Winter Mathematical Competition
Pleven, February 3-5, 2006

Problem 9.1. (Peter Boyvalenkov) Find all pairs (a, b) of non-negative numbers such that the equations $x^2 + a^2x + b^3 = 0$ and $x^2 + b^2x + a^3 = 0$ have a common real root.

Solution: If x_0 is a common root of the equations, then

$$x_0(a^2 - b^2) = a^3 - b^3.$$

Case 1. For $a \neq b$ we have $x_0 = \dfrac{a^2 + ab + b^2}{a + b}$. Since $x_0 > 0$, it follows by the first equation that $x_0^2 + a^2x_0 + b^3 > 0$, a contradiction.

Case 2. If $a = b$, then the equations coincide. They have a real root when $D = a^4 - 4a^3 = a^3(a - 4) \geq 0$. Since $a \geq 0$, we conclude that the solutions of the problem are the pairs (a, a), where $a \in \{0\} \cup [4, +\infty)$.

Problem 9.2. (Stoyan Atanasov) Let b and c be real numbers such that the equation $x^2 + bx + c = 0$ has two distinct real roots x_1 and x_2 with $x_1 = x_2^2 + x_2$.

a) Find b and c if $b + c = 4$.

б) Find b and c if they are coprime integers.

Solution: It follows from the condition $x_1 = x_2^2 + x_2$ and Vieta's formulas that

$$\left|\begin{array}{rcr} x_1 + (b-1)x_2 & = & -c \\ x_1 + x_2 & = & -b \\ x_1x_2 & = & c \end{array}\right. ,$$

and hence $c^2 + 4(1 - b)c + b^3 - b^2 = 0$, $b \neq 2$.

a) Since $c = 4 - b$, we obtain $b^3 + 4b^2 - 28b + 32 = 0 \iff (b-2)^2(b+8) = 0$. Therefore $b = -8$ and $(b, c) = (-8, 12)$.

b) Consider $c^2 + 4(1 - b)c + b^3 - b^2 = 0$ as a quadratic equation of c. It follows that $D = 16(1 - b)^2 - 4(b^3 - b^2) = 4(1 - b)(b - 2)^2$ is a perfect square. Thus $b = 2$ or $1 - b = k^2$, where k is an integer. Then $(b, c) = (2, 2)$ or $(b, c) = (1 - k^2, k(k - 1)^2)$. Obviously the first pair is not a solution of the problem. The integers in the second pair are coprime when $k - 1 = \pm 1$, that is, $k = 2$ or $k = 0$. So $(b, c) = (-3, 2)$ or $(b, c) = (1, 0)$. In both cases the roots of the given equation are real and distinct.

Problem 9.3. (Stoyan Atanasov) Given a $\triangle ABC$, let BL, $L \in AC$, be the bisector of $\sphericalangle ABC$ and AH, $H \in BC$, the altitude to BC. Prove that $\sphericalangle AHL = \sphericalangle ALB$ if and only if $\sphericalangle BAC = \sphericalangle ACB + 90^\circ$.

Solution: $(\Rightarrow)$ Let $\sphericalangle AHL = \sphericalangle ALB = \varphi$. Denote by I the incenter of $\triangle ABH$. Then $\sphericalangle AHI = \frac{1}{2} \sphericalangle AHB = 45^\circ$ and $\sphericalangle AIL = 180^\circ - \sphericalangle AIB = 45^\circ$. Hence

$$\begin{aligned} \sphericalangle LAI + \sphericalangle LHI &= (180^\circ - \sphericalangle ALI - \sphericalangle AIL) + (\sphericalangle AHL + \sphericalangle AHI) \\ &= (180^\circ - \varphi - 45^\circ) + (\varphi + 45^\circ) = 180^\circ. \end{aligned}$$

It follows that the quadrilateral $AIHL$ is cyclic. Then $\varphi = 45^\circ$ and $\sphericalangle BAC = 90^\circ + \sphericalangle BAI = 90^\circ + \frac{1}{2}(90^\circ - \sphericalangle ABC)$. Since $\sphericalangle ABC = 180^\circ - \sphericalangle BAC - \sphericalangle ACB$, we conclude that

$$\sphericalangle BAC = \sphericalangle ACB + 90^\circ.$$

$(\Leftarrow)$ Let $\alpha = 90^\circ + \gamma$. Then AL is an external angular bisector for $\triangle ABH$. Hence L is the center of the excircle of $\triangle ABH$ tangent to the side AH. Then $\sphericalangle AHL = \frac{1}{2} \sphericalangle CHA = 45^\circ$. On the other hand,

$$\begin{aligned} \sphericalangle ALB &= 180^\circ - \sphericalangle BAL - \sphericalangle ABL = 180^\circ - \alpha - \frac{\beta}{2} \\ &= 180^\circ - \alpha - \frac{180^\circ - \alpha - \gamma}{2} = 90^\circ - \frac{\alpha - \gamma}{2} = 45^\circ. \end{aligned}$$

