308, Prop. 4]) *Let* $\alpha \in \mathbf{C}$ $(\mathrm{Re}\,\alpha > 0)$, $\nu \in \mathbf{C}$ *and* $\rho > 0$. *Then*

$$\mathcal{D}^{\alpha}_{2,w} N_{\rho,\nu}(t) = 2^{-\alpha} N_{\rho,\nu-\alpha}(t). \tag{3.3}$$

Corollary 3.1 *Let* $\alpha \in \mathbf{C}$ $(\mathrm{Re}\,\alpha > 0)$, $\beta \in \mathbf{C}(\mathrm{Re}\,\beta > 0)$, $\nu \in \mathbf{C}$ *and* $\rho > 0$. *Then*

$$\left(\mathcal{D}^{\alpha}_{2,w} I^{\beta}_{2,w} N_{\rho,\nu}\right)(t) = 2^{\beta-\alpha} N_{\rho,\nu+\beta-\alpha}(t), \tag{3.4}$$

$$\left(I^{\beta}_{2,w} \mathcal{D}^{\alpha}_{2,w} N_{\rho,\nu}\right)(t) = 2^{\beta-\alpha} N_{\rho,\nu+\beta-\alpha}(t). \tag{3.5}$$

Corollary 3.2 *Let* $\nu \in \mathbf{C}$, $\rho > 0$ *and* $m = 1, 2, \ldots$. *Then*

$$D^{m}_{2} N_{\rho,\nu}(t) = (-1)^m 2^{-m} N_{\rho,\nu-m}(t). \tag{3.6}$$

Corollary 3.3 *Let* α, $\beta \in \mathbf{C}$ $(\mathrm{Re}\,\alpha > 0,\ \mathrm{Re}\,\beta > 0)$, $\nu \in \mathbf{C}$ *and* $\rho > 0$. *Then*

$$\mathcal{D}^{\alpha}_{2,w}\mathcal{D}^{\beta}_{2,w} N_{\rho,\nu}(t) = \mathcal{D}^{\beta}_{2,w}\mathcal{D}^{\alpha}_{2,w} N_{\rho,\nu}(t) = \mathcal{D}^{\alpha+\beta}_{2,w} N_{\rho,\nu}(t).$$

Theorem 3.1 *If we denote* $\mathcal{L}^{\nu}_{\rho} = 2^{-1+2\rho} t^{\nu-\gamma-2\rho+1} D t^{2\rho-2\nu} \mathcal{D}^{\rho}_{2,w} t^{\nu+\gamma}$, *where* $\nu, \gamma \in \mathbf{C}$ *and* $\rho > 0$, *then*

$$\mathcal{L}^{\nu}_{\rho}(t^{-\gamma} K_{\rho,\nu}(t)) = -\rho t^{-\gamma} K_{\rho,\nu}(t). \tag{3.7}$$

P r o o f. By definition, we have

$$\begin{aligned}
\mathcal{L}^{\nu}_{\rho}\left(t^{-\gamma} K_{\rho,\nu}(t)\right) &= 2^{-1+2\rho} t^{\nu-\gamma-2\rho+1} D t^{2\rho-2\nu} \mathcal{D}^{\rho}_{2,w}\, t^{\nu} K_{\rho,\nu}(t) \\
&= 2^{-1+2\rho} t^{\nu-\gamma-2\rho+1} D t^{2\rho-2\nu} \mathcal{D}^{\rho}_{2,w} N_{\rho,\nu}(t) \\
&= 2^{2\rho} t^{-\nu-\gamma} \left[(\rho-\nu)\mathcal{D}^{\rho}_{2,w} - t^2 \mathcal{D}^{\rho+1}_{2,w}\right] N_{\rho,\nu}(t).
\end{aligned}$$

By (3.3), [11, p.308,(1.15)] and integrating by parts, we obtain

$$\begin{aligned}
\mathcal{L}^{\nu}_{\rho}\left(t^{-\gamma} K_{\rho,\nu}(t)\right) &= -2^{2\rho} t^{-\nu-\gamma} 2^{-\nu-1} \int_0^{\infty} \frac{d}{dx}\left(x^{\nu-\rho} e^{-\frac{t^2}{x}}\right) e^{-\left(\frac{x}{4}\right)^{\rho}} dx \\
&= -\rho t^{-\nu-\gamma} N_{\rho,\nu}(t) = -\rho t^{-\gamma} K_{\rho,\nu}(t).
\end{aligned}$$

