Classical invariant theory and its applications

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Outline

- Classical invariant theory
- Locally nilpotent derivations
- Combinatorics
- Graph theory

CLASSICAL INVARIANT THEORY

Let k be an algebraicaly closed field and V a finite dimensional vector space over k. Let $e_1, e_2, \dots, e_n$ be a basis of V and denote by $f_i \in V^*$ the linear function on V with

$$f_i(x_1 e_1 + \ldots + x_n e_n) = x_i.$$

The f_i generate an algebra $S(V)$ of polynomial functions on V. There is an isomorphism $S(V) = k[f_1, f_2, \dots, f_n]$. Also the algebra $S(V)$ is graded algebra.

Definition

Let $GL(V)$ be the group of all invertible linear transformations of V. If $g \in GL(V)$, $f \in S(V)$ define $g \cdot f \in S(V)$ by

$$g \cdot f(v) = f(g^{-1}v).$$

It is easy to check that it is the action

$$g(hf) = (gh)(f).$$

If G is a subgroup of $GL(V)$

Definition

We say that $f \in S(V)$ is a G-invariant if $g \cdot f = f$ for all $g \in G$.

The G-invariant polynomial functions form a graded subalgebra $S(V)^G$ of $S(V)$.
In invariant theory one studies the properties of such algebras $S(V)^G$.
The classical invariant theory is concerned with the cases when G is a classical group.

Main problems

All problems of the classical invariants theory divides into two main parts:
A.Finitely generation. Is the algebra $k[V]^G$ finitely generated?
For SL_2 Gordan (1868), for SL_n Hilbert(1890, 1893),
Reductive group
B. Constructive problem. Describe the algebra $k[V]^G$.
Compute a minimal generating set for $k[V]^G$.

Finite groups

Let G be a finite group.

Theorem (Hilbert, Noether)

The algebra $k[V]^G$ is generatedby not more than $\binom{|G|+\dim V}{\dim V}$ homogeneous invariants, of degree not exceeding $|G|$.

the Reynolds operator $S(V) \to S(V)^G$

$$R(f) = \frac{1}{|G|} \sum_{g \in G} g \cdot f, f \in S(V).$$

$k[x_1, x_2, \ldots, x_n]^{S_n} = k[R(x_1), R(x_2), \ldots, R(x_n)]$,

$$R(x_1) = x_1 + x_2 + \cdots + x_n,$$
$$\ldots$$
$$R(x_1^n) = x_1^n + x_2^n + \cdots + x_n^n.$$

Molien formula

$S(V)^G$ is graded algebra:

$$S(V)^G = (S(V)^G)_0 \oplus (S(V)^G)_1 \oplus \cdots$$

The Poincare series

$$\mathcal{P}(S(V)^G, z) = \sum_{i=0}^{\infty} \dim(S(V)^G)_i z^i,$$

is a rational function.A classical theorem of Molien gives an explicit expression for the rational function and ties together invariant theory with generating functions.

Theorem (Molien, 1897)

$$\mathcal{P}(S(V)^G, z) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1_V - z \cdot g)}.$$

Stanley, Richard P. Invariants of finite groups and their applications to combinatorics. Bull. Amer. Math. Soc. (N.S.) 1 (1979), no. 3, 475–511

$$S_k(n) = \sum_{w^n=1, w\neq 1} \frac{1}{|1-w|^{2k}}$$

$$S_1(n) = \frac{n^2-1}{12}$$

$$\sum_{k=1}^{\infty} 4^k S_k(n) x^{2k} = 1 - \frac{nx\cot(n\sin^{-1}x)}{\sqrt{1-x^2}}.$$

Invariants of binary forms

$V_1 = k^2 = \langle x, y \rangle$.
$G = SL(V_1) = SL_2$ and $V_d = S^d(V_1)$ $V = V_d \oplus V_1$.
Algebras of invariants $I_d = k[V_d]^{SL_2}$ and $C_d = k[V_d \oplus k^2]^{SL_2}$
the algebra of invariants of binary form and the algebra of covariants of binary forms.
The algebra $k[V_{d_1} \oplus V_{d_2} \oplus \cdots \oplus V_{d_n}]^{SL_2}$ — the algebra of joint invariant of binary forms of degrees d_1, d_2, $\ldots d_n$.
Note: The symbolic method $k[V_1 \oplus V_1 \oplus \cdots \oplus V_1 \cdots]^{SL_2} \to k[V_d]^{SL_2}$

Derivations

A Derivation of a ring is an additive map D satisfying the Leibniz rule

$$D(f\,g) = f\,D(g) + D(f)\,g, \text{ for all } f, g \in R.$$

The subring

$$\ker D := \{f \in k[x_0, x_1, \ldots, x_n] \mid D(f) = 0\},$$

the kernel of the derivations.

