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Union of Mathematicians in Bulgaria, Institute of Mathematics and Informatics

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COUNTEREXAMPLES, PARADOXES AND SURPRISES IN MATHEMATICS

Remarkable occasions in 2022:

- 75 years of the foundation of the Math. Institute, Bulgarian Acad. of Sciences;
- 50 years ago, 15 May 1972, the Institute moved to its 'new home', Block 8.

On that day, in the well-organized Library, 3rd floor, among the newest materials, I found the paper ***How to write Mathematics?***, by Paul Halmos. This forever valuable master piece was translated into BG and distributed around. The main: ***what, to whom,*** and ***how***!;

- *International Center for Mathematical Sciences-Sofia*, Inauguration, July'22;
- UN: 2022 'International Year of Basic Sciences for Sustainable Development.'

Today: Discussion on traditional objects: numbers, functions, equations, etc.

Emphasis: On three notions: **counterexamples, paradoxes, surprises,** only in the context of Mathematics. No Philosophy, no pathologies.

If a result is of the type **iff**, the situation is clear: change the condition, lose the result. Hence think of statements which are proved under specified conditions. Ask if a sufficient condition is necessary, and visa versa. Can you relax a condition? Can you prove a stronger statement under the same condition?

- ***Counterexample*** = statement which is related to another statement, perhaps being opposite, or clarifying the role of the conditions, etc.
- ***Paradox*** = statement which is true despite thinking it is wrong.
- ***Surprise*** = true statement, or method of proof, which is unexpected.

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***Counterexamples,* when and why ?** (abbr. CE; star * means 'no Math')
Sometimes the name is conditional. Important is the content!

STATEMENT*: All men in this room are with clean shoes. Sorry ... **CE !**

STATEMENT*: No one in this room has been accused as a killer. True? **CE !**

STATEMENT*: The impact factor is the greatest criterion ... ? **CEs !**

STATEMENT: If there are more than 23 people, with probability more than 50%, at least two people have the same birthday! This is the **Birthday paradox**.

STATEMENT: Given a unit square in the plane. Is it possible all four distances from any point in the square to the four vertices to be rational numbers? **Surprise** ! The answer is 'no', and, can you imagine, this is an open problem!

Classical items for rehearsal, or refreshing our memory:

• Any differentiable function is continuous. Is the converse true? Answer: No !
Weierstrass function:

$W_{a,b}(x) = \sum_{n=0}^{\infty} a^n \cos(\pi b^n x)$, $0 < a < 1$, b is an odd integer, $ab > 1 + \frac{3\pi}{2}$.

Takagi function: $\tau(x) := \sum_{n=0}^{\infty} \frac{1}{2^n}\langle\langle 2^n x\rangle\rangle$, $\langle\langle x\rangle\rangle$ is the distance from x to the nearest integer; can change 2 by any integer r. All are continuous, but not differentiable.

• Knowing and enjoying properties of functions which are continuous ⇒
Question: Is the set of *semicontinuous* functions a linear space? Answer 'no'! There are F, G, H, each s.c., however, their sum $F + G + H$ is nowhere s.c.

• A seq. $f_n(x), x \in \mathbb{R}, n \in \mathbb{N}$ may converge on $\mathbb{R}$, however, non-uniformly on any closed bounded interval $[a, b]$. Hint: Take $f_n(x) = nx\,\mathrm{e}^{-n^2 x^2}$, $x \in \mathbb{R}$.

• Given: a seq. $f_n \in \mathcal{C}^\infty$ converges uniformly to f. Is it true that f_n' converge to f' as $n \to \infty$? Answer: not always! Hint: $f_n(x) = \frac{\sin(nx)}{\sqrt{n}}$, $f(x) = 0$, $x \in \mathbb{R}$.