Theorem 3.2 *Let* $\mathcal{J}_\rho^\nu = 2^{-2+4\rho} t^{\nu-\gamma-2\rho+1} D t^{1-2\rho} D t^{4\rho-2\nu} \mathcal{D}_{2,w}^{2\rho} t^{\nu+\gamma}$, *where* $\nu, \gamma \in \mathbf{C}$ *and* $\rho > 0$. *Then,*

$$\mathcal{J}_\rho^\nu (t^{-\gamma} K_{\rho,\nu}(t)) = \rho^2 t^{-\gamma} K_{\rho,\nu}(t). \tag{3.8}$$

P r o o f. By the definition we have

$$\begin{aligned}\mathcal{J}_\rho^\nu (t^{-\gamma} K_{\rho,\nu}(t)) &= 2^{4\rho} t^{-\gamma-\nu} \Big((\nu-\rho)(\nu-2\rho) \mathcal{D}_{2,w}^{2\rho} t^{\nu+\gamma} t^{-\gamma} K_{\rho,\nu}(t) \\ &-(1-2\nu+3\rho) t^2 \mathcal{D}_{2,w}^{2\rho+1} t^{\nu+\gamma} t^{-\gamma} K_{\rho,\nu}(t) + t^4 \mathcal{D}_{2,w}^{2\rho+2} t^{\nu+\gamma} t^{-\gamma} K_{\rho,\nu}(t) \Big) \\ &= 2^{4\rho} t^{-\gamma-\nu} \Big((\nu-\rho)(\nu-2\rho) \mathcal{D}_{2,w}^{2\rho} N_{\rho,\nu}(t) \\ &-(1-2\nu+3\rho) t^2 \mathcal{D}_{2,w}^{2\rho+1} N_{\rho,\nu}(t) + t^4 \mathcal{D}_{2,w}^{2\rho+2} N_{\rho,\nu}(t) \Big) \\ &= 2^{4\rho} t^{-\gamma-\nu} \Big((\nu-\rho)(\nu-2\rho) 2^{-2\rho} N_{\rho,\nu-2\rho}(t) \\ &-(1-2\nu+3\rho) t^2 2^{-2\rho-1} N_{\rho,\nu-2\rho-1}(t) + t^4 2^{-2\rho-2} N_{\rho,\nu-2\rho-2}(t) \Big) \\ &= 2^{-\nu-1+4\rho} t^{-\gamma-\nu} \int_0^\infty \left(\left(\nu - \rho + \frac{t^2}{x} \right) x^{\nu-2\rho} e^{-\frac{t^2}{x}} \right)' e^{-\left(\frac{x}{4}\right)^\rho} dx.\end{aligned}$$

The integration by parts gives

$$\mathcal{J}_\rho^\nu (t^{-\gamma} K_{\rho,\nu}(t)) = 2^{-\nu-1+2\rho} \rho t^{-\gamma-\nu} \int_0^\infty \left(\nu - \rho + \frac{t^2}{x} \right) x^{\nu-\rho-1} e^{-\frac{t^2}{x}} e^{-\left(\frac{x}{4}\right)^\rho} dx.$$

From the formula $\left(x^{\nu-\rho} e^{-\frac{t^2}{x}} \right)' = \left(\nu - \rho + \frac{t^2}{x} \right) x^{\nu-\rho-1} e^{-\frac{t^2}{x}}$ we obtain

$$\mathcal{J}_\rho^\nu (t^{-\gamma} K_{\rho,\nu}(t)) = 2^{-\nu-1+2\rho} \rho\, t^{-\gamma-\nu} \int_0^\infty \left(x^{\nu-\rho} e^{-\frac{t^2}{x}} \right)' e^{-\left(\frac{x}{4}\right)^\rho} dx.$$

Again, the integration by parts gives

$$\begin{aligned}\mathcal{J}_\rho^\nu (t^{-\gamma} K_{\rho,\nu}(t)) &= \rho^2 t^{-\gamma-\nu} 2^{-\nu-1} \int_0^\infty x^{\nu-1} e^{-\frac{t^2}{x}} e^{-\left(\frac{x}{4}\right)^\rho} dx \\ &= \rho^2 t^{-\gamma-\nu} N_{\rho,\nu}(t) = \rho^2 t^{-\gamma} K_{\rho,\nu}(t).\end{aligned}$$

