Chain Ring Analogues of Some Theorems from Extremal Set Theory

IVAN LANDJEV
Institute of Mathematics and Informatics,
Bulgarian Academy of Sciences,
8 Acad. G. Bonchev str., 1113 Sofia

We present chain ring analogues of two theorems from classical extremal set theory – the theorems of Sperner and Erdős-Ko-Rado.

1. Let R be a finite chain ring with $|R| = q^m$, $R/\text{rad}R \cong F_q$. Every submodule M of ${}_RR^n$ is isomorphic to a direct sum of cyclic modules:

$$M \cong R/N^{\lambda_1} \oplus R/N^{\lambda_2} \oplus \ldots \oplus R/N^{\lambda_n},$$

where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$ are integers and $N = \text{rad}R$. The n-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is called the shape of M. We consider the partially ordered set $\mathcal{P}$ of all submodules contained in a module of shape $\lambda = (\lambda_1, \ldots, \lambda_n)$. We prove the following anlogue of Sperner's theorem [4]:

Theorem A. Let $M = {}_RR^n$. The size of a maximal antichain in $\mathcal{P}$ is equal to

$$\begin{bmatrix} \boldsymbol{m}^n \\ \boldsymbol{m}^{n/2} \end{bmatrix}_{q^m},$$

where $\boldsymbol{m}^n = (\underbrace{m, \ldots, m}_{n})$, and $\begin{bmatrix} \lambda \\ \mu \end{bmatrix}_{q^m}$ denotes the number of all modules of shape μ contained in a module of shape λ.

2. The next theorem is an analogue of classical results by Erdős-Ko-Rado, Hsieh, Frankl, Wilson and Tanaka [1, 3, 6, 2, 5].

Theorem B. Let R be a finite chain ring with nilpotency index m and residue field of order q. Denote by Σ the n-dimensional (left) projective Hjelmslev geometry over R. Let $\mathcal{F}$ be a family of k-dimensional Hjelmslev subspaces every two of which meet in at least one point. If $n \leq 2k + 1$ then

$$|\mathcal{F}| \leq \begin{bmatrix} \boldsymbol{m}^n \\ \boldsymbol{m}^k \end{bmatrix}_{q^m}.$$

In case of equality S is one of the following:

- all the Hjelmslev subspaces through a fixed point,
- in case of $n = 2k + 1$, all Hjelmslev k-subspaces in a fixed hyperplane of Σ.

References

[1] P. Erdős, C. Ko, R. Rado, Intersection theorems for systems of finite sets, *Duart J. Math. Oxford Ser. (2)* 12(1961), 313–320.

[2] P. Frankl, R. M. Wilson, the Erdős-Ko-Rado theorem for vector spaces, *J. Combin. Theory Ser. A* 43(2)(1986), 228–236.

[3] W. N. Hsieh, Intersection systems for systems of finite vector spaces, *Discrete Math.* 12(1)(1975), 1–16.

[4] E. Sperner, Ein Satz über Untermengen einer endlichen Menge, *Math. Z.* **27**(1)(1928), 544–548.

[5] H. Tanaka, Classification of subsets with minimal width and dual width in Grassmann, bilinear forms and dual, polar graphs, *J. Combin. Theory Ser. A* 113(5)(2006), 903–910.

[6] R. M. Wilson, The exact bound in the Erdős-Ko-Rado theorem, *Combinatorica* 4(2-3)(1984), 247–257.