A derivation D is called locally nilpotent if for every $r \in R$ there is an $n \in \mathbb{N}$ such that $D^n(r) = 0$.

The problems **A** and **B** for $\ker D$

Weitzenböck derivations

A linear locally nilpotent derivation $\mathcal{D}$ of the polynomial algebra $k[x_1, x_2, \ldots, x_n]$ is called a *Weitzenböck derivation*.

$$\mathcal{D}(x_i) = \sum_{i=0}^{n} a_{i,j} x_i$$

The basic Weitzenböck derivation $\mathcal{D}_n$:$\mathcal{D}_n(x_i) = ix_{i-1}$.
Denote by $\mathcal{D}_{\boldsymbol{d}}$, $\boldsymbol{d} := (d_1, d_2, \ldots, d_s)$ the Weitzenbök derivation of the polynomial algebra if its matrix consists of s Jordan blocks of size $d_1 + 1$, $d_2 + 1$, $\ldots$, $d_s + 1$, respectively.
The matrix $\{a_{i,j}\}$ is nilpotent.

Kernel of Weitzenböck derivation

Theorem

Let $\mathcal{D}_{\mathbf{d}}$ be a Weitzenböck derivation. Then

$$\ker \mathcal{D}_{\mathbf{d}} \cong k[V_{d_1} \oplus \cdots \oplus V_{d_s} \cdots \oplus k^2]^{SL_2} \cong k[V_{d_1} \oplus \cdots \oplus V_{d_s}]^{\mathbb{G}_a}$$

$\mathbb{G}_a = (k, +)$ — the additive group of a field. Not a reductive group!
The second isomorphism is the Grosshans principle.
Clasical result - – Robert's theorem, 1861.

Theorem

Let $\mathcal{D}_d$ be a basic Weitzenböck derivation. Then

$$\ker \mathcal{D}_d \cong k[V_d \oplus k^2]^{SL_2} \cong k[V_d]^{\mathbb{G}_a}$$

The symbolic method

Let $\mathcal{D}_d$ be basic Weitzenböck derivation.
The kernel is a graded algebra

$$\ker \mathcal{D}_d = (\ker \mathcal{D}_d)_0 \oplus (\ker \mathcal{D}_d)_1 \oplus \cdots \oplus (\ker \mathcal{D}_d)_m \oplus \cdots$$

Theorem

Then there is a surjective map $\ker \mathcal{D}_{(1,1,\ldots,1)} \to (\ker \mathcal{D}_d)_m$

L. Bedratyuk, Weitzenböck derivations and the classical invariant theory, II: The symbolic method, Serdica Math. J.–2011.–V.37.–2–P.87–106.

Center the universal enveloping algebra

G — compact Lie group, $\mathfrak{g}$– its Lie aldebra acting by derivations on $k[V]$.
$k[V]^G = k[V]^{\mathfrak{g}}$.

Theorem

If $\mathfrak{g} = \langle D_1, D_2, \ldots D_n \rangle$, *then*

$$k[V]^G = \ker D_1 \cap \ker D_2 \cap \cdots \cap \ker D_n.$$

D_i — *locally nilpotent derivations* $k[V]$.

Theorem

Let $Z(\mathfrak{g})$ *be the center the universal enveloping algebra* $U(\mathfrak{g})$. *Then*

$$Z(\mathfrak{g}) \cong k[\mathfrak{g}]^{\mathfrak{g}}.$$

Semi-tranvectant

Let $D_n(x_i) = i x_{i-1}$ — the basic Weitzenböck derivation. Define $D_n^*(x_i) = (n-i)x_{i+1}$. Let $\mathrm{ord}(q)$ be the minimal power D_n^* of such that $(D_n^*)^{\mathrm{ord}(q)}(q) = 0$.