4. ${}_{\gamma}K_{\nu}^{(\rho)}$-transform on spaces of generalized functions

A. McBride [7] defined $\mathcal{F}_{p,\mu}$ as follows: let $\mu \in \mathbf{C}$,

$$\mathcal{F}_{p,\mu} = \left\{\varphi \in C^{\infty}(\mathbf{R}^{+}) : x^{k}\frac{d^{k}}{dx^{k}}(x^{-\mu}\varphi(x)) \in L^{p}(\mathbf{R}^{+}), \forall k \in \mathbf{N}\right\},$$

with $1 \leq p < \infty$ and

$$\mathcal{F}_{\infty,\mu} = \left\{\varphi \in C^{\infty}(\mathbf{R}^{+}) : x^{k}\frac{d^{k}}{dx^{k}}(x^{-\mu}\varphi(x)) \to 0, \text{ where } x \to 0 \text{ and } x \to \infty, \forall k \in \mathbf{N}\right\}$$

if $p = \infty$. From In [8] it is seen that the space $\mathcal{F}_{p,\mu}$ is closely connected to the Banach space $L_{p,\mu}$ of Lebesgue measurable functions $f(x)$ such that

$$\|f\|_{p,\mu} = \left(\int_{0}^{\infty} |x^{\mu}f(x)|\,\frac{dx}{x}\right)^{1/p} < \infty.$$

Proposition 4.1 *Let* $1 \leq p \leq \infty$, $\mu \in \mathbf{C}$, $\nu \in \mathbf{C}$, $\gamma \in \mathbf{C}$, $\rho > 0$, $1/p + 1/p' = 1$ *and*

(4.1) $$\operatorname{Re}\mu > -\frac{1}{p'} + |\operatorname{Re}\nu| + \operatorname{Re}\gamma.$$

Then ${}_{\gamma}K_{\nu}^{(\rho)}$ *is a continuous linear mapping from* $L_{p,\mu}$ *into* $L_{p,2/p-\mu-1}$ *and from* $\mathcal{F}_{p,\mu}$ *into* $\mathcal{F}_{p,2/p-\mu-1}$.

Proof. By (1.4) and (1.5) the integral

$$\int_{0}^{\infty} x^{\operatorname{Re}\mu-1/p}\,|\mathcal{K}(x)|\,dx = \int_{0}^{\infty} x^{\operatorname{Re}\mu-1/p}\,|K_{\nu,\rho}(x)|\,dx$$

converges provided that (4.1) is satisfied. Then Proposition 4.1 follows from [7, pp. 158-159, Th. 8.1 and Cor. 8.2] and the proof is over. ∎

The Mellin transform $(\mathcal{M}\varphi)(s)$ of a suitable function $\varphi(t)$, $t > 0$, is defined by

(4.2) $$(\mathcal{M}\varphi)(s) = \int_{0}^{\infty} t^{s-1}\varphi(t)dt.$$

Lemma 4.1 *Let* $\rho > 0$, $\nu \in \mathbf{C}$, $\gamma \in \mathbf{C}$, $s \in \mathbf{C}$ *and*

(4.3) $$\operatorname{Re}s > \operatorname{Re}\gamma + |\operatorname{Re}\nu|.$$

Then

(4.4) $$\mathcal{M}(t^{-\gamma}K_{\rho,\nu}(t))(s) = \rho^{-1}2^{s-\gamma-2}\Gamma\left(\frac{s+\nu-\gamma}{2\rho}\right)\Gamma\left(\frac{s-\nu-\gamma}{2}\right).$$

Proof. The asymptotic behaviour of $K_{\rho,\nu}$ guarantees (4.4). By (4.2) and (1.2) we have after changing the order of integration

$$\begin{aligned}\mathcal{M}(t^{-\gamma}K_{\rho,\nu}(t))(s) &= 2^{-\nu-1}\int_0^\infty t^{s-1}t^{\nu-\gamma}\int_0^\infty \tau^{-\nu-1}e^{-\tau}e^{-\left(\frac{t^2}{4\tau}\right)^\rho}d\tau\, dt\\ &= 2^{-\nu-1}\int_0^\infty \tau^{-\nu-1}e^{-\tau}\int_0^\infty t^{s-1}t^{\nu-\gamma}e^{-\left(\frac{t^2}{4\tau}\right)^\rho}dt\, d\tau\\ &= 2^{-\nu-1}\int_0^\infty \tau^{-\nu-1}e^{-\tau}(2\rho)^{-1}(4\tau)^{\frac{s+\nu-\gamma}{2}}\Gamma\left(\frac{s+\nu-\gamma}{2\rho}\right)d\tau.\end{aligned}$$

The relation $\Gamma(z)=\int_0^\infty \tau^{z-1}e^{-\tau}d\tau$ $(\operatorname{Re} z>0)$ yields

$$\mathcal{M}(t^{-\gamma}K_{\rho,\nu}(t))(s)=\rho^{-1}2^{s-2-\gamma}\Gamma\left(\frac{s+\nu-\gamma}{2\rho}\right)\Gamma\left(\frac{s-\nu-\gamma}{2}\right)$$

and (4.4) is proved.