It follows that $\sphericalangle AHL = \sphericalangle ALB$.

Problem 9.4. (Peter Boyvalenkov) Tokens are placed in some of the cells of a table of size 8×8 such that:

(1) there is at least one token in any rectangle of size 2×1 or 1×2;

(2) there are two neighboring tokens in any rectangle of size 7×1 or 1×7.

Find the minimum possible number of tokens.

Solution: It follows from fig. 1 that 37 tokens can be placed in a way to satisfy conditions (1) and (2). We shall prove that 37 is the desired number.

It follows from (1) that there are at least 4 tokens in every column of the table. Consider the columns of the table of size 6×6 obtained by cutting outmost rows and columns of the given table. It follows from (1) that there are at least 3 tokens in every such column. If there are 3 tokens in a column 6×1 with no neighbors we have a contradiction to (2).

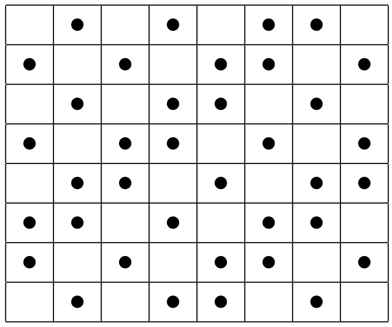

fig. 1

Therefore in a column with 3 tokens they are placed in second, third and fifth cell or in second, forth and fifth cell.

Denote by k the number of columns with 3 tokens each. There are at least 4 tokens in each of the remaining $6-k$ columns of a table 6×6 and the two outmost columns of the initial table. Note that by (1) there are 5 tokens in each column of the initial table with 3 tokens in the table 6×6.

Suppose there are two neighboring columns having 3 tokens each. Then there exists a rectangle 2×1 without a token, a contradiction. Therefore there are at most 3 columns having 3 tokens each, i.e. $k \leq 3$.

Consider the two rectangles 6×1 above and under the table 6×6. There are two cases:

Case 1. There are at most 3 tokens in one of these rectangles. Now, there are at least 5 tokens in the outmost columns of the initial table and there are at least

$$5k + 2 \cdot 5 + 4(6-k) + 2(3-k) = 40 - k \geq 37$$

tokens.

Case 2. There are at least 4 tokens in both rectangles. Then the total number of tokens is at least

$$5k + 4(8-k) + 2(4-k) = 40 - k \geq 37.$$

Problem 10.1. (Kerope Chakaryan) Consider the inequality $\sqrt{x}+\sqrt{2-x} \geq \sqrt{a}$, where a is a real number.

a) Solve the inequality for $a = 3$.

б) Find all a, for which the set of solutions of the inequality is a segment (possibly, a point) with length less than or equal to $\sqrt{3}$.

Solution: a) For $a = 3$ and $x \in [0, 2]$ the inequality is equivalent to $2\sqrt{x(2-x)} \geq 1$, that is $4x^2 - 8x + 1 \leq 0$. Hence its solutions are

$$x \in \left[\frac{2-\sqrt{3}}{2}, \frac{2+\sqrt{3}}{2}\right].$$

б) For $a \geq 0$ and $x \in [0, 2]$ the inequality is equivalent to $2\sqrt{x(2-x)} \geq a - 2$. If $a \leq 2$, then any $x \in [0, 2]$ is a solution and the condition of the problem does not hold. Let $a > 2$. Then $4x(2-x) \geq (a-2)^2$ (in particular, $x \in [0, 2]$), that is, $4x^2 - 8x + a^2 - 4a + 4 \leq 0$. It follows that $D = 16a(4-a) \geq 0$ and hence $a \in (2, 4]$. In this case the solutions of the inequality are $x \in [x_1, x_2]$, where $x_1 \leq x_2$ are the roots of the respective quadratic equation. The given condition becomes $x_2 - x_1 \leq \sqrt{3}$. Since $x_2 - x_1 = \frac{\sqrt{D}}{4} = \sqrt{a(4-a)}$, we obtain $a^2 - 4a + 3 \geq 0$. In virtue of $a \in (2, 4]$ we conclude that $a \in [3, 4]$.