• Rational points on elliptic curves (Richard Guy and co.). Which integers n can be represented in terms of rationals in the form:

$$(x+y+z+t)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)=n\,?$$

Shown, infinitely many ± rational solutions. The LHS has a minimum = 16. *Question*: Is there always a positive solution for any $n \geq 16$? Observation: $16 \sim (1,1,1,1), 17 \sim (2,3,3,4), 18 \sim (1,1,2,2), 19 \sim (8,9,18,21), 20 \sim (1,3,3,3)$. **Conjecture:** This is true for any $n \geq 16$. But! Allan MacLeod made a search for $16 \leq n \leq 1000$, found there are **exceptions**: $n = 36, 40, 64, 68, 100, ..., 200, ...$ Thus, 36 is the smallest **CE** ! MacLeod proved: no positive solution exists for any n of the form $4k^2, 4k^2+4$, if k is odd, and for all $k \equiv 0 (\text{mod } 4)$.

Moral: ***Ask Questions***. Any specific detailed answer, 'yes' or 'no', is useful. ***Be brave, make Conjectures***. Sometime they will be correct, sometime not. You, or somebody else, either confirm the Conjecture, or provide an explicit CE.

• **Case which is both surprising and paradoxical:**

Start with a sequence $X_1, X_2, \ldots$ of i.i.d. r.v.s. and assign a *rank* to each term. The rank of X_n is equal to r, if there are exactly r indexes $i \leq n$ s.t. $X_i \geq X_n$. Take $\mathbf{Y}_k = (\mathbf{Y}_{k,1}, \mathbf{Y}_{k,2}, \mathbf{Y}_{k,3}, \ldots)$, a stochastic process consisting of the terms X_i having initial rank equal to k; $\mathbf{Y}_k$ is called a *sequence of kth partial records.*

Ignatov's Theorem:

The sequences $\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3, \ldots$ *are independent and identically distributed!*

Surprising result, indeed! With interesting implications.

Why paradoxical? Zv. I. announced this result in 1971, submitted a paper, in German, to 'Ann. Univ. Sofia Fac. Math. Mech.' It appeared ... only in 1977. Important: For his achievements in Probability theory, including the theorem above, Zvetan Ignatov became Doctor Honoris Causa of Sofia University in 2013. (Another BG probabilist with this highest degree from SU is Captain N. Yanev.)

B O O K S

1964 Gelbaum-Olmsted ***Counterexamples in Analysis***, Holden Day; Dover **2003**; Mir **1967**
1970 Steen-Seebach ***Counterexamples in Topology***, Springer; Dover **1995**
1978 Capobianco-Maluzzo ***Examples and Counterexamples in Graph Theory***, N-H
1982 Khaleelulla ***Counterexamples in Topological Vector Spaces***, Springer, LNM 936
1986 Székely ***Paradoxes in Probability Theory and Mathematical Statistics***, Ak Kiado
1986 Romano-Siegel ***Counterexamples in Probability and Statistics***, Wadsworth
1987 Stoyanov ***Counterexamples in Probability***, Wiley; Dover **2013**; Moscow **1999, 2012**
1990 Gelbaum-Olmsted ***Counterexamples in Mathematics***, Springer
1991 Rassias ***Counter Examples in Differential Equations, Related Topics***, World Sci
1993 Wise-Hall ***Counterexamples in Probability and Real Analysis***, OUP
2004 Klymchuk ***Counterexamples in Calculus***, Maths Press (NZ); MAA **2010**
2007 Fornaess ***Lectures on Counterexamples in Several Complex Variables***, AMS
2007 Rajwade ***Surprises and Counterexamples in Real Function Theory***, Hindustan
2009 Boss ***Lectures in Mathematics. Counterexamples and Paradoxes*** (R), Librocom
2010 Borwein ***Convex Functions: Constructions, Charact. Counterexamples***, CUP
2017 Bourchtein ***Counterexamples on Uniform Convergence***, Wiley
2019 Charalambous ***Dimension Theory: Theorems and Counterexamples***, Springer
2019 Xiaoxiang-Jianlong ***Counterexamples in Ring Theory***, Sci Press
2021 Schilling-Kühn ***Counterexamples in Measure and Integration***, CUP

Comments:

- Until now, 19 essential (!) titles. Only 3 books are published by Dover.
- Copies of any edition of my books are donated to the Libraries of IMI-BAN, FMI-SU, and the National Library "Cyril and Methodius".
- The Rassias's title is a ***funny counterexample*** to the following statement:

All titles above include the word 'counterexample', as one word !