Theorem (Bedratyuk, 2012)

Let $p, q \in \ker D_n$, *then and* $[p,q]^r \in \ker D_n$

$$[p,q]^r = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{(\mathcal{D}^*)^i(p)}{[\mathrm{ord}(p)]_i} \frac{(\mathcal{D}^*)^{r-i}(q)}{[\mathrm{ord}(q)]_{r-i}},$$

$[m]_i = m(m-1)\dots(m-(i-1))$ *is the falling factorial.*

$x_0 \in \ker D_n$ construct

$$[x_0, x_0]^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} x_i x_{n-i}.$$

Diximier map

For arbitrary locally nilpotent derivation D the following theorem holds:

Theorem

Suppose that there exists a polynomial h such that $D(h) \neq 0$ but $D^2(h) = 0$. Then

$$\ker D = k[\sigma(x_0), \sigma(x_1), \ldots, \sigma(x_n)][D(h)^{-1}] \cap k[x_0, x_1, \ldots, x_n],$$

where σ is the Diximier map

$$\sigma(x_i) = \sum_{m=0}^{\infty} D^m(x_i) \frac{\lambda^m}{m!}, \lambda = -\frac{h}{D(h)}, D(\lambda) = -1.$$

COMBINATORICS:Identities for Appell polynomials

The Appell polynomials $\mathcal{A} = \{A_n(x)\}$, where $\deg(A_n(x)) = n$ and the polynomials satisfy the identity

$$A_n'(x) = nA_{n-1}(x), n = 0, 1, 2, \ldots .$$

Polynomials Bernoulli $B_n(x)$, Euler $E_n(x)$ Hermite $H_n(x)$, $n = 0, 1, 2, \ldots$

$$\frac{te^{xt}}{e^t - 1} = \sum_{i=0}^{\infty} B_n(x)\frac{t^n}{n!}, \frac{2e^{xt}}{e^t + 1} = \sum_{i=0}^{\infty} E_n(x)\frac{t^n}{n!}, e^{xt - \frac{t^2}{2}} = \sum_{i=0}^{\infty} H_n(x)\frac{t^n}{n!}.$$

Polynomial Identities for the Appell polynomials

$$F(A_0(x), A_1(x), \dots, A_n(x)) = 0,$$

where F is some polynomial of $n+1$ variables.
Main Idea:

$$\text{if } \frac{d}{dx}F(A_0(x), A_1(x), \dots, A_n(x)) = 0 (= \mathrm{const})$$

$$\text{then} F(A_0(x), A_1(x), \dots, A_n(x)) = \mathrm{const} = F(A_0(0), A_1(0), \dots, A_n(0)).$$

COMBINATORICS:Identities for Appell polynomials

Let now $k[a_0, a_1, \ldots, a_n]$ and $k[x]$ be algebras of polynomials, $\mathcal{D}_n(a_i) = ia_i$ - Weitzenböck derivation. Let us consider the substitution homomorphism $\varphi_{\mathcal{A}} : k[a_0, a_1, \ldots, a_n] \to k[x]$ defined by $\varphi_{\mathcal{A}}(a_i) = A_i(x)$. the homomorphism $\varphi_{\mathcal{A}}$ intertwines with the derivative operator $\dfrac{d}{dx}$,

Theorem

$$\varphi_{\mathcal{A}} \circ \mathcal{D}_n = \frac{d}{dx}\varphi_{\mathcal{A}}.$$

Theorem

If $F[a_0, a_1, \ldots, a_n] \in \ker \mathcal{D}_n$ *then we have the identity*

$$F(A_0(x), A_1(x), \ldots, A_n(x)) = F(A_0(0), A_1(0), \ldots, A_n(0)).$$

Example

$$[a_0, a_0]^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} a_i a_{n-i}.$$

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} A_i(x) A_{n-i}(x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} A_i(0) A_{n-i}(0).$$

For Bernoulli

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} B_i(x) B_{n-i}(x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} B_i B_{n-i}.$$