Problem 10.2. (Ivailo Kortezov) Let $ABCD$ be a parallelogram. The points E and F on the sides AB and BC, respectively, are such that DE is the bisector of $\sphericalangle ADF$ and $AE + CF = DF$. The line through C and perpendicular to DE meets the side AD at L and the diagonal BD at H. Let $N = DE \cap AC$. Prove that:

a) $AE = DL$;

b) $BC = CD$ if $HN || AD$;

c) $ABCD$ is a square if $HN || AD$.

Solution: a) Let $M = DE \cap CL$ and $K \in DF \cap CL$. Then DM is an altitude and an angular bisector in $\triangle LKD$, hence $DL = DK$. Since $\triangle LKD \sim \triangle CKF$, it follows that $KF = CF$ and $AE = DF - CF = DF - KF = DK = DL$.

b) Since $\triangle ANE \sim \triangle CND$, $\triangle HNC \sim LAC$ and $\triangle LHD \sim \triangle CHB$, we obtain

$$\frac{AE}{CD} = \frac{AN}{NC} = \frac{LH}{HC} = \frac{DL}{BC}.$$

Then the equality $AE = DL$ implies that $BC = CD$.

c) It follows by b) that $ABCD$ is a rhombus, Then $DB \perp AC$ and hence H is the orthocenter of $\triangle DNC$. So $HN \perp DC$, which implies $AD \perp DC$. Thus, $ABCD$ is a square.

Problem 10.3. (Kerope Chakaryan) Find all positive integers t, x, y, z such that

$$2^t = 3^x 5^y + 7^z.$$

Solution: It follows that $2^t \equiv 1 \pmod 3$ which means that t is even. Also $2^t \equiv 2^z \pmod 5$, that is, $2^{t-z} \equiv 1 \pmod 5$ (obviously $t > z$). Then 4 divides $t - z$ and hence 2 divides z. Further, it is clear that $t \geq 6 > 2$ and therefore $0 \equiv 3^x(-3)^y + (-1)^z \pmod 8$ or, equivalently, $3^{x+y} \equiv (-1)^{y+1} \pmod 8$. If y is even, then $3^{x+y} \equiv -1 \pmod 8$, a contradiction. So y is odd and $3^{x+y} \equiv 1 \pmod 8$. It follows that $x + y$ is even and hence x is odd. Set $t = 2m$ ($m \geq 3$), $z = 2n$ ($n \geq 1$) and write the equation in the form

$$(2^m - 7^n)(2^m + 7^n) = 3^x 5^y.$$

Since $(2^m - 7^n, 2^m + 7^n) = 1$, three cases are possible:

1) $2^m - 7^n = 3^x, 2^m + 7^n = 5^y$;

2) $2^m - 7^n = 5^y, 2^m + 7^n = 3^x$;

3) $2^m - 7^n = 1, 2^m + 7^n = 3^x 5^y$.

In the first two cases we have $2^m \mp 7^n = 3^x$. Having in mind that $m \geq 3$ and x is odd, we get $\mp(-1)^n \equiv 3 \pmod 8$, that is, $3 \equiv \pm 1 \pmod 8$, a contradiction.

In the third case the equality $2^m - 7^n = 1$ implies that $2^m \equiv 1 \pmod 7$. Then 3 divides m. Let $m = 3k$. It follows that $(2^k - 1)(2^{2k} + 2^k + 1) = 7^n$. It is easy to see that that $(2^k - 1, 2^{2k} + 2^k + 1)$ equals 1 or 3. Hence $2^k - 1 = 1, 2^{2k} + 2^k + 1 = 7^n$. Then $k = 1, n = 1, m = 3, t = 6, z = 2$ and using that we find $x = y = 1$.

In conclusion, the only solution of the problem is $t = 6, x = 1, y = 1, z = 2$.

Problem 10.4. (Ivailo Kortezov) There are 40 knights in a kingdom. Every morning they fight in pairs (everyone has exactly one enemy to fight with) and every evening they sit around a table (during the evening they do not change their sits).

a) Find the least number of days such that the fights can be arranged in a way that every two knights have fought at least once.

b) Find the least number of days such that the evenings can be arranged in a way that every two knights have been neighbors around the table.