People still ask about My Book, when, why, how, wow. Some details:

- 1967 MGU, Shilov: '*If you want to understand Analysis, you have to know all in this book*!', showing the translation of Gelbaum & Olmsted (cost 99 kop.).
- 1972, Budapest: Public comment of Kolmogorov: '*Either I prove a statement, or provide a counterexample. This usually takes a day or two, up to a week*'.

• 1979, Varna, EMS: David Kendall (Cambridge) liked my collection and suggested to 'John Wiley & Sons' to contact me and make a contract for a book.
• 1981, Vilnius: Kolmogorov, '*An excellent idea. Better to publish such a challenging book instead of proving theorems, for which nobody can tell, they will be useful or useless*'. Good insights came also from Gnedenko, Shiryaev, Doob, Itô.
• Long BG story before signing/realizing the contract with Wiley. It took a year.
• Great illustrations prepared by AT Fomenko, for the text and the super jacket.
• My invited talk, 'Intern. Conference on Teaching Probability and Statistics', Victoria (BC), Canada. The Chair introduced me: '*Dear Colleagues! There is a surprise. We have today JS, coming from behind the iron curtain*!' Room full!
• 01 December 1987, I received from Wiley 3 copies of the book just published! Kolmogorov died on 25 October 1987. He would have been the first to get a gift copy. A copy was given to the Library of Stats Laboratory, MGU; later stolen!
• 1994: Luc Devroye (McGill) has noticed that Wise-Hall used several items from my 1987 book, no reference. My reaction, with a smile: 'God will punish them!' Hmm, $\frac{1}{2}$ of this has happened. Gary Wise was sent to prison for a crime.
• Special semesters for MSc students based on My Book: Yale, MGU, TU-Dresden, Shandong University. Seminars given at 44 universities, worldwide. In 2015, Mini-course 'Use of CEs for teaching and research in Probability', Humboldt-Universität Berlin and University of Potsdam. (SU ?)

New journal: ***Examples and Counterexamples***, Elsevier, started in 2021. Some 10 years ago, I distributed a letter among publishers proposing to establish and publish regularly a new *Journal of Counterexamples in Mathematics.*
Publishers, however, follow too strong commercial approach. And, over the years, I redirected my enthusiasm to other challenging project, hence no wish to be involved in such a business. 'Examples and Counterexamples' is available online, temporarily for free. There are some interesting papers.

Diverse specific counterexamples:

• Is it true that a bounded function on a compact set achieves its maximum and minimum? We may think that 'yes'. The answer is 'no'.

Details: Consider the set Q of all rational points in $[0,1]$ and the function:

$f(x) = \frac{(-1)^n m}{n+1}, \ x \in Q, \ x = \frac{m}{n}$, and $f(x) = 0$, otherwise.

Then, on $[0,1]$, $\liminf f(x) = -1 < f(x) < 1 = \limsup f(x) = 1$. Hence f is bounded, f does not achieve either its maximum or its minimum on the compact set $[0,1]$. More, f is nowhere semicontinuous.

• Every orthonormal basis $\{\phi_n\}_{n=1}^{\infty}$ for a Hilbert space $\mathcal{H}$ is also an unconditional basis in the sense that if $\sum a_n \phi_n$ converges, then for any $\varepsilon_n = \pm 1, \ n \in \mathbb{N}$, the series $\sum \varepsilon_n a_n \phi_n$ also converges. There is in $\mathcal{H}$ a conditional basis $\{b_n\}$ such that $\sum a_n b_n$ converges, but $\sum \varepsilon_n a_n b_n$ diverges.

• **Distributional equation:** If X and Y are real or complex numbers, we easily 'solve' the eq. $X + Y = XY$. What if matrices? It is quite nontrivial to deal with r.v.s, say X and Y, and consider the following equation in distribution:

$$X + Y \overset{\mathrm{d}}{=} XY.$$

Question: What is X and Y? (Range of values and the d.f. of each.) Simple model: X and Y are independent, $\perp\!\!\!\perp$, with the same unknown distribution, F.

Surprise: The answer is unexpected and depends on the type of F. Easy to see, the values of X and Y must be in the interval $(-2,2)$, hence bounded.