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} B_i B_{n-i} = (1-n) B_n.$$

Thus

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} B_i(x) B_{n-i}(x) = (1-n) B_n.$$

Discriminant

$$\mathrm{Discr}_n(a_0) :=$$

$$= \begin{vmatrix} a_0 & na_1 & \cdots & a_n & 0 & \cdots & \cdots & 0 \\ 0 & a_0 & \cdots & na_{n-1} & a_n & 0 & \cdots & 0 \\ & & \multicolumn{4}{c}{\dotfill} & & \\ 0 & \cdots & 0 & a_0 & na_1 & \binom{n}{2}a_2 & \cdots & a_n \\ na_0 & (n-1)na_1 & \ldots & na_{n-1} & 0 & 0 & \cdots & 0 \\ 0 & na_0 & \cdots & \ldots & na_{n-1} & 0 & \cdots & 0 \\ & & \multicolumn{4}{c}{\dotfill} & & \\ 0 & 0 & \cdots & na_0 & (n-1)na_1 & (n-2)\binom{n}{2}a_2 & \ldots & na_{n-1} \end{vmatrix}.$$

The corresponding identities has the form $\mathrm{Discr}_n(\mathcal{A}) = \mathrm{Discr}_n(\mathcal{A})_0$.
Conjecture. $\mathrm{Discr}_n(\mathcal{H})_0 = \prod_{k=1}^{n} k^k$

Catalecticant

The catalecticant of a binary form of even degree, $n = 2k$, can be written as the determinant of the $(k+1) \times (k+1)$ matrix

$$\mathrm{Cat}_n(a_0) := \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_k \\ a_1 & a_2 & a_3 & \cdots & a_{k+1} \\ & \multicolumn{3}{c}{\dotfill} & \\ a_{k-1} & a_k & a_{k-1} & \cdots & a_{2k-1} \\ a_k & a_{k+1} & a_{k+2} & \cdots & a_{2k} \end{vmatrix}.$$

The corresponding identities has the form $\mathrm{Cat}_n(\mathcal{A}) = \mathrm{Cat}_n(\mathcal{A})_0$.
Conjecture. $\mathrm{Cat}_n(\mathcal{H})_0 = (-1)^n n!!$.

Joint Identities

Derivation D_{d_1,d_2}.

$$\mathrm{Dv}_n(a_0, b_0) := [a_0, b_0]^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} a_i b_{n-i} \in \ker D_{n,n}.$$

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} E_i(x) B_{n-i}(x) = \mathrm{const}.$$

Resultant

$$
\mathrm{sRes}_n(a_0, b_0) := \begin{vmatrix}
a_0 & na_1 & \cdots & a_n & 0 & \cdots & \cdots & 0 \\
0 & a_0 & \cdots & na_{n-1} & a_n & 0 & \cdots & 0 \\
 & & \multicolumn{4}{c}{\dotfill} & & \\
0 & \cdots & 0 & a_0 & na_1 & \binom{n}{2} a_2 & \cdots & a_n \\
b_0 & nb_1 & \cdots & b_n & 0 & \cdots & \cdots & 0 \\
0 & b_0 & \cdots & nb_{n-1} & b_n & 0 & \cdots & 0 \\
 & & \multicolumn{4}{c}{\dotfill} & & \\
0 & \cdots & 0 & b_0 & b_1 & \binom{n}{2} b_2 & \cdots & b_n
\end{vmatrix}
$$

L. Bedratyuk, Semi-invariants of binary forms and identities for Bernoulli, Euler and Hermite polynomials, Acta Arith. 151 (2012), pp. 361-376.

Identity, general cases.

Let $\{P_n(x)\}, n = \deg P_n(x)$ — family of polynomials, bases of $k[x]$.
Then $P_n(x)'$ can be expressed via $P_0(x), P_1(x), \ldots, P_{n-1}(x)$:

$$P_n(x)' = a_{n,0}P_0(x) + a_{n,1}P_1(x) + \ldots + a_{n,n-1}P_{n-1}(x), P_0(x)' = 0.$$

Assign to the family $\{P_n(x)\}$ the derivation

$$D_P(x_n) = a_{n,0}x_0 + a_{n,1}x_1 + \ldots + a_{n,n-1}x_{n-1}, D_P(x_0) = 0.$$

Since $D_P(x_n)^{n+1} = 0$, then D_P is LND.