Solution: a) There are $40.39/2 = 20.39$ pairs of knights. Since there are 20 pairs every morning we need at least 39 days. The arrangement can be done in the following way: place 39 of the knights A_1, A_2, ..., A_{39} at the vertices of a regular 39-gon and place the last knight B at the center. Let B fights A_i on the day i and the remaining fights be $A_{i-j}A_{i+j}$ (the chord $A_{i-j}A_{i+j}$ is perpendicular to BA_i and the indices are taken modulo 39). Since 39 is an odd number every chord is perpendicular to one radius and therefore every pair fights in a certain day.

б) The necessary pairs are $40.39.2/2 = 40.39$. Since there are 40 pairs we need at least 39 evenings. Using a) the arrangement can be done in the following way: connect all segments corresponding to the fights on days i and $i+1$ (the days are numbered modulo 39). We obtain the closed broken line

$$BA_iA_{i+2}A_{i-2}A_{i+4}A_{i-4}...A_{i+38}A_{i-38}$$

(note that $A_{i-38} = A_{i+1}$), which includes 40 points without repetition (no two indices differ by 39 because of parity argument and since the largest difference equals $38 - (-38) < 2.39$). Therefore this broken line contains all 40 points and we take this distribution of the knights around the table at evening i. According to a) every two knights are neighbors on the day before their fight and on the day of the fight.

Problem 11.1. (Emil Kolev) Let a be a real number. Solve the equation

$$\log_a(a^{2(x^2+x)} + a^2) = x^2 + x + \log_a(a^2 + 1).$$

Solution: It is clear that $a > 0$, $a \neq 1$. We have $a^{x^2+x}(a^2+1) = a^{2(x^2+x)} + a^2$. Setting $u = a^{x^2+x}$, we obtain the equation $u^2 - (a^2+1)u + a^2 = 0$ with roots 1 и a^2. Then $x^2 + x = 0$ and $x^2 + x - 2 = 0$, respectively. Thus, for any $a > 0$, $a \neq 1$, the equation has four roots $x = -2, -1, 0, 1$.

Problem 11.2. (Aleksandar Ivanov) Given a $\triangle ABC$ with $\sphericalangle ACB = 60^\circ$, define the sequence of points $A_0, A_1, \ldots, A_{2006}$ in the following way: $A_0 =$

A, A_1 is the orthogonal projection of A_0 on BC, A_2 is the orthogonal projection of A_1 on AC, $\ldots$, A_{2005} is the orthogonal projection of A_{2004} on BC and A_{2006} is the orthogonal projection of A_{2005} on AC. The sequence of points $B_0, B_1, \ldots, B_{2006}$ is defined in a similar way: $B_0 = B$, B_1 is the orthogonal projection of B_0 on AC, B_2 is the orthogonal projection of B_1 on BC and so on. Prove that the line $A_{2006}B_{2006}$ is tangent to the incircle of $\triangle ABC$ if and only if

$$\frac{AC+BC}{AB} = \frac{2^{2006}+1}{2^{2006}-1}.$$

Solution: We have $CA_1 = \frac{1}{2}CA_0$, $CA_2 = \frac{1}{4}CA_0$ and so on. Then $CA_{2006} = \frac{1}{2^{2006}}CA_0 = \frac{1}{2^{2006}}CA$ and analogously $CB_{2006} = \frac{1}{2^{2006}}CB$. It follows that $A_{2006}B_{2006} \parallel AB$ and $A_{2006}B_{2006} = \frac{1}{2^{2006}}AB$. Since the line $A_{2006}B_{2006}$ is tangent to the incircle of $\triangle ABC$ if and only if the quadrilateral $ABB_{2006}A_{2006}$ is cyclic, we have

$$\begin{aligned} & AB + A_{2006}B_{2006} = AA_{2006} + BB_{2006} \\ & \Leftrightarrow AB + \frac{AB}{2^{2006}} = \frac{(2^{2006}-1)(AC+BC)}{2^{2006}} \Leftrightarrow \frac{AC+BC}{AB} = \frac{2^{2006}+1}{2^{2006}-1}. \end{aligned}$$

