Classical invariant theory and its applications

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Outline

- Classical invariant theory
- Locally nilpotent derivations
- Combinatorics
- Graph theory
Let $k$ be an algebraically closed field and $V$ a finite dimensional vector space over $k$. Let $e_1, e_2, \ldots, e_n$ be a basis of $V$ and denote by $f_i \in V^*$ the linear function on $V$ with
\[ f_i(x_1 e_1 + \ldots + x_n e_n) = x_i. \]

The $f_i$ generate an algebra $S(V)$ of polynomial functions on $V$. There is an isomorphism $S(V) = k[f_1, f_2, \ldots, f_n]$. Also the algebra $S(V)$ is graded algebra.
Definition

Let $GL(V)$ be the group of all invertible linear transformations of $V$. If $g \in GL(V)$, $f \in S(V)$ define $g \cdot f \in S(V)$ by

$$g \cdot f(v) = f(g^{-1}v).$$

It is easy to check that it is the action

$$g(hf) = (gh)(f).$$

If $G$ is a subgroup of $GL(V)$

**Definition**

We say that $f \in S(V)$ is a $G$-invariant if $g \cdot f = f$ for all $g \in G$.

The $G$-invariant polynomial functions form a graded subalgebra $S(V)^G$ of $S(V)$. In invariant theory one studies the properties of such algebras $S(V)^G$. The classical invariant theory is concerned with the cases when $G$ is a classical group.
Main problems

All problems of the classical invariants theory divides into two main parts:

**A. Finitely generation.** Is the algebra $k[V]^G$ finitely generated? For $SL_2$ Gordan (1868), for $SL_n$ Hilbert(1890, 1893),


Finite groups

Let $G$ be a finite group.

**Theorem (Hilbert, Noether)**

The algebra $k[V]^G$ is generated by not more than $(|G| + \dim V)$ homogeneous invariants, of degree not exceeding $|G|$.

The Reynolds operator $S(V) \to S(V)^G$

$$R(f) = \frac{1}{|G|} \sum_{g \in G} g \cdot f, f \in S(V).$$

$$k[x_1, x_2, \ldots, x_n]^{S_n} = k[R(x_1), R(x_2), \ldots, R(x_n)],$$

$$R(x_1) = x_1 + x_2 + \cdots + x_n,$$

$$\ldots$$

$$R(x_1^n) = x_1^n + x_2^n + \cdots + x_n^n.$$
Molien formula

\( S(V)^G \) is graded algebra:

\[
S(V)^G = (S(V)^G)_0 \oplus (S(V)^G)_1 \oplus \cdots
\]

The Poincare series

\[
P(S(V)^G, z) = \sum_{i=0}^{\infty} \dim(S(V)^G)_i z^i,
\]

is a rational function. A classical theorem of Molien gives an explicit expression for the rational function and ties together invariant theory with generating functions.

**Theorem (Molien, 1897)**

\[
P(S(V)^G, z) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1_V - z \cdot g)}.
\]
\[ S_k(n) = \sum_{w^n = 1, w \neq 1} \frac{1}{|1 - w|^{2k}} \]

\[ S_1(n) = \frac{n^2 - 1}{12} \]

\[ \sum_{k=1}^{\infty} 4^k S_k(n) x^{2k} = 1 - \frac{nx \cot(n \sin^{-1} x)}{\sqrt{1 - x^2}}. \]
Invariants of binary forms

\[ V_1 = k^2 = \langle x, y \rangle. \]

\[ G = SL(V_1) = SL_2 \text{ and } V_d = S^d(V_1) \quad V = V_d \oplus V_1. \]

Algebras of invariants \( I_d = k[V_d]^{SL_2} \) and \( C_d = k[V_d \oplus k^2]^{SL_2} \)

the algebra of invariants of binary form and the algebra of covariants of binary forms.

The algebra \( k[V_{d_1} \oplus V_{d_2} \oplus \cdots \oplus V_{d_n}]^{SL_2} \) — the algebra of joint invariant of binary forms of degrees \( d_1, d_2, \ldots d_n \).

**Note:** The symbolic method \( k[V_1 \oplus V_1 \oplus \cdots \oplus V_1 \cdots]^{SL_2} \rightarrow k[V_d]^{SL_2} \)
A derivation of a ring is an additive map $D$ satisfying the Leibniz rule

$$D(fg) = fD(g) + D(f)g, \text{ for all } f, g \in R.$$ 

The subring

$$\ker D := \{ f \in k[x_0, x_1, \ldots, x_n] \mid D(f) = 0 \},$$

the kernel of the derivations.