Theorem

If $F(x_0, x_1, \ldots, x_n) \in \ker D_P$ *then*

$$F(P_0(x), P_1(x), \ldots, P_n(x)) = \mathrm{const}.$$

Reduce to Appell polynomials. Interwining map.

$\ker D_P$?
Build isomorphism $\varphi_{AP} : \ker D_A \to \ker D_P$.

Definition

A linear map ψ_{AF} is called a $(\mathcal{D}_{\mathcal{A}}, \mathcal{D}_{\mathcal{F}})$-intertwining map if the following condition holds:

$$\psi_{AF}\mathcal{D}_{\mathcal{A}} = \mathcal{D}_{\mathcal{F}}\psi_{AF}.$$

Any such map induces an isomorphism from $\ker \mathcal{D}_{\mathcal{A}}$ to $\ker \mathcal{D}_{\mathcal{F}}$.
Any identity for Appell polynomials generates an identity for the family $\{P_n(x)\}, n = \deg P_n(x)$.

Fibbonacci and Lucas polynomials

The Fibonacci $F_n(x)$ and Lucas $L_n(x)$ polynomials. The generating functions:

$$\frac{t}{1-xt-t^2} = \sum_{n=0}^{\infty} F_n(x)t^n,$$

$$\frac{1+t^2}{1-xt-t^2} = \sum_{n=0}^{\infty} L_n(x)t^n.$$

The derivatives of the polynomials can be expressed in terms of the polynomials as follows:

$$\frac{d}{dx}F_n(x) = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k (n-1-2k) F_{n-1-2k}(x),$$

$$\frac{d}{dx}L_n(x) = n \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k L_{n-1-2k}(x).$$

Fibbonacci and Lucas derivations

The expressions motivate the following definitions

Definition

Let $D_{\mathcal{F}}$, $D_{\mathcal{L}}$ be the derivations of $k[x_0, x_1, x_2, \ldots, x_n]$ defined by:

$$D_{\mathcal{F}}(x_n) = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k (n-1-2k) x_{n-1-2k}, i = 2, 3, \ldots, n,$$

$$D_{\mathcal{F}}(x_0) = D_{\mathcal{F}}(x_1) = 0,$$

$$D_{\mathcal{L}}(x_n) = n \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k x_{n-1-2k}, i = 2, 3, \ldots, n,$$

$$D_{\mathcal{L}}(x_0) = D_{\mathcal{L}}(x_1) = 0.$$

$D_{\mathcal{F}}$, $D_{\mathcal{L}}$ are called **the Fibonacci derivation** and **the Lucas derivation** respectively.

A Appel-Lucas intertwining map

Theorem

A Appel-Lucas intertwining map has the form

$$\psi_{AL}(x_n) = x_n + \alpha_n^{(1)} x_{n-2} + \alpha_n^{(2)} x_{n-4} + \ldots + \alpha_n^{(i)} x_{n-2i} + \ldots + \alpha_n^{([\frac{n-1}{2}])} x_{n-2[\frac{n-1}{2}]},$$

where

$$\alpha_n^{(s)} = \frac{(-1)^s}{s!} b_0 n^{\underline{s}} + \cdots + \frac{(-1)^{s-i}}{(s-i)!} b_i n^{\underline{s+i}} + \cdots + b_s n^{\underline{2s}},$$

and the generating function for $b_0, b_1, \ldots, b_n, \ldots$ is

$$\sum_{i=0}^{\infty} b_i z^i = J_0^{-1}(\sqrt{4z}).$$

$n^{\underline{a}} := n(n-1)(n-2)\cdots(n-(a-1))$.