Problem 11.3. (Aleksandar Ivanov) Let a be an integer. Find all real numbers x, y, z such that

$$a(\cos 2x + \cos 2y + \cos 2z) + 2(1-a)(\cos x + \cos y + \cos z) + 6 = 9a.$$

Solution: Using the formula $\cos 2\alpha = 2\cos^2\alpha - 1$, the equation becomes

$$a(\cos^2 x + \cos^2 y + \cos^2 z) + (1-a)(\cos x + \cos y + \cos z) + 3 - 6a = 0.$$

Consider the function $f(t) = at^2 + (1-a)t + 1 - 2a$, $t \in [-1, 1]$. The roots of the equation $f(t) = 0$ are $t_1 = -1$ and $t_2 = \frac{2a-1}{a}$, $a \neq 0$. Three cases are possible:

1) $a < 0$. Since $\frac{2a-1}{a} > 1$, it follows that $f(t) \geq 0$ for any $t \in [-1, 1]$ and $f(t) = 0$ if and only if $t = -1$.

2) $a = 0$. Then $f(t) = t + 1 \geq 0$ for any $t \in [-1, 1]$ and $f(t) = 0$ if and only if $t = -1$.

3) $a > 0$. Then $a \geq 1$ and hence $\frac{2a-1}{a} \geq 1$ with equality for $a = 1$. It follows that $f(t) \geq 0$ for any $t \in [-1, 1]$. Moreover, if $a > 1$ then $f(t) = 0$ for $t = -1$, and if $a = 1$, then $f(t) = 0$ for $t = \pm 1$.

Since our equation has the form $f(\cos x) + f(\cos y) + f(\cos z) = 0$, we conclude that:

– if $a \neq 1$, then $\cos x = \cos y = \cos z = -1$. Hence the solutions of the problem are $x = (2k+1)\pi$, $y = (2l+1)\pi$, $z = (2m+1)\pi$, where $k, l, m \in \mathbb{Z}$.

– if $a = 1$, then in addition to the above solutions we also have $\cos x = \cos y = \cos z = 1$, that is, $x = 2r\pi$, $y = 2s\pi$, $z = 2t\pi$, where $r, s, t \in \mathbb{Z}$.

Problem 11.4. (Emil Kolev) A positive integer a with decimal representation of 2006 digits is called "bad" if all its digits are equal to 1, 2 or 3 and 3 does not divide any integer formed by three consecutive digits of a.

a) Find the number of all bad integers.

b) Let a and b be different bad integers such $a+b$ is also a bad integer. Denote by k the number of positions, where the digits of a and b coincide. Find all possible values of k.

Solution: a) Let $n > 1$ and $a = \overline{a_1 a_2 \dots a_n}$ be a positive integer with digits 1, 2 or 3. Since 3 divides exactly one of the integers $\overline{a_{n-1}a_n 1}$, $\overline{a_{n-1}a_n 2}$ and $\overline{a_{n-1}a_n 3}$, then exactly two of them are bad. It follows that adding 1, 2 or 3 to a one obtains exactly two bad integers with $n+1$ digits. Since the number of the two-digit integers whose decimal representations contain only the digits 1, 2 or 3 equals 9, the answer of a) is $9 \cdot 2^{2004}$.

b) The integers $122122\dots12212$ and $233233\dots23323$ are bad and their sum $355355\dots35535$ is also a bad integer. Thus 0 is one of the possible values of k.

Let now $a = \overline{a_1 a_2 \dots a_n}$ and $b = \overline{b_1 b_2 \dots b_n}$ be different bad integers and suppose that their sum is also a bad integer. Then 3 does not divide $a_i + a_{i+1} + a_{i+2}$, $b_i + b_{i+1} + b_{i+2}$ and $a_i + a_{i+1} + a_{i+2} + b_i + b_{i+1} + b_{i+2}$. This means that $a_i + a_{i+1} + a_{i+2} \equiv b_i + b_{i+1} + b_{i+2} \equiv 1, 2 \pmod 3$. Assume that two of the digits a_i, a_{i+1}, a_{i+2} coincide with the respective digits b_i, b_{i+1}, b_{i+2}. It follows from above that the third digits also coincide. Continuing in the same way, we conclude that $a = b$, a contradiction.