The Mellin transform $\mathcal{M}$ for $\varphi\in\mathcal{F}_{p,\mu}$ is defined by

$$(\mathcal{M}\varphi)(s)=\int_0^\infty t^{s-1}\varphi(t)dt,\qquad \operatorname{Re}s=1/p-\operatorname{Re}\mu. \tag{4.5}$$

By [8], we have for $1\le p\le 2$ and $\mu\in\mathbf{C}$ that $\mathcal{M}$ is a continuous linear mapping from $\mathcal{F}_{p,\mu}$ into $L_{p'}(\mathbf{R}^+)$.

Proposition 4.2 *Let $1\le p\le 2$, $\mu\in\mathbf{C}$, $\nu\in\mathbf{C}$, $\gamma\in\mathbf{C}$, $\rho>0$ and*

$$\operatorname{Re}\mu>-1/p'+|\operatorname{Re}\nu|+\operatorname{Re}\gamma,\ \operatorname{Re}s=1/p'+\operatorname{Re}\mu. \tag{4.6}$$

Then, for $\varphi\in\mathcal{F}_{p,\mu}$, the Mellin transform of ${}_\gamma K_\nu^{(\rho)}\varphi$ equals

$$\mathcal{M}\left({}_\gamma K_\nu^{(\rho)}\varphi\right)(s)=\rho^{-1}2^{s-\gamma-2}\Gamma\left(\frac{s+\nu-\gamma}{2\rho}\right)\Gamma\left(\frac{s-\nu-\gamma}{2}\right)(\mathcal{M}\varphi)(1-s), \tag{4.7}$$

where $s=\frac{1}{p}-\operatorname{Re}\mu+it$.

Proof. By Fubini's theorem and (4.4), for a sufficiently good function $\varphi\in C_0^\infty(\mathbf{R}^+)$, we have

$$\begin{aligned}\mathcal{M}\left({}_\gamma K_\nu^{(\rho)}\varphi\right)(s) &= \int_0^\infty t^{s-1}\int_0^\infty (t\tau)^{-\gamma}K_{\rho,\nu}(t\tau)\varphi(\tau)d\tau\, dt\\ &= \int_0^\infty \varphi(\tau)d\tau\int_0^\infty t^{s-1}(t\tau)^{-\gamma}K_{\rho,\nu}(t\tau)dt\\ &= \int_0^\infty \tau^{-s}\varphi(\tau)d\tau\int_0^\infty y^{s-1}y^{-\gamma}K_{\rho,\nu}(y)dy\\ &= \mathcal{M}\left(y^{-\gamma}K_{\nu,\rho}(y)\right)(s)(\mathcal{M}\varphi)(1-s)\\ &= \rho^{-1}2^{s-\gamma-2}\Gamma\left(\frac{s+\nu-\gamma}{2\rho}\right)\Gamma\left(\frac{s-\nu-\gamma}{2}\right)(\mathcal{M}\varphi)(1-s)\end{aligned}$$

and (4.7) is proved for $\varphi \in C_0^\infty(\mathbf{R}^+)$. By [7, p.18, Cor. 2.7], $C_0^\infty(\mathbf{R}^+)$ is dense in $\mathcal{F}_{p,\mu}$ and hence, the relation (4.7) holds for $\varphi \in \mathcal{F}_{p,\mu}$.

Proposition 4.3 *For* $1 \leq p \leq \infty$, $\mu \in \mathbf{C}$, $\nu \in \mathbf{C}$, $\gamma \in \mathbf{C}$, $\rho > 0$ *and* $\operatorname{Re}\mu > -1/p' + |\operatorname{Re}\nu| + \operatorname{Re}\gamma$, *we have*

$$\int_0^\infty \left({}_\gamma K_\nu^{(\rho)} f\right)(x)\varphi(x)dx = \int_0^\infty f(x)\left({}_\gamma K_\nu^{(\rho)}\varphi\right)(x)dx \tag{4.8}$$

holds for $\varphi \in \mathcal{F}_{p,\mu}$, $f \in \mathcal{F}_{p',\mu-1+2/p'}$ *and* $\varphi \in L_{p,\mu}$, $f \in L_{p',\mu-1+2/p'}$.