Appel-Fibonacci intertwining map

Theorem

A Appel-Fibonacci ψ_{AF} intertwining map has the form

$$\psi_{AL}(x_n) = x_{n+1} + \alpha_n^{(1)} x_{n-1} + \alpha_n^{(2)} x_{n-3} + \ldots + \alpha_n^{([\frac{n-1}{2}])} x_{n+1-2[\frac{n-1}{2}]},$$

$$\alpha_n^{(s)} = (n-2s+1)\left(\frac{(-1)^s}{s!} b_0 \underline{n^{s-1}} + \cdots + \frac{(-1)^{s-i}}{(s-i)!} b_i \underline{n^{s+i}} + \cdots + b_s \underline{n^{2s-1}}\right),$$

and the ordinary generating function for $b_0, b_1, \ldots, b_n, \ldots$ is as follows

$$\sum_{i=0}^{\infty} b_i z^i = \frac{\sqrt{z}}{J_1(\sqrt{4z}\,)}.$$

L. Bedratyuk, Derivations and Identitites for Fibonacci and Lucas Polynomials, Fibonacci Quart. 51 (2013), no. 4, 351–366

Kravchuk polynomials

$\{K_n(x,a), n = 0, 1, \ldots\}$ — Kravchuk polynomials.

$$K_n(x,a) := \sum_{i=0}^{n} (-1)^i \binom{x}{i} \binom{a-x}{n-i},$$

Generating function:

$$\sum_{i=0}^{\infty} K_i(x,a) z^i = (1+z)^a \left(\frac{1-z}{1+z} \right)^x .$$

$$\frac{d}{dx} K_n(x,a) = -2 \sum_{i=1}^{n} \frac{1-(-1)^i}{2i} K_{n-i}(x,a).$$

$$\frac{d}{da} K_n(x,a) = \sum_{i=0}^{n-1} \frac{(-1)^{n+1+i}}{n-i} K_i(x,a).$$

Kravchuk derivations

Form $\frac{d}{dx}K_n(x, a)$ and $\frac{d}{da}K_n(x, a)$ implies:

Definition

Define $D_{\mathcal{K}_1}$ $D_{\mathcal{K}_2}$ by

$$D_{\mathcal{K}_1}(x_0) = 0, D_{\mathcal{K}}(x_n) = \sum_{i=1}^{n} \frac{1-(-1)^i}{2i} x_{n-i},$$

$$D_{\mathcal{K}_2}(x_0) = 0, D_{\mathcal{K}_2}(x_n) = \sum_{i=0}^{n-1} \frac{(-1)^{n+1+i}}{n-i} x_i, n = 1, 2, \ldots, n, \ldots,$$

is called the first and the second Kravchuk derivations.

Conjecture 1. For $n > 0$

$$\sum_{i=0}^{n} K_i(x,a) \sum_{k=0}^{n-i} \frac{(-1)^k}{k!} K_1(x,a)^k S^{(k)}(n-i) =$$

$$= \begin{cases} 0, & n \text{ odd }, \\ (-1)^m (2m-1)!!\, a(a-2)(a-4)\dots(a-2(m-1)), & n = 2m, \end{cases},$$

$$S^{(k)}(n) = \sum_{m=k}^{n} \binom{n-1}{m-1} \frac{2^m k!}{m!} s\,(m,k)\,, s\,(m,k)\,\text{Stirling numbers}$$

Conjecture 2.

$$\sum_{i=0}^{n} K_i(x,a) \sum_{k=0}^{n-i} \frac{(-1)^k}{(n-i)!} K_1(x,a)^k s(n-i,k) = \begin{cases} 0, & n \text{ odd}, \\ (-1)^m \binom{x}{m}, & n = 2\,m, \end{cases}$$

Appel-Kravchuk intertwining map

Theorem

A Appel-Kravchuk ψ_{DK_1} intertwining map has the form

$$\psi_{DK_1}(x_0) = x_0, \psi_{DK_1}(x_n) = \sum_{i=1}^{n} T(n,i)x_i.$$

where

$$T(n,i) = \sum_{j=i}^{n}(-1)^{j-i}2^{n-j}j!S(n,j)\binom{j-1}{i-1},$$

$S(n,j)$– Stigling number of the second kind.