A derivation $D$ is called locally nilpotent if for every $r \in R$ there is an $n \in \mathbb{N}$ such that $D^n(r) = 0$.

The problems A and B for $\ker D$
Weitzenböck derivations

A linear locally nilpotent derivation $\mathcal{D}$ of the polynomial algebra $k[x_1, x_2, \ldots, x_n]$ is called a Weitzenböck derivation.

$$\mathcal{D}(x_i) = \sum_{i=0}^{n} a_{i,j}x_i$$

The basic Weitzenböck derivation $\mathcal{D}_n$; $\mathcal{D}_n(x_i) = ix_{i-1}$.

Denote by $\mathcal{D}_d$, $d := (d_1, d_2, \ldots, d_s)$ the Weitzenböck derivation of the polynomial algebra if its matrix consists of $s$ Jordan blocks of size $d_1 + 1$, $d_2 + 1$, $\ldots$, $d_s + 1$, respectively.

The matrix $\{a_{i,j}\}$ is nilpotent.
Kernel of Weitzenböck derivation

**Theorem**

Let $\mathcal{D}_d$ be a Weitzenböck derivation. Then

$$\ker \mathcal{D}_d \cong k[V_{d_1} \oplus \cdots \oplus V_{d_s} \oplus k^2]^{SL_2} \cong k[V_{d_1} \oplus \cdots \oplus V_{d_s}]^{G_a}$$

$G_a = (k, +)$ — the additive group of a field. Not a reductive group!
The second isomorphism is the Grosshans principle.
Classical result — Robert’s theorem, 1861.

**Theorem**

Let $\mathcal{D}_d$ be a basic Weitzenböck derivation. Then

$$\ker \mathcal{D}_d \cong k[V_d \oplus k^2]^{SL_2} \cong k[V_d]^{G_a}$$
The symbolic method

Let $D_d$ be basic Weitzenböck derivation. The kernel is a graded algebra

$$\ker D_d = (\ker D_d)_0 \oplus (\ker D_d)_1 \oplus \cdots \oplus (\ker D_d)_m \oplus \cdots$$

**Theorem**

Then there is a surjective map $\ker D_{(1,1,\ldots,1)} \rightarrow (\ker D_d)_m$

Center the universal enveloping algebra

$G$ — compact Lie group, $\mathfrak{g}$— its Lie algebra acting by derivations on $k[V]$. 
$k[V]^G = k[V]^\mathfrak{g}$.

**Theorem**

If $\mathfrak{g} = \langle D_1, D_2, \ldots D_n \rangle$, then

$$k[V]^G = \ker D_1 \cap \ker D_2 \cap \cdots \cap \ker D_n.$$ 

$D_i$ — locally nilpotent derivations $k[V]$.

**Theorem**

Let $Z(\mathfrak{g})$ be the center the universal enveloping algebra $U(\mathfrak{g})$. Then

$$Z(\mathfrak{g}) \cong k[\mathfrak{g}]^\mathfrak{g}.$$
Let $D_n(x_i) = ix_{i-1}$ — the basic Weitzenböck derivation. Define $D^*_n(x_i) = (n - i)x_{i+1}$. Let $\text{ord}(q)$ be the minimal power $D^*_n$ of such that $(D^*_n)^{\text{ord}(q)}(q) = 0$.

**Theorem (Bedratyuk, 2012)**

Let $p, q \in \ker D_n$, then and $[p, q]^r \in \ker D_n$

$$[p, q]^r = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{(D^*)^i(p)}{[\text{ord}(p)]_i} \frac{(D^*)^{r-i}(q)}{[\text{ord}(q)]_{r-i}},$$

$[m]_i = m(m - 1) \ldots (m - (i - 1))$ is the falling factorial.