Continuous Model: $X \sim F$, density $f(x) = F'(x), \ x \in \mathbb{R}, \ Y \perp\!\!\!\perp X, \ Y \overset{\mathrm{d}}{=} X$.

STATEMENT **C**: *There is a unique solution:*

$$X + Y \overset{\mathrm{d}}{=} XY \iff f(x) = \frac{1}{\pi\sqrt{4 - x^2}}, \ x \in (-2,2) \quad (\textbf{Arc - Sine Law}).$$

Trick: Start with a r.v. $X \sim$ Arc-Sine Law on $(-2,2)$ and let $Y \perp\!\!\!\perp X$, same law. For the moments $a_k = \mathsf{E}[X^k]$, and similarly for $b_k = \mathsf{E}[Y^k]$:

$a_{2k-1} = \mathsf{E}[X^{2k-1}] = 0$, and $a_{2k} = \mathsf{E}[X^{2k}] = \int_{-2}^{2} x^{2k} f(x)\,\mathrm{d}x = \ldots = \binom{2k}{k}$.

Odd order moments $\mathsf{E}[(X+Y)^{2k-1}]$ and $\mathsf{E}[(XY)^{2k-1}]$ are all zero.

$$\mathsf{E}[(XY)^{2n}] = \binom{2n}{n}^2, \quad \mathsf{E}[(X+Y)^{2n}] = \sum_{k=0}^{n} \binom{2n}{2k} a_{2k} b_{2n-2k} = \ldots = \binom{2n}{n}^2.$$

Hausdorff's Moment Problem: *Two distributions on the* **same** *bounded interval with the* **same** *moments, are the* **same**. Hence $X + Y \overset{\mathrm{d}}{=} XY$.

Discrete Model:. X and Y are discrete, common d.f. F, step-wise, no density.

STATEMENT **D**: *For any n there are purely discrete r.v.s $X \perp\!\!\!\perp Y$ with values in a set of $n+1$ points, $\{x_0, x_1, \ldots, x_n\}$ in $[-2, 2]$ such that* $X + Y \overset{\mathrm{d}}{=} XY$.

Solution: The values $x_0,\ x_j, j = 1, 2, \ldots, n$ and the masses are as follows:

$$x_0 = 2, \quad x_j = 2\cos\left(\frac{2\pi}{2n+1}\, j\right), \ j = 1, 2, \ldots, n;$$

$$\mathsf{P}(X = 2) = \frac{1}{2n+1}, \ j = 0; \quad \mathsf{P}(X = x_j) = \frac{2}{2n+1}, \ j = 1, 2, \ldots, n.$$

We can show directly that indeed $X + Y \overset{\mathrm{d}}{=} XY$.

Comment: Try to extend, $X_1 + X_2 + \ldots \overset{\mathrm{d}}{=} X_1 X_2 \cdots$. No chance.

Question: $X_1 \neq X_2$ such that $X_1 + X_2 \overset{\mathrm{d}}{=} X_1 X_2$? **Conjecture:** No!

- **Normal + Log-normal distributions:** $Z \sim \mathcal{N}(0,1),\ X = \mathrm{e}^Z \sim Log\mathcal{N}(0,1)$.

$$\varphi(x) = \frac{1}{\sqrt{2}\,\pi} \exp(-\frac{x^2}{2}), \ x \in \mathbb{R}; \quad f(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{x} \exp\left[-\frac{1}{2}(\ln x)^2\right], \ x > 0.$$

For Z, all is o'k. For X, no m.g.f. = heavy tail, $m_k = \mathsf{E}[X^k] = \mathrm{e}^{k^2/2},\ k = 1, 2, \ldots$

Stieltjes classes: (one *absolutely continuous*, one *discrete*):

$X_\varepsilon, \varepsilon \in [-1, 1]:$ density $f_\varepsilon(x) = f(x)\,[1 + \varepsilon \sin(2\pi \ln x)],\ x > 0,\ X_0 = X;$

$Y_a,\ a > 0:\ \mathbf{P}[Y_a = a\mathrm{e}^n] = a^{-n}\,\mathrm{e}^{-n^2/2}/A,\ n = 0, \pm 1, \pm 2, \ldots.$

Shocking property: $\mathsf{E}[X_\varepsilon^k] = \mathsf{E}[Y_a^k] = \mathsf{E}[X^k] = \mathrm{e}^{k^2/2},\ k = 1, 2, \ldots.$

$\Rightarrow$ **Log$\mathcal{N}$ is M-indeterminate!** So 'many' others, same moments. Equivalently: The *Stieltjes moment problem* for $Log\mathcal{N}$ has a ***non-unique solution***!