Proof. By Proposition 4.1, ${}_\gamma K_\nu^{(\rho)} f$ and ${}_\gamma K_\nu^{(\rho)}\varphi$ exists for $f \in \mathcal{F}_{p',\mu-1+2/p'}$ and $\varphi \in \mathcal{F}_{p,\mu}$, respectively, provided that (4.1) is valid. It is easily seen that the equality (4.8) is true for functions of $C_0^\infty(\mathbf{R}^+)$. Then, to prove (4.8) for $\varphi \in \mathcal{F}_{p,\mu}$, $f \in \mathcal{F}_{p',\mu-1+2/p'}$ and $\varphi \in L_{p,\mu}$, $f \in L_{p',\mu-1+2/p'}$, it is sufficient to show that both sides of (4.8) are bounded linear functional on $L_{p,\mu} \times L_{p',\mu-1+2/p'}$. Applying the Hölder inequality and the definition of the norm of $L_{p,\mu}$ we obtain

$$\int_0^\infty \left|\left({}_\gamma K_\nu^{(\rho)} f\right)(x)\varphi(x)\right| dx = \int_0^\infty |x^{-\mu}\varphi(x)| \left|x^\mu \left({}_\gamma K_\nu^{(\rho)} f\right)(x)\right| dx$$

$$\leq \left(\int_0^\infty |x^{-\mu}\varphi(x)|^p dx\right)^{1/p} \left(\int_0^\infty \left|x^\mu \left({}_\gamma K_\nu^{(\rho)} f\right)(x)\right|^{p'} dx\right)^{1/p'}$$

$$= \|\varphi\|_{p,\mu} \left\|{}_\gamma K_\nu^{(\rho)} f\right\|_{p',-\mu}.$$

By Proposition 4.1 with p replaced by p' and μ by $\mu - 1 + 2/p'$,

$$\left\|{}_\gamma K_\nu^{(\rho)} f\right\|_{p',-\mu} \leq k\, \|f\|_{p',\mu-1+2/p'} \quad (k > 0)$$

and hence

$$\left|\int_0^\infty \left({}_\gamma K_\nu^{(\rho)} f\right)(x)\varphi(x)dx\right| \leq k\, \|\varphi\|_{p,\mu} \|f\|_{p',\mu-1+2/p'}.$$

This shows that the left hand side of (4.8) is a bounded linear functional on $L_{p,\mu} \times L_{p',\mu-1+2/p'}$. The same result for the right hand side of (4.8) is proved similarly. This completes the proof of Proposition 4.3.

Proposition 4.3 allows to define the generalized ${}_\gamma K_\nu^{(\rho)} f$-transform on $\mathcal{F}'_{p,\mu}$ when $1 \leq p \leq \infty$, $\mu, \gamma, \nu \in \mathbf{C}$, as follows. For every $f \in \mathcal{F}'_{p,\mu}$ the generalized ${}_\gamma K_\nu^{(\rho)} f$-transform is defined through

$$< {}_\gamma K_\nu^{(\rho)} f, \varphi > = < f, {}_\gamma K_\nu^{(\rho)} \varphi > \tag{4.9}$$

with $\varphi \in \mathcal{F}_{p,2/p-\mu-1}$.

Then by Proposition 4.1 and by (4.9), we arrive at the following result.

Proposition 4.4 *Let* $1 \leq p \leq \infty$, $\mu \in \mathbf{C}$, $\nu \in \mathbf{C}$, $\gamma \in \mathbf{C}$, $\rho > 0$ *and* $\mathrm{Re}\,\mu < 1/p - |\mathrm{Re}\,\nu| - \mathrm{Re}\,\gamma$. *Then the operator* ${}_\gamma K_\nu^{(\rho)}$ *is a continuous linear mapping of* $\mathcal{F}'_{p,\mu}$ *into* $\mathcal{F}'_{p,2/p-\mu-1}$.

5. Composition of the ${}_\gamma K_\nu^{(\rho)}$-transform with some operators

Next, we investigate the composition of the ${}_\gamma K_\nu^{(\rho)} f$- transform with some differential operators and fractional calculus operators on McBride's spaces $\mathcal{F}_{p,\mu}$ and $\mathcal{F}'_{p,\mu}$.

Proposition 5.1. *Suppose* $1 \leq p \leq \infty$, $\mu, \nu, \gamma \in \mathbf{C}$, $\rho > 0$. *For every* $\varphi \in \mathcal{F}_{p,\mu}$ *we have*

(a) $D_2^m \left(x^{\nu+\gamma}\, {}_\gamma K_\nu^{(\rho)}(t^{-2m}\varphi(t))(x)\right) = (-1)^m 2^{-m} x^{\gamma+\nu-m}\, {}_\gamma K_{\nu-m}^{(\rho)}\,(t^{-m}\varphi(t))\,(x)$ *for* $m \in \mathbf{N}$ *and* $\mathrm{Re}\,\mu > -1/p' + |\mathrm{Re}\,\nu| + \mathrm{Re}\,\gamma + 2m$.