$x_0 \in \ker D_n$ construct

$$[x_0, x_0]^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} x_i x_{n-i}.$$
Diximier map

For arbitrary locally nilpotent derivation \( D \) the following theorem holds:

**Theorem**

Suppose that there exists a polynomial \( h \) such that \( D(h) \neq 0 \) but \( D^2(h) = 0 \). Then

\[
\ker D = k[\sigma(x_0), \sigma(x_1), \ldots, \sigma(x_n)] [D(h)^{-1}] \cap k[x_0, x_1, \ldots, x_n],
\]

where \( \sigma \) is the Diximier map

\[
\sigma(x_i) = \sum_{m=0}^{\infty} D^m(x_i) \frac{\lambda^m}{m!}, \lambda = -\frac{h}{D(h)}, D(\lambda) = -1.
\]
The Appell polynomials $\mathcal{A} = \{A_n(x)\}$, where $\deg(A_n(x)) = n$ and the polynomials satisfy the identity

$$A'_n(x) = nA_{n-1}(x), \quad n = 0, 1, 2, \ldots.$$ 

Polynomials Bernoulli $B_n(x)$, Euler $E_n(x)$ Hermite $H_n(x)$, $n = 0, 1, 2, \ldots$

$$\frac{te^{xt}}{e^t - 1} = \sum_{i=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad \frac{2e^{xt}}{e^t + 1} = \sum_{i=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad e^{xt} - \frac{t^2}{2} = \sum_{i=0}^{\infty} H_n(x) \frac{t^n}{n!}.$$
Polynomial Identities for the Appell polynomials

\[ F(A_0(x), A_1(x), \ldots, A_n(x)) = 0, \]

where \( F \) is some polynomial of \( n + 1 \) variables.

Main Idea:

\[
\text{if } \frac{d}{dx} F(A_0(x), A_1(x), \ldots, A_n(x)) = 0 (= \text{const}) \\
\text{then } F(A_0(x), A_1(x), \ldots, A_n(x)) = \text{const} = F(A_0(0), A_1(0), \ldots, A_n(0)).
\]
Let now $k[a_0, a_1, \ldots, a_n]$ and $k[x]$ be algebras of polynomials, $D_n(a_i) = ia_i$ - Weitzenböck derivation. Let us consider the substitution homomorphism $\varphi_A : k[a_0, a_1, \ldots, a_n] \to k[x]$ defined by $\varphi_A(a_i) = A_i(x)$. The homomorphism $\varphi_A$ intertwines with the derivative operator $\frac{d}{dx}$.

**Theorem**

$$\varphi_A \circ D_n = \frac{d}{dx} \varphi_A.$$ 

**Theorem**

*If $F[a_0, a_1, \ldots, a_n] \in \ker D_n$ then we have the identity*

$$F(A_0(x), A_1(x), \ldots, A_n(x)) = F(A_0(0), A_1(0), \ldots, A_n(0)).$$
Example

\[ [a_0, a_0]^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} a_ia_{n-i}. \]

\[ \sum_{i=0}^{n} (-1)^i \binom{n}{i} A_i(x)A_{n-i}(x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} A_i(0)A_{n-i}(0). \]

For Bernoulli

\[ \sum_{i=0}^{n} (-1)^i \binom{n}{i} B_i(x)B_{n-i}(x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} B_iB_{n-i}. \]

\[ \sum_{i=0}^{n} (-1)^i \binom{n}{i} B_iB_{n-i} = (1 - n)B_n. \]

Thus

\[ \sum_{i=0}^{n} (-1)^i \binom{n}{i} B_i(x)B_{n-i}(x) = (1 - n)B_n. \]
Discriminant

\[
\text{Discr}_n(a_0) := \\
\begin{vmatrix}
a_0 & na_1 & \cdots & a_n & 0 & \cdots & \cdots & 0 \\
0 & a_0 & \cdots & na_{n-1} & a_n & 0 & \cdots & 0 \\
0 & \cdots & 0 & a_0 & na_1 & \binom{n}{2}a_2 & \cdots & a_n \\
n_{a_0} & (n-1)na_1 & \cdots & na_{n-1} & 0 & 0 & \cdots & 0 \\
0 & na_0 & \cdots & \cdots & na_{n-1} & 0 & \cdots & 0 \\
0 & 0 & \cdots & na_0 & (n-1)na_1 & (n-2)\binom{n}{2}a_2 & \cdots & na_{n-1}
\end{vmatrix}.
\]

The corresponding identities has the form \( \text{Discr}_n(A) = \text{Discr}_n(A)_0 \).