Appel-Kravchuk intertwining map

Theorem

A Appel-Kravchuk ψ_{DK_2} intertwining map has the form

$$\psi_{DK_2}(x_0) = x_0, \psi_{DK_2}(x_n) = \sum_{i=1}^{n} B(n,i)x_i, n > 0.$$

where

$$B(n,k) = k!\, S(n,k),$$

Appel-Kravchuk intertwining map

$$H_n := \det(x_{i+j-2}) = \begin{vmatrix} x_0 & x_1 & x_2 & \cdots & x_n \\ x_1 & x_2 & x_3 & \cdots & x_{n+1} \\ & \multicolumn{3}{c}{\dots\dots\dots\dots\dots} & \\ x_{n-1} & x_n & x_{n-1} & \cdots & x_{2n-1} \\ x_n & x_{n+1} & x_{n+2} & \cdots & x_{2n} \end{vmatrix}.$$

$\psi_{DK_1}(H_n) \in \ker \mathcal{D}_{\mathcal{K}_1}$ and $\psi_{DK_2}(H_n) \in \mathcal{D}_{\mathcal{K}_2}$.

Identities for Kravchuk polynomials

By using the substition homomorphism $\varphi_{\mathcal{K}}(x_i) = K_i(x, a)$ we get the two polynomial identities.

$$\varphi_{\mathcal{K}}(\psi_{DK_1}(H_n)) = \varphi_1(a), \varphi_{\mathcal{K}}(\psi_{DK_2}(H_n)) = \varphi_1(x).$$

Conjecture 3.

$$(i) \quad \varphi_1(a) = (-1)^{\frac{n(n+1)}{2}} \prod_{i=0}^{n-1} i! \prod_{i=0}^{n-2} (a+i)^{n-1-i},$$

$$(ii) \quad \varphi_2(x) = (-1)^{\frac{n(n+1)}{2}} \prod_{i=0}^{n} 2^i i! \prod_{i=0}^{n-2} (x-i)^{n-1-i}.$$

Graph invariants

Graph — $G = (V, E), E \subseteq V^{(2)}$, where $V^{(2)} = \{\{u, v\} \mid u, v \in V, u \neq v\}$.
Graphs $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ is called isomorphic if exists a bijection $\alpha : V_1 \to V_2$, such that $\{u, v\} \in E_1$ iff $\{\alpha(u), \alpha(v)\} \in E_2$.

GI decision problem

Computational problem GI.

Problem (GI)

Two graphs G_1 and G_2. Are they isomorphic?

Problem

If $\boldsymbol{GI} \in \boldsymbol{P}$?

Complexity of known GI-algorithm $O(2^{\sqrt{n \log n}})$, (1983).
For many graphs (planar) there is a polynomial algorithm for GI.

Graphs and theory of invariants

Labelled, undirected graphs as vector space.
Let $\mathbf{e}_{\{i,j\}}$ be the simple graph with one single edge $\{i,j\}$, and by $x_{\{i,j\}}\mathbf{e}_{\{i,j\}}$ a graph $\{i,j\}$, of a weight $x_{\{i,j\}}$. The set of all graphs on n nodes is a vector space $\mathcal{V}_n$ of the dimension $\binom{n}{2}$ with the basis $\mathbf{e}_{\{i,j\}}$.
The symmetric group S_n acts on $\mathcal{V}_n$:

$$g\mathbf{e}_{\{i,j\}} = \mathbf{e}_{\{g(i),g(j)\}}.$$

$x_{\{i,j\}} \to x_k$.
Group $S_n^{(2)} \cong S_n$ is a subgroup of $S_{\binom{n}{2}}$.

Algebra of invariants $k[\mathcal{V}_n]^{S_n^{(2)}}$.

Theorem

Let $k[\mathcal{V}_n]^{S_n^{(2)}} = k[f_1, f_2, \ldots, f_s]$. Then k-weighted graphs G_1 i G_2 are isomorphic iff $f_i(G_1) = f_i(G_2), i = 1, \ldots, s$.

Main results

Algebra of invariants $k[\mathcal{V}_n]^{S_n^{(2)}}$ calculated only for $n \leq 5$.

Nicolas Thiery, Algebraic invariants of graphs; a study based on computer exploration,SIGSAM Bulletin (ACM Special Interest Group on Symbolic and Algebraic Manipulation), 34(3): 9-20, September 2000

A minimal generating set for $k[\mathcal{V}_5]^{S_5^{(2)}}$ consists of 57 polynomials, $\deg f_i \leq 10$.