(b) $x^{-2m}\, {}_\gamma K_\nu^{(\rho)}\,(t^{\gamma+\nu}(2^{-1}Dt^{-1})^m\varphi(t))\,(x) = 2^{-m}x^{-m}\, {}_\gamma K_{\nu-m}^{(\rho)}\,(t^{\nu+\gamma-m}\varphi(t))\,(x)$ *for* $m \in \mathbf{N}$ *and* $\mathrm{Re}\,\mu > -1/p' + |\mathrm{Re}\,\nu| - \mathrm{Re}\,\nu + 2m$.

(c) $I_{2,w}^{\alpha}(t^{\nu+\gamma}\, {}_\gamma K_\nu^{(\rho)}\varphi)(x) = x^{\nu+\gamma+\alpha}\, {}_\gamma K_{\nu+\alpha}^{(\rho)}(t^{-\alpha}\varphi(t))(x)$ *when* $\mathrm{Re}\,\alpha > 0$ *and* $\mathrm{Re}\,\mu > -1/p' + \mathrm{Re}\,\gamma + |\mathrm{Re}\,\nu| + 2\,\mathrm{Re}\,\alpha$.

(d) $\mathcal{D}_2^{\alpha}\left(x^{\nu+\gamma}\, {}_\gamma K_\nu^{(\rho)}\varphi(x)\right) = (-1)^{1+[\alpha]}2^{-(1+[\alpha])}x^{\nu+\gamma-\alpha}\, {}_\gamma K_{\nu-\alpha}^{(\rho)}\,(t^{\alpha}\varphi(t))\,(x)$ *when* $\mathrm{Re}\,\alpha > 0$ *and* $\mathrm{Re}\,\mu > -1/p' + \mathrm{Re}\,\gamma + |\mathrm{Re}\,\nu| + 2\,\mathrm{Re}\,\alpha$.

P r o o f. We shall prove (a) and (c). The equalities in (b) and (c) can be proved in a similar way. According to Proposition 4.1 and [7, p.21, Th. 2.11 and p.26, Cor. 2.15] the left and right hand sides of the equality of (a) are continuous linear mappings from $\mathcal{F}_{p,\mu}$ into $\mathcal{F}_{p,2/p-\mu+\nu+\gamma-1}$ provided that thecondition of (a) holds, applying (1.1) and (3.6), we have

$$D_2^m\left(\int_0^\infty x^{\nu+\gamma}(xt)^{-\gamma}K_{\rho,\nu}(xt)t^{-2m}\varphi(t)dt\right)$$

$$= D_2^m\left(\int_0^\infty x^{\nu}t^{\nu}K_{\rho,\nu}(xt)t^{-\nu-\gamma}t^{-2m}\varphi(t)dt\right)$$

$$= \int_0^\infty t^{-2m}D_2^m N_{\rho,\nu}(xt)t^{-\nu-\gamma}\varphi(t)dt$$

$$= \int_0^\infty t^{-2m}\left(2^{-1}x^{-1}\frac{d}{dx}\right)^m N_{\rho,\nu}(xt)t^{-\nu-\gamma}\varphi(t)dt.$$

Changing the variables $xt = u$, we obtain

$$\begin{aligned}
&= \int_0^\infty D_2^m N_{\rho,\nu}(u)\,(u/x)^{-\nu-\gamma}\,\varphi(u/x)\,\frac{du}{x}\\
&= (-1)^m 2^{-m}\int_0^\infty N_{\rho,\nu-m}(xt)t^{-\nu-\gamma}\varphi(t)dt\\
&= (-1)^m 2^{-m}\int_0^\infty (xt)^{\nu-m}K_{\rho,\nu-m}(xt)t^{-\nu-\gamma}\varphi(t)dt\\
&= (-1)^m 2^{-m}x^{\nu+\gamma-m}\int_0^\infty (xt)^{-\gamma}K_{\rho,\nu-m}(xt)t^{-m}\varphi(t)dt
\end{aligned}$$

which proves (a).