**Conjecture.** \( \text{Discr}_n(H)_0 = \prod_{k=1}^{n} k^k \)
The catalecticant of a binary form of even degree, \( n = 2k \), can be written as the determinant of the \((k + 1) \times (k + 1)\) matrix

\[
\text{Cat}_n(a_0) := \begin{vmatrix}
a_0 & a_1 & a_2 & \cdots & a_k \\
a_1 & a_2 & a_3 & \cdots & a_{k+1} \\
& & & & \\
& & & & \\
a_{k-1} & a_k & a_{k-1} & \cdots & a_{2k-1} \\
a_k & a_{k+1} & a_{k+2} & \cdots & a_{2k}
\end{vmatrix}.
\]

The corresponding identities has the form \( \text{Cat}_n(A) = \text{Cat}_n(A)_0 \).

**Conjecture.** \( \text{Cat}_n(\mathcal{H})_0 = (-1)^n n!! \).
Joint Identities

Derivation $D_{d_1,d_2}$.

$$Dv_n(a_0, b_0) := [a_0, b_0]^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} a_i b_{n-i} \in \ker D_{n,n}.$$ 

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} E_i(x) B_{n-i}(x) = \text{const.}$$
Resultant

\[ s\text{Res}_n(a_0, b_0) := \begin{vmatrix}
  a_0 & na_1 & \cdots & a_n & 0 & \cdots & \cdots & 0 \\
  0 & a_0 & \cdots & na_{n-1} & a_n & 0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & \cdots & 0 & a_0 & na_1 & \binom{n}{2} a_2 & \cdots & a_n \\
  b_0 & nb_1 & \cdots & b_n & 0 & \cdots & \cdots & 0 \\
  0 & b_0 & \cdots & nb_{n-1} & b_n & 0 & \cdots & 0 \\
  0 & \cdots & 0 & b_0 & b_1 & \binom{n}{2} b_2 & \cdots & b_n \\
\end{vmatrix} \]

Identity, general cases.

Let \( \{ P_n(x) \} \), \( n = \deg P_n(x) \) — family of polynomials, bases of \( k[x] \).
Then \( P_n(x)' \) can be expressed via \( P_0(x), P_1(x), \ldots, P_{n-1}(x) \):

\[
P_n(x)' = a_{n,0}P_0(x) + a_{n,1}P_1(x) + \ldots + a_{n,n-1}P_{n-1}(x), \quad P_0(x)' = 0.
\]

Assign to the family \( \{ P_n(x) \} \) the derivation

\[
D_P(x) = a_{n,0}x_0 + a_{n,1}x_1 + \ldots + a_{n,n-1}x_{n-1}, \quad D_P(x_0) = 0.
\]

Since \( D_P(x_n)^{n+1} = 0 \), then \( D_P \) is LND.

**Theorem**

If \( F(x_0, x_1, \ldots, x_n) \in \ker D_P \) then

\[
F(P_0(x), P_1(x), \ldots, P_n(x)) = \text{const}.
\]
Reduce to Appell polynomials. Interwining map.

\[ \text{ker } D_P? \]
\[ \text{Build isomorphism } \varphi_{AP} : \text{ker } D_A \to \text{ker } D_P. \]

**Definition**

A linear map \( \psi_{AF} \) is called a \((D_A, D_F)\)-intertwining map if the following condition holds:

\[ \psi_{AF} D_A = D_F \psi_{AF}. \]

Any such map induces an isomorphism from \( \text{ker } D_A \) to \( \text{ker } D_F \).

Any identity for Appell polynomials generates an identity for the family \( \{P_n(x)\}, n = \deg P_n(x) \).
Fibonacci and Lucas polynomials

The Fibonacci $F_n(x)$ and Lucas $L_n(x)$ polynomials. The generating functions:

$$\frac{t}{1 - xt - t^2} = \sum_{n=0}^{\infty} F_n(x) t^n,$$

$$\frac{1 + t^2}{1 - xt - t^2} = \sum_{n=0}^{\infty} L_n(x) t^n.$$

The derivatives of the polynomials can be expressed in terms of the polynomials as follows:

$$\frac{d}{dx} F_n(x) = \left[ \frac{n-1}{2} \right] \sum_{k=0}^{[n-1/2]} (-1)^k (n-1-2k) F_{n-1-2k}(x),$$

$$\frac{d}{dx} L_n(x) = n \sum_{k=0}^{[n-1/2]} (-1)^k L_{n-1-2k}(x).$$
Fibonacci and Lucas derivations

