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A B S T R A C T S

Communications

A Lyusternik-Graves theorem for the proximal point method

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We consider a generalized version of the proximal point algorithm for solving the perturbed inclusion $y \in T(x)$, where y is a perturbation element near 0 and T is a set-valued mapping acting from a Banach space X to a Banach space Y which is metrically regular around some point $(\bar{x}, 0)$ in its graph. We study the behavior of the convergent iterates generated by the algorithm and we prove that they inherit the regularity properties of T, and vice versa. We analyze the cases when the mapping T is metrically regular and strongly regular.

More specifically, choose a sequence of Lipschitz continuous function $g_n : X \to Y$ and consider the following algorithm:

$$0 \in g_n(x_{n+1} - x_n) + T(x_{n+1}) \;\text{ for } n = 0, 1, 2, \ldots. \tag{1}$$

If the Lipschitz constants λ_n are upper bounded by one over twice the regularity modulus of T around the reference solution, then for any initial point sufficiently close to the solution, there exists a sequence satisfying (1) which is linearly convergent to this solution (see [1, Theorem 3.1]). Furthermore, the convergence is superlinear when λ_n converges to 0. When T happens to be strongly regular, the sequence is unique (within a certain neighborhood). In [3] the authors prove something more, that for any y close to 0, if one considers the perturbed problem

$$y \in T(x), \tag{2}$$

then under metric regularity there exist a solution to this equation and a proximal point sequence satisfying

$$y \in g_n(x_{n+1} - x_n) + T(x_{n+1}) \;\text{ for } n = 0, 1, 2, \ldots \tag{3}$$

which converges (super)linearly to that solution as long as the sequence λ_n is sufficiently small for all n. Similarly, local uniqueness of the sequence is guaranteed under strong regularity.

In this talk, based on [2], we consider the exact generalized proximal point algorithm (3), as in [1, 3]. We will follow the same idea from [4, 6], where the authors extend the paradigm of the Lyusternik–Graves theorem (see, e.g., [5]) to the framework of a mapping acting from the pair initial point-parameter to the set of convergent Newton sequences associated with them. Under some surjectivity assumption, known as *ample parameterization*, the (strong) metric regularity of the generalized equation is proved to be equivalent to the (strong) metric regularity of the inverse mapping associated with convergent Newton sequences. These results can be understood as some sort of Lyusternik–Graves theorem, where instead of considering a metrically regular mapping which is perturbed by some Lipschitz function, we iteratively perturb a mapping by several Lipschitz functions whose Lipschitz moduli are small enough to preserve, no only the metric regularity of their sum, but the Lipschitzian properties of the mapping that associates to each pair initial iteration-perturbation the set of converging sequences satisfying the algorithm.

The main results in [2] that will be presented show the equivalence between the metric regularity of T and the Aubin property of the set of convergent proximal point sequences generated by the proximal point method, and the strong regularity of T and the existence of a Lipschitz single-valued localization of the mapping associated with the convergent sequences.

References

[1] Aragón Artacho, F.J., Dontchev, A.L., Geoffroy, M.H.: Convergence of the proximal point method for metrically regular mappings. ESAIM Proc. **17**, 1–8 (2007)

[2] Aragón Artacho, F.J., Gaydu, M.: A Lyusternik-Graves theorem for the proximal point method. To appear in Comput. Optim. Appl. (2011)

[3] Aragón Artacho, F.J., Geoffroy, M.H.: Uniformity and inexact version of a proximal method for metrically regular mappings. J. Math. Anal. Appl. **335**, no. 1, 168–183 (2007)

[4] Aragón Artacho, F.J., Dontchev, A.L., Gaydu, M., Geoffroy, M.H., Veliov, V.M.: Metric regularity of Newton's iteration. SIAM J. Control Optim. **49**, no. 2, 339–362 (2011)

[5] Dontchev, A.L., Rockafellar, R.T.: Implicit functions and solution mappings. A view from variational analysis. Springer Mathematics Monographs, Dordrecht (2009)

[6] Dontchev, A.L., Rockafellar, R.T.: Newton's method for generalized equations: a sequential implicit function theorem. Math. Program. **123**, no. 1, Ser. B, 139–159 (2010)

[7] Rockafellar, R.T.: Monotone operators and the proximal point algorithm. SIAM J. Control Optim. **14**, no. 5, 877-898 (1976)

On Solutions of Perturbed Optimization Problems

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By a space we understand a completely regular topological Hausdorff space. We use the terminology from [2, 5]. Let $\mathbb{R}$ be the space of reals and $\mathbb{R}^\infty = \mathbb{R} \cup \{+\infty\}$.

Let X be a topological space.

Denote by $C(X)$ the Banach space of all bounded continuous functions $f : X \longrightarrow \mathbb{R}$ with the sup-norm $\| f