By Proposition 4.1, [7, p.21, Th. 2.11 and p.56, Th. 3.23] and the condition of (c), the left and right hand sides of the equality of (c) are continuous linear mappings from $\mathcal{F}_{p,\mu}$ into $\mathcal{F}_{p,2/p-\mu+\gamma+\nu+2\alpha-1}$. For $\varphi \in C_0^\infty(\mathbf{R}^+)$, we get

$$I_{2,w}^\alpha(t^{\nu+\gamma}\,{}_\gamma K_\nu^{(\rho)}\varphi)(x) = \frac{2}{\Gamma(\alpha)}\int_x^\infty t(t^2-x^2)^{\alpha-1}t^{\nu+\gamma}\,{}_\gamma K_\nu^{(\rho)}\varphi(t)dt$$

$$\begin{aligned}
&= \frac{2}{\Gamma(\alpha)}\int_x^\infty t(t^2-x^2)^{\alpha-1}\int_0^\infty (t\tau)^{-\gamma}t^{\nu+\gamma}K_{\rho,\nu}(t\tau)\varphi(\tau)d\tau dt\\
&= \frac{2}{\Gamma(\alpha)}\int_x^\infty t(t^2-x^2)^{\alpha-1}\int_0^\infty \tau^{-\gamma}(t\tau)^{\nu}K_{\rho,\nu}(t\tau)\tau^{-\nu}\varphi(\tau)d\tau dt\\
&= \frac{2}{\Gamma(\alpha)}\int_x^\infty t(t^2-x^2)^{\alpha-1}\int_0^\infty 2^{-\nu-1}\int_0^\infty s^{\nu-1}e^{-\frac{(t\tau)^2}{s}}e^{-(s/4)^\rho}ds\,\tau^{-\gamma-\nu}\varphi(\tau)d\tau dt.
\end{aligned}$$

Invoking $I_{2,w}^\alpha e^{-\frac{\tau^2}{s}t^2} = \left(\frac{\tau^2}{s}\right)^{-\alpha}e^{-\frac{\tau^2}{s}x^2}$ and changing the order of integration,

$$\begin{aligned}
&= \int_0^\infty \tau^{-\gamma-\nu}\varphi(\tau)d\tau\,2^{-\nu-1}\int_0^\infty s^{\nu-1}\frac{2}{\Gamma(\alpha)}\int_x^\infty t(t^2-x^2)^{\alpha-1}e^{-\frac{(t\tau)^2}{s}}dt\,e^{-(s/4)^\rho}ds\\
&= \int_0^\infty \tau^{-\gamma-\nu}\varphi(\tau)d\tau\,2^{-\nu-1}\int_0^\infty s^{\nu-1}\tau^{-2\alpha}s^\alpha e^{-\frac{(x\tau)^2}{s}}e^{-(s/4)^\rho}ds\\
&= \int_0^\infty \tau^{-\gamma-\nu-2\alpha}\varphi(\tau)\,N_{\rho,\nu+\alpha}(x\tau)d\tau\\
&= \int_0^\infty \tau^{-\gamma-\nu-2\alpha}\varphi(\tau)\,(x\tau)^{\nu+\alpha}K_{\rho,\nu+\alpha}(x\tau)d\tau\\
&= x^{\gamma+\nu+\alpha}\int_0^\infty (x\tau)^{-\gamma}K_{\rho,\nu+\alpha}(x\tau)\tau^{-\alpha}\varphi(\tau)d\tau = x^{\gamma+\nu+\alpha}\,{}_\gamma K_{\nu+\alpha}^{(\rho)}(\tau^{-\alpha}\varphi(\tau))(x).
\end{aligned}$$

Since by [7, p.18, Cor. 2.7], $C_0^\infty(\mathbf{R}^+)$ is dense in $\mathcal{F}_{p,\mu}$, the result is obtained for $\varphi \in \mathcal{F}_{p,\mu}$. ∎

For the next result we first recall the definition of the Erdélyi-Kober operators of fractional calculus $I^{\alpha}_{2,l} f$, (1.7), for $f \in \mathcal{F}'_{p,\mu}$ given in [7, p.77, Def. 3.51 and Th. 3.52].

For $\alpha \in \mathbf{C}$ and $2 - \operatorname{Re}\mu \neq 1/p' - 2l$, $l = 0, 1, 2, \ldots$, we define $I^{\alpha}_{2,l} f$ as

$$< I^{\alpha}_{2,l} f, \varphi > = < f, x I^{\alpha}_{2,w} x^{-1} \varphi > .$$

Moreover, $I^{\alpha}_{2,l}$ is continuous linear mapping from $\mathcal{F}'_{p,\mu}$ into $\mathcal{F}'_{p,\mu-2\alpha}$.

Proposition 5.2. *Let* $1 \leq p \leq \infty$, $\mu, \nu, \gamma \in \mathbf{C}$, $\rho > 0$. *For every* $f \in$ *and* $\mathcal{F}'_{p,\mu}$ *we have*

(a) $x^{-2m}\, {}_{\gamma}K^{(\rho)}_{\nu} (t^{\gamma+\nu}(2^{-1}Dt^{-1})^m f(t))(x) = 2^{-m} x^{-m}\, {}_{\gamma}K^{(\rho)}_{\nu-m} (t^{\gamma+\nu-m} f(t))(x)$ *for* $m \in \mathbf{N}$ *provided* $\operatorname{Re}\mu < 1/p - 2m - |\operatorname{Re}\nu| + \operatorname{Re}\nu$.