The expressions motivate the following definitions

**Definition**

Let $D_F$, $D_L$ be the derivations of $k[x_0, x_1, x_2, \ldots, x_n]$ defined by:

$$D_F(x_n) = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^k (n-1-2k)x_{n-1-2k}, i = 2, 3, \ldots, n,$$

$$D_F(x_0) = D_F(x_1) = 0,$$

$$D_L(x_n) = n \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^k x_{n-1-2k}, i = 2, 3, \ldots, n,$$

$$D_L(x_0) = D_L(x_1) = 0.$$

$D_F$, $D_L$ are called the **Fibonacci derivation** and the **Lucas derivation** respectively.
A Appel-Lucas intertwining map has the form

\[ \psi_{AL}(x_n) = x_n + \alpha_n^{(1)} x_{n-2} + \alpha_n^{(2)} x_{n-4} + \ldots + \alpha_n^{(i)} x_{n-2i} + \ldots + \alpha_n^{([\frac{n-1}{2}])} x_{n-2\left[\frac{n-1}{2}\right]}, \]

where

\[ \alpha_n^{(s)} = \frac{(-1)^s}{s!} b_0 n^s + \cdots + \frac{(-1)^{s-i}}{(s-i)!} b_i n^{s+i} + \cdots + b_s n^{2s}, \]

and the generating function for \( b_0, b_1, \ldots, b_n, \ldots \) is

\[ \sum_{i=0}^{\infty} b_i z^i = J_0^{-1}(\sqrt{4z}). \]

\[ n^a := n(n-1)(n-2) \cdots (n-(a-1)). \]
A Appel-Fibonacci $\psi_{AF}$ intertwining map has the form

$$\psi_{AL}(x_n) = x_{n+1} + \alpha_n^{(1)} x_{n-1} + \alpha_n^{(2)} x_{n-3} + \ldots + \alpha_n^{([n-1]/2)} x_{n+1-2[n-1]/2},$$

$$\alpha_n^{(s)} = (n-2s+1) \left( \frac{(-1)^s}{s!} b_0 n^{s-1} + \ldots + \frac{(-1)^{s-i}}{(s-i)!} b_i n^{s+i} + \ldots + b_s n^{2s-1} \right),$$

and the ordinary generating function for $b_0, b_1, \ldots, b_n, \ldots$ is as follows

$$\sum_{i=0}^\infty b_i z^i = \frac{\sqrt{z}}{J_1(\sqrt{4z})}.$$

L. Bedratyuk, Derivations and Identities for Fibonacci and Lucas Polynomials, Fibonacci Quart. 51 (2013), no. 4, 351–366
{\( K_n(x, a), n = 0, 1, \ldots \) — Kravchuk polynomials.}

\[
K_n(x, a) := \sum_{i=0}^{n} (-1)^i \binom{x}{i} \binom{a-x}{n-i},
\]

Generating function:

\[
\sum_{i=0}^{\infty} K_i(x, a) z^i = (1 + z)^a \left( \frac{1-z}{1+z} \right)^x.
\]

\[
\frac{d}{dx} K_n(x, a) = -2 \sum_{i=1}^{n} \frac{1 - (-1)^i}{2i} K_{n-i}(x, a).
\]

\[
\frac{d}{da} K_n(x, a) = \sum_{i=0}^{n-1} \frac{(-1)^{n+1+i}}{n-i} K_i(x, a).
\]
Kravchuk derivations

Form $\frac{d}{dx} K_n(x, a)$ and $\frac{d}{da} K_n(x, a)$ implies:

**Definition**

Define $D_{K_1}$, $D_{K_2}$ by

$$D_{K_1}(x_0) = 0, \quad D_{K_1}(x_n) = \sum_{i=1}^{n} \frac{1 - (-1)^i}{2i} x_{n-i},$$

$$D_{K_2}(x_0) = 0, \quad D_{K_2}(x_n) = \sum_{i=0}^{n-1} \frac{(-1)^{n+1+i}}{n-i} x_i, \quad n = 1, 2, \ldots, n, \ldots,$$

is called the first and the second Kravchuk derivations.
Conjecture 1. For $n > 0$

$$\sum_{i=0}^{n} K_i(x, a) \sum_{k=0}^{n-i} \frac{(-1)^k}{k!} K_1(x, a)^k S^{(k)}(n - i) =$$

$$= \begin{cases} 
0, & n \text{ odd}, \\
(-1)^m(2m - 1)!! a(a - 2)(a - 4) \ldots (a - 2(m - 1)), & n = 2m,
\end{cases}$$

$$S^{(k)}(n) = \sum_{m=k}^{n} \binom{n - 1}{m - 1} \frac{2^m k!}{m!} s(m, k), s(m, k) \text{ Stirling numbers}$$