So, among any three consecutive digits of a, at most one coincides with the respective digit of b. On the other hand, if $a_i = b_i$, then $a_{i+3} = b_{i+3}$

(and analogously $a_{i-3} = b_{i-3}$). Indeed $a_i + a_{i+1} + a_{i+2} \equiv b_i + b_{i+1} + b_{i+2} \pmod 3$ implies that $a_{i+1} + a_{i+2} \equiv b_{i+1} + b_{i+2} \pmod 3$. If $a_{i+3} \neq b_{i+3}$, then $a_i + a_{i+1} + a_{i+2} \equiv b_i + b_{i+1} + b_{i+2} \pmod 3$ which is impossible.

Thus, if $k > 0$, then among any three consecutive digits of a exactly one coincides with the respective digit of b. It follows that $k = 669$ or $k = 668$.

Problem 12.1. (Oleg Mushkarov) Consider the function

$$f(x) = \frac{x^2 - 2006x + 1}{x^2 + 1}.$$

a) Solve the inequality $f'(x) \geq 0$.

b) Prove that $|f(x) - f(y)| \leq 2006$ for any real numbers x and y.

Solution: a) We have

$$f'(x) = \frac{(2x - 2006)(x^2 + 1) - 2x(x^2 - 2006x + 1)}{(x^2 + 1)^2} = \frac{2006(x^2 - 1)}{(x^2 + 1)^2}.$$

Hence $f'(x) \geq 0$ if and only if $x \in (-\infty, -1] \cup [1, +\infty)$.

b) It follows from a) that $f(x)$ increases for $x \in (-\infty, -1) \cup (1, +\infty)$ and decreases for $x \in (-1, 1)$. Hence its maximum equals $f(-1) = 1004$ and its minimum equals $f(1) = -1002$. Then $|f(x) - f(y)| \leq |1004 - (-1002)| = 2006$ for any x and y.

Problem 12.2. (Oleg Mushkarov) Let k be a circle with center O and radius $\sqrt{5}$. Points M and N lie on a diameter of k and $MO = NO$. Chords AB and AC, passing through M and N, respectively, are such that

$$\frac{1}{MB^2} + \frac{1}{NC^2} = \frac{3}{MN^2}.$$

Find the length of MO.

Solution: Let M and N lie on the diameter PQ ($M \in PO, N \in QO$). Set $x = MO = NO$, $0 \leq x \leq \sqrt{5}$. Then

$$MA.MB = MP.MQ = (\sqrt{5} - x)(\sqrt{5} + x) = 5 - x^2.$$

Analogously $NA.NC = 5 - x^2$. It follows that

$$\frac{1}{MB^2} + \frac{1}{NC^2} = \frac{MA^2 + NA^2}{(5 - x^2)^2}.$$

Using the median formula we get

$$5 = AO^2 = \frac{1}{4}\left[2(MA^2 + NA^2) - 4x^2\right],$$

i.e. $MA^2 + NA^2 = 2(5 + x^2)$. Therefore

$$\frac{1}{MB^2} + \frac{1}{NC^2} = \frac{2(5 + x^2)}{(5 - x^2)^2} = \frac{3}{4x^2}.$$

Hence $x^4 + 14x^2 - 15 = 0$, i.e. $x = 1$.

Problem 12.3. (Ivan Landgev) Find the maximal cardinality of a set of phone numbers satisfying the following three conditions:

a) all of them are five-digit numbers (the first digit can be 0);

b) each phone number contains at most two different digits;

c) the deletion of an arbitrary digit in two arbitrary phone numbers (possibly in different positions) does not lead to identical sequences of digits of length 4.

Solution: Let C be a set of phone numbers satisfying the above three conditions. Assume that C has maximal cardinality. Denote by A the set of phone numbers in C which have four or five equal digits, and by B the set of phone numbers in C which have exactly three equal digits. Obviously, $C = A \cup B$. Also $|A| \leq 10$, since any digit can appear four or five times in at most one number in C.

Denote by $B_{i,j}$, $0 \leq i, j \leq 9$, $i \neq j$, the set of phone numbers which contain three digits i and two digits j. We shall prove that the maximal cardinality of $B_{i,j} \cup B_{j,i}$ is 4. It is enough to consider the case $i = 0$, $j = 1$. Let a_i be the number of phone numbers in $B_1 = B_{0,1} \cup B_{1,0}$ with i blocks (a sequence $a_i, \ldots, a_j$ is called a block if $a_{i-1} \neq a_i = \cdots = a_j \neq a_{j+1}$.) Assume that $|B_1| = 5$. Then

$$\begin{aligned} a_2 + a_3 + a_4 + a_5 &= 5 \\ 2a_2 + 3a_3 + 4a_4 + 5a_5 &\leq 14 \end{aligned}$$

since any two phone numbers have no common subsequence of length four. Moreover, it is easy to see that $a_2 \leq 2$ и $a_3 \leq 2$. Hence $a_2 = a_3 = 2$, $a_4 = 1$. Then $01110, 10001 \in B_1$ and it follows that B_1 does not contain a phone number with two blocks.