(b) $D^m_2 x^{\nu+\gamma}\, {}_{\gamma}K^{(\rho)}_{\nu} (t^{-2m} f(t))(x) = (-1)^m 2^{-m} x^{\nu+\gamma-m}\, {}_{\gamma}K^{(\rho)}_{\nu-m} (t^{-m} f(t))(x)$ *for* $m \in \mathbf{N}$ *when* $\operatorname{Re}\mu < 1/p - 2m - |\operatorname{Re}\nu| - \operatorname{Re}\gamma$.

(c) ${}_{\gamma}K^{(\rho)}_{\nu} \left(t^{\nu+\gamma+1} I^{\alpha}_{2,l} f(t)\right)(x) = x^{-\alpha}\, {}_{\gamma}K^{(\rho)}_{\nu+\alpha} (t^{\nu+\gamma+\alpha+1} f(t))(x)$ *when* $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\mu < 1 + 1/p - |\operatorname{Re}\nu| + \operatorname{Re}\nu$, *and* $\operatorname{Re}\mu \neq 1 + 1/p + 2l$, $l \in \mathbf{N}$.

(d) ${}_{\gamma}K^{(\rho)}_{\nu} \left(t^{\nu+\gamma+1} I^{1+[\alpha]-\alpha}_{2,l} t^{-1} (2^{-1}Dt^{-1})^{1+[\alpha]} f(t)\right)(x) = (-1)^{1+[\alpha]} 2^{-1-[\alpha]} t^{\alpha}\, {}_{\gamma}K^{(\rho)}_{\nu-\alpha} x^{\nu+\gamma-\alpha} f$ *for* $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\mu < 1/p - 2\operatorname{Re}\alpha + \operatorname{Re}\nu - |\operatorname{Re}\nu|$, *and* $\operatorname{Re}\mu \neq 1/p + 2(l - 1 - [\alpha])$, $l \in \mathbf{N}$.

Proof. As in the previous proposition, we shall prove (a) and (c) since the equalities in (b) and (c) can be proved in a similar way. By the condition of (a), [7, p. 32, Th. 2.22] and Proposition 4.4, the equalities of (a) are continuous linear mappings from $\mathcal{F}'_{p,\mu}$ into $\mathcal{F}'_{p,2/p-\mu+\gamma+\nu-1}$.

By (4.9) and [7, p.32, Th. 2.22] we have

$$< x^{-2m}\, {}_{\gamma}K^{(\rho)}_{\nu} t^{\gamma+\nu} (2^{-1}Dt^{-1})^m f, \varphi > = < f, (-1)^m D^m_2 x^{\nu+\gamma}\, {}_{\gamma}K^{(\rho)}_{\nu} t^{-2m} \varphi >$$

and by Proposition 5.1, (c) and [7, p.32, Th. 2.22] we get

$$= < f, (-1)^{2m} 2^{-m} x^{\gamma+\nu-m}\, {}_{\gamma}K^{(\rho)}_{\nu-m}\, t^{-m} \varphi > = < 2^{-m} x^{-m}\, {}_{\gamma}K^{(\rho)}_{\nu-m}\, t^{\gamma+\nu-m} f, \varphi > .$$

This proves (a).

By the condition of (c), Proposition 4.4 and [7, p.32, Th. 2.22 and p.77, Th. 3.52], the left and right hand sides of the equality of (c) are continuous linear mappings from $\mathcal{F}'_{p,\mu}$ into $\mathcal{F}'_{p,2/p-\mu+2\alpha+\nu+\gamma}$.

By (4.9) and [7, p.32, Th. 2.22 and p.77, Def.3.51] we have

$$< {}_\gamma K_\nu^{(\rho)} t^{\nu+\gamma+1} I_{2,l}^\alpha f, \varphi > = < f, x I_{2,w}^\alpha x^{-1} x^{\nu+\gamma+1} {}_\gamma K_\nu^{(\rho)} \varphi >,$$

and by Proposition 5.1 (c), (4.9) and [7, p.32, Th. 2.22 and p.77, Def.3.51] it follows that

$$= < f, x\, x^{\nu+\gamma+\alpha} {}_\gamma K_{\nu+\alpha}^{(\rho)} (t^{-\alpha}\varphi(t)) > = < x^{-\alpha} {}_\gamma K_{\nu+\alpha}^{(\rho)} x^{\nu+\alpha+1} f, \varphi > .$$

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