Conjecture 2.

$$\sum_{i=0}^{n} K_i(x, a) \sum_{k=0}^{n-i} \frac{(-1)^k}{(n-i)!} K_1(x, a)^k s(n - i, k) =$$

$$= \begin{cases} 
0, & n \text{ odd}, \\
(-1)^m \binom{x}{m}, & n = 2m,
\end{cases}$$
A Appel-Kravchuk $\psi_{DK_1}$ intertwining map has the form

$$\psi_{DK_1}(x_0) = x_0, \psi_{DK_1}(x_n) = \sum_{i=1}^{n} T(n, i)x_i.$$ 

where

$$T(n, i) = \sum_{j=i}^{n} (-1)^{j-i} 2^{n-j} j! S(n, j) \binom{j-1}{i-1},$$

$S(n, j)$– Stigling number of the second kind.
A Appel-Kravchuk $\psi_{DK_2}$ intertwining map has the form

$$\psi_{DK_2}(x_0) = x_0, \psi_{DK_2}(x_n) = \sum_{i=1}^{n} B(n, i)x_i, \quad n > 0.$$ 

where

$$B(n, k) = k! S(n, k),$$
Appel-Kravchuk intertwining map

\[ H_n := \det(x_{i+j-2}) = \begin{vmatrix} x_0 & x_1 & x_2 & \cdots & x_n \\ x_1 & x_2 & x_3 & \cdots & x_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n-1} & x_n & x_{n-1} & \cdots & x_{2n-1} \\ x_n & x_{n+1} & x_{n+2} & \cdots & x_{2n} \end{vmatrix} \]

\[ \psi_{DK_1}(H_n) \in \ker D_{K_1} \text{ and } \psi_{DK_2}(H_n) \in D_{K_2}. \]
By using the substitution homomorphism $\varphi_K(x_i) = K_i(x, a)$ we get the two polynomial identities.

$$\varphi_K(\psi_{DK_1}(H_n)) = \varphi_1(a), \varphi_K(\psi_{DK_2}(H_n)) = \varphi_1(x).$$

**Conjecture 3.**

(i) $\varphi_1(a) = (-1)^{\frac{n(n+1)}{2}} \prod_{i=0}^{n-1} i! \prod_{i=0}^{n-2} (a + i)^{n-1-i}$,

(ii) $\varphi_2(x) = (-1)^{\frac{n(n+1)}{2}} \prod_{i=0}^{n} 2^i i! \prod_{i=0}^{n-2} (x - i)^{n-1-i}$. 
Graph invariants

Graph \( G = (V, E) \), \( E \subseteq V^{(2)} \), where \( V^{(2)} = \{ \{u, v\} \mid u, v \in V, u \neq v \} \). Graphs \( G_1 = (V_1, E_1) \), \( G_2 = (V_2, E_2) \) is called isomorphic if exists a bijection \( \alpha : V_1 \rightarrow V_2 \), such that \( \{u, v\} \in E_1 \) iff \( \{\alpha(u), \alpha(v)\} \in E_2 \).
GI decision problem

Computational problem GI.

**Problem (GI)**

*Two graphs* $G_1$ and $G_2$. *Are they isomorphic?*

**Problem**

*If* $GI \in P$?

Complexity of known $GI$-algorithm $O(2^{\sqrt{n \log n}})$, (1983).

For many graphs (planar) there is a polynomial algorithm for $GI$. 
Graphs and theory of invariants

Labelled, undirected graphs as vector space. Let $e_{\{i,j\}}$ be the simple graph with one single edge $\{i,j\}$, and by $x_{\{i,j\}} e_{\{i,j\}}$ a graph $\{i,j\}$, of a weight $x_{\{i,j\}}$. The set of all graphs on $n$ nodes is a vector space $\mathcal{V}_n$ of the dimension $\binom{n}{2}$ with the basis $e_{\{i,j\}}$.

The symmetric group $S_n$ acts on $\mathcal{V}_n$:

$$ge_{\{i,j\}} = e_{\{g(i),g(j)\}}.$$

$x_{\{i,j\}} \rightarrow x_k$.

Group $S_n^{(2)} \cong S_n$ is a subgroup of $S_{\binom{n}{2}}$.

Algebra of invariants $k[\mathcal{V}_n]^{S_n^{(2)}}$.