Simple graph

Graph G, with the set of weights $\{0, 1\}$.

$$G = \varepsilon_1 \mathbf{e}_{\{1,2\}} + \varepsilon_2 \mathbf{e}_{\{1,3\}} + \cdots + \varepsilon_m \mathbf{e}_{\{n-1,n\}} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m), m = \binom{n}{2}.$$

Number of simple graphs on n nodes is the sequence $A000088$ в The On-Line Encyclopedia of Integer Sequences:

$$1, 1, 2, 4, 11, 34, 156, 1044, 12346, 274668, 12005168, 1018997864, \ldots$$

Cycle index and Molien series

Permutation α Let $j_i(\alpha)$ be the number of cycles of length $1 \leq i \leq n$ in the disjoint cycle decomposition of α. Then the cycle index of G denoted $Z(G, s_1, s_2, \ldots, s_n)$, is the polynomial in the variables $s_1, s_2, \ldots, s_n$ defined by

$$Z(G, s_1, s_2, \ldots, s_n) = \frac{1}{|G|} \sum_{\alpha \in G} \prod_{i=0}^{n} s_i^{j_i(\alpha)},$$

Theorem (Harary, 1955, (The generating function of *A000088*))

$$g_n(z) = Z(S_n^{(2)}, 1+z)(s_k \to (1+z)^k).$$

The set of all simple graph on n nodes is a finite ring.

Theorem

$$g_n(z) = \frac{1}{m!} \sum_{\alpha \in S_n^{(2)}} \frac{\det(1 - \alpha \cdot z^2)}{\det(1 - \alpha \cdot z)}, m = \binom{n}{2}.$$

Reduction GI for simple graph

Let $I = (x_1^2 - x_1, x_2^2 - x_2, \ldots, x_m^2 - x_m)$ be an ideal in $k[x_1, x_2, \ldots, x_m]$. Consider the

$$R_n = k[x_1, x_2, \ldots, x_m]/I) \cong k[x_1, x_2, \ldots, x_m \mid x_1^2 = x_1, x_2^2 = x_2, \ldots, x_m^2 = x_m].$$

$\deg f \leq m$ (!).
The action of $S_n^{(2)}$ on R_n.

Theorem

Let $R_n^{S_n^{(2)}} = k[f_1, f_2, \ldots, f_t]$. The the simple graphs G_1 and G_2 are isomorphic iff $f_i(G_1) = f_i(G_2), i = 1, \ldots, t$.

Mimimal generating set for $n = 5$.

Theorem

Minimal generating set of algebra invariants of the simple graphs on 5 *nodes consists of* 5 *invariants.*

$$
\begin{gathered}
R(x_1), \\
R(x_1 x_{10}), \\
R(x_1 x_2 x_5), R(x_1 x_2 x_3), \\
R(x_1 x_2 x_3 x_9)
\end{gathered}
$$

Mimimal generating set for $n = 6$.

Minimal generating set of algebra invariants for weighted graph of 6 nodes is unknown.

Theorem

Minimal generating set of algebra invariants of the simple graphs on 6 *nodes consists of* 12 *invariants.*

$$R(x_1),$$
$$R(x_1 x_2),$$
$$R(x_1 x_3 x_7), R(x_1 x_2 x_7), R(x_1 x_2 x_3)$$
$$R(x_1 x_2 x_3 x_9), R(x_1 x_2 x_{13} x_{14}), R(x_1 x_2 x_3 x_{15}), R(x_1 x_2 x_3 x_5),$$
$$R(x_1 x_2 x_3 x_8 x_9), R(x_1 x_2 x_3 x_6 x_8), R(x_1 x_2 x_3 x_9 x_{10}).$$

Conjecture

Let $R_n = k[x_1, x_2, \ldots, x_n]$, $x_1^2 = x_1, x_2^2 = x_2, \ldots, x_n^2 = x_n$. Let G be a subgroup of S_n,.

Гіпотеза

Let

$$R_n^G = k[f_1, f_2, \ldots, f_N], f_i \in R_n.$$

Then exists a constant a, such that

$$N = O(n^a).$$

Thank you for your attention!