On the other hand, it is possible to find four phone numbers in B_1 which satisfy the condition c). Take, for example,

$$B_1 = \{10001,\ 01010,\ 11100,\ 00111\}.$$

The set C can be written as

$$C = A \cup B = A \cup \left(\cup_{0\leq i<j\leq 9} B_{i,j} \cup B_{j,i}\right).$$

It is clear that the choices of phone numbers in $B_{i,j} \cup B_{j,i}$ and $B_{k,l} \cup B_{l,k}$ are independent for $(i,j) \neq (k,l)$. Moreover, we may choose ten phone numbers in A which are not in conflict with any choice of the other numbers in C. Take, for example,

$$A = \{00000, 11111, \ldots, 99999\}.$$

Thus the maximal cardinality of C equals

$$\begin{aligned} |C| &= |A| + \sum_{0\leq i<j\leq 9} |B_{i,j} \cup B_{j,i}| \\ &= 10 + \binom{10}{2} \cdot 4 \\ &= 10 + 45 \cdot 4 = 190. \end{aligned}$$

Problem 4. (Nikolai Nikolov) Let O be the circumcenter of a triangle ABC with $AC = BC$. The line AO meets the side BC at D. If $|BD|$ and $|CD|$ are integers, and $|AO| - |CD|$ is a prime number, find these three numbers.

Solution: Set $AO = R$, $BD = b$, $CD = c$ and $OD = d$. Since CO is the bisector of $\sphericalangle ACD$, then

$$\frac{d}{R} = \frac{c}{b+c}.$$

Let the line AO meet the circumcircle of $\triangle ABC$ at E. Then $AD.DO = BD.CD$, i.e.

$$(R+d)(R-d) = bc.$$

Since $d = \dfrac{cR}{b+c}$, it follows that

$$R^2 = \frac{(b+c)^2 c}{b+2c}.$$

Set $k = (b, c, R),\ m = \left(\dfrac{b}{k}, \dfrac{c}{k}\right),\ R_1 = \dfrac{R}{k},\ b_1 = \dfrac{b}{km}$ и $c_1 = \dfrac{c}{km}$. Then

$$R_1^2 = \frac{m^2(b_1+c_1)^2 c_1}{b_1+2c_1}.$$

Since $(m, R_1) = 1$ and $(b_1+2c_1, b_1+c_1) = (b_1+2c_1, c_1) = (b_1, c_1) = 1$, we obtain $R_1^2 = (b_1+c_1)^2 c_1$ and $m^2 = b_1 + 2c_1$. Hence c_1 is a perfect square, say $c_1 = n^2$. Now $c = kmc_1 = kmn^2,\ b = kmb_1 = km(m^2-2n^2)$ and $R = kR_1 = kn(m^2-n^2)$.

The inequality $1 > \sin \measuredangle BAC = \dfrac{b+c}{2R} = \dfrac{m}{2n}$ shows that $\sqrt{2}n < m < 2n$. (Conversely, this condition implies that such a $\triangle ABC$ exists, it is acute and the line AO meets side BC.) In particular, $n \geq 2$. Since $R - c = kn(m^2 - n^2 - mn)$ is a prime number, it follows that n is a prime number, $k = 1$ and $m^2 - n^2 - mn = 1$, i.e. $(m-1)(m+1) = n(m+n)$. Hence n divides either $m-1$ or $m+1$.

1) Let $m - 1 = ln$. Then $l(ln+2) = ln + 1 + n$, i.e.

$$n = \frac{1-2l}{l^2-l-1}.$$

Since $n < 0$ for $l \geq 2$ we get $l = 1$ and $n = 1$, a contradiction.

2) Let $m + 1 = ln$. Then $l(ln-2) = ln - 1 + n$, i.e.

$$n = \frac{2l-1}{l^2-l-1}.$$

Since $n \leq 1$ for $l \geq 3$ and $n = -1$ for $l = 1$ we get $l = 2$. Then $n = R - c = 3,\ m = 5,\ b = 35$ and $c = 45$.