• **Peano's theorem for ODEs**, valid for finite-dimensional spaces, fails to hold for spaces of infinite dimension. Counterexamples found by Dieudonne (1950), Godunov (1973, 1975), and others.

• **Osgood's theorem for PDEs:** Conditions for validity cannot be relaxed.

• **Krylov-Evans Saga:** For a convex and fully nonlinear parabolic or elliptic PDEs, by using different methods, Krylov and Evans proved existence and regularity of the solution. This was done in 1982, followed further by other important developments. Krylov is a leader in Stochastic Calculus (Itô) and PDEs, Evans is a well-known Analyst, in PDEs. In 2004, Lawrence Evans and Nicolai Krylov received from AMS the very prestigious *Leroy Steel Prize.*

• **Independence in Probability:** Collection of r.v.s $X_1, X_2, \ldots, X_n$ is $\perp\!\!\!\perp$ **iff**

$$\mathsf{P}(X_1 \le x_1, \ldots, X_n \le x_n) = \mathsf{P}(X_1 \le x_1) \cdots \mathsf{P}(X_n \le x_n) \quad \text{for all} \quad x_j \in \mathbb{R}.$$

Corollary: For $k = 2, \ldots, n-1$, any sub-collection of k r.v.s is also $\perp\!\!\!\perp$.
Question: For random events A, B, C, D, is it true that

$$\mathsf{P}(ABCD) = \mathsf{P}(A)\mathsf{P}(B)\mathsf{P}(C)\mathsf{P}(D) \quad \Longrightarrow \quad \perp\!\!\!\perp \quad A, B, C, D\ ?$$

Answer: No! Much more is available. Realizable is any finite configuration of random events with preliminary given i/d-structure!

• **Conjecture about the Wiener Process** (My Book, 1987, 1997).
If for a process $X = (X_t, t \ge 0)$ with $X_0 = 0$, independent are only a finite number of increments, and each increment is distributed $\mathcal{N}$, it does not follow that X is a standard Wiener process. Marc Yor and Hans Föllmer published a paper in 'Ann. Inst. Henri Poincaré', 42 pp. At the EMS Congress in Barcelona, 2000, Yor gave a plenary talk on ... 'Stoyanov's Conjecture'. Weak Wiener Process of order k was introduced, studied and applied to stochastic finance models.

• **Powers of 2:** *The numbers 2, 4, 8, 64 and 2048 are the only powers of 2 which are written by even digits. Give a proof, or provide a counterexample.*

I heard about this from my Math Teacher, Ivan Tyufekchiev (Pernik) many years ago. I have discussed this case with many colleagues, e.g., with Andrew Wiles, during a visit to the Newton Institute, Cambridge. The question is still open!

- ***Surprise:*** **Probabilistic Method in Analysis:**

Uspensky (AMM, 1932): Show that

$$\int_0^1 \cdots \int_0^1 \frac{x_1^2 + \cdots + x_n^2}{x_1 + \cdots + x_n} \, \mathrm{d}x_1 \cdots \mathrm{d}x_n \to \frac{2}{3}, \quad \text{as} \quad n \to \infty.$$

Several solutions rely essentially on the powers 2 and 1. What if else ?