**Theorem**

Let $k[\mathcal{V}_n]^{S_n^{(2)}} = k[f_1, f_2, \ldots, f_s]$. Then $k$-weighted graphs $G_1$ and $G_2$ are isomorphic iff $f_i(G_1) = f_i(G_2), i = 1, \ldots, s$. 
Main results

Algebra of invariants $k[\mathcal{V}_n]^{S_n(2)}$ calculated only for $n \leq 5$.


A minimal generating set for $k[\mathcal{V}_5]^{S_5(2)}$ consists of 57 polynomials, $\deg f_i \leq 10$. 
Graph $G$, with the set of weights $\{0, 1\}$.

$$G = \varepsilon_1 e_{\{1,2\}} + \varepsilon_2 e_{\{1,3\}} + \cdots + \varepsilon_m e_{\{n-1,n\}} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m), \quad m = \binom{n}{2}. $$

Number of simple graphs on $n$ nodes is the sequence A000088 in The On-Line Encyclopedia of Integer Sequences:

$$1, 1, 2, 4, 11, 34, 156, 1044, 12346, 274668, 12005168, 1018997864, \ldots$$
Cycle index and Molien series

Permutation $\alpha$ Let $j_i(\alpha)$ be the number of cycles of length $1 \leq i \leq n$ in the disjoint cycle decomposition of $\alpha$. Then the cycle index of $G$ denoted $Z(G, s_1, s_2, \ldots, s_n)$, is the polynomial in the variables $s_1, s_2, \ldots, s_n$ defined by

$$Z(G, s_1, s_2, \ldots, s_n) = \frac{1}{|G|} \sum_{\alpha \in G} \prod_{i=0}^{n} s_i^{j_i(\alpha)},$$

Theorem (Harary, 1955, (The generating function of A000088 ))

$$g_n(z) = Z(S_n^{(2)}, 1 + z)(s_k \to (1 + z)^k).$$

The set of all simple graph on $n$ nodes is a finite ring.

Theorem

$$g_n(z) = \frac{1}{m!} \sum_{\alpha \in S_n^{(2)}} \frac{\det(1 - \alpha \cdot z^2)}{\det(1 - \alpha \cdot z)}, m = \binom{n}{2}.$$
Reduction GI for simple graph

Let \( I = (x_1^2 - x_1, x_2^2 - x_2, \ldots, x_m^2 - x_m) \) be an ideal in \( k[x_1, x_2, \ldots, x_m] \). Consider the

\[ R_n = k[x_1, x_2, \ldots, x_m]/I \cong k[x_1, x_2, \ldots, x_m | x_1^2 = x_1, x_2^2 = x_2, \ldots, x_m^2 = x_m]. \]

\( \deg f \leq m \) (!).

The action of \( S_n^{(2)} \) on \( R_n \).

**Theorem**

Let \( R_n^{S_n^{(2)}} = k[f_1, f_2, \ldots, f_t] \). The simple graphs \( G_1 \) and \( G_2 \) are isomorphic iff \( f_i(G_1) = f_i(G_2), i = 1, \ldots, t \).
Theorem

Minimal generating set of algebra invariants of the simple graphs on 5 nodes consists of 5 invariants.

\[ R(x_1), \]
\[ R(x_1x_{10}), \]
\[ R(x_1x_2x_5), R(x_1x_2x_3), \]
\[ R(x_1x_2x_3x_9) \]
Minimal generating set for \( n = 6 \).

Minimal generating set of algebra invariants for weighted graph of 6 nodes is unknown.

**Theorem**

*Minimal generating set of algebra invariants of the simple graphs on 6 nodes consists of 12 invariants.*

\[
R(x_1), \quad R(x_1x_2), \\
R(x_1x_3x_7), R(x_1x_2x_7), R(x_1x_2x_3) \\
R(x_1x_2x_3x_9), R(x_1x_2x_13x_14), R(x_1x_2x_3x_15), R(x_1x_2x_3x_5), \\
R(x_1x_2x_3x_8x_9), R(x_1x_2x_3x_6x_8), R(x_1x_2x_3x_9x_10).
\]
Let $R_n = k[x_1, x_2, \ldots, x_n]$, $x_1^2 = x_1, x_2^2 = x_2, \ldots, x_n^2 = x_n$. Let $G$ be a subgroup of $S_n$.

Then exists a constant $a$, such that

$$N = O(n^a).$$
Thank you for your attention!