E.g., with π and e, the known classical constants, show that

$$\int_0^1 \cdots \int_0^1 \frac{x_1^\pi + \cdots + x_n^\pi}{x_1^{\mathrm{e}} + \cdots + x_n^{\mathrm{e}}} \, \mathrm{d}x_1 \cdots \mathrm{d}x_n \to \frac{\mathrm{e}+1}{\pi+1}, \quad \text{as} \quad n \to \infty.$$

My Result: *For functions f, g and φ, all positive and integrable, as $n \to \infty$,*

$$\frac{1}{c_0^n} \int \cdots \int \frac{f(x_1) + \cdots + f(x_n)}{g(x_1) + \cdots + g(x_n)} \varphi(x_1) \cdots \varphi(x_n) \, \mathrm{d}x_1 \cdots \mathrm{d}x_n \to \frac{\int f(x)\varphi(x)\, \mathrm{d}x}{\int g(x)\varphi(x)\, \mathrm{d}x}.$$

Hint: Here $c_0 = \int \varphi(x)\, \mathrm{d}x$, so $\tilde{\varphi} = \varphi / c_0$ is a probability density. We deal with a sequence of i.i.d. r.v.s $X_1, \ldots, X_n$ each with density $\tilde{\varphi}$, apply the LLN to two sequences, $\xi_n := \frac{1}{n}(f(X_1) + \ldots + f(X_n))$ and $\eta_n := \frac{1}{n}(g(X_1) + \ldots + g(X_n))$, and use the convergence implication $Y_n \overset{P}{\to} Y \Rightarrow \mathsf{E}[H(Y_n)] \to \mathsf{E}[H(Y)]$, valid for continuous bounded function H. Here $Y_n = \xi_n / \eta_n$, and the limit $Y = \mathit{const}$, as it follows from the LLN. In the integrals $\int_0^1$, see Uspensky's problem, $\varphi \equiv 1$ is the uniform density on $[0, 1]$. The joint density of $(X_1, \ldots, X_n)$ is hidden, it is 1.

- ***Probabilistic Method*, Erdös - Rényi:** Combinatorics, Graph Theory, NT.

- ***Probabilistic Solutions of PDEs*:** Kolmogorov-Feynman-Kac Formulas.

- **Fréchet–Shohat Theorem:** Powerful method to prove limit theorems. Among them are the CLT, Wigner's semicircle law for the distribution of the eigenvalues of random matrices (recent talk by M. Kuntsevich, July 2022, ICMS), Terence Tao's progress in Sendov's Conjecture (talk online, January 2022, ICMS).

Absurd: **1*.** Real case: A student was asked to find $S := \frac{a}{b} + \frac{c}{d}$, he suggested a 'genius' way, by finding the common nominator, his answer: $S = \frac{ac}{bc+ad}$. WOW! To all class: 'My younger granddaughter can do this correctly'. Story! Interestingly, there are sets of mixtures of real and complex numbers for which $\frac{a}{b} + \frac{c}{d} = \frac{ac}{bc+ad}$.

2*. Funny story: On arXiv Math, on the same day, two papers on the Riemann Hypothesis (RH) were posted, maybe randomly, but ironically, as 'neighbors':

'*Short proof of the RH*' and '*It is very likely that the RH is not true*'.

Clearly, at least one of these papers is wrong, or both are wrong!

3*. I know a respected institution in a Balkan country where a person can get a PhD degree 'Doctor of Mathematics' without knowing what is 'MathSciNet' and what is 'arXiv Math'. Do not even think to ask: What is MGP?

Last item: On Bertrand's Paradox: A new variation appeared ***today***, see M. Klazar's paper on arXiv Math, 16 November 2022.

In 1889, Bertrand asked: *What is the probability that a random chord T in a plane circle C is longer than the side of the inscribed equilateral triangle*?

If C has radius 1, then T has to be longer than $\sqrt{3}$. The 'trick' is how you understand 'random chord'. Bertrand suggested 3 ways, and 3 different solutions.

Klazar 'approximates' the unit plane circle with small boxes/squares of size $\frac{1}{n} \times \frac{1}{n}$ and counts the pairs of boxes separated by distance more than $\sqrt{3}$. For $n \to \infty$, he shows that the proportion of such pairs converges to the number

$$\frac{1+\sqrt{3}}{8} - \frac{\pi\,(2-\sqrt{2})}{96} \approx 0.33272.$$

Different interpretations, different answers! The challenge remains!

Final: The accent today was a little more on Probability and Analysis.

You, my Fellows, working in any area of Mathematics !

You can make your scientific work and life more interesting and successful if paying good attention to available counterexamples, paradoxes and surprises, and bravely look for finding new and/or missing ones! Good luck!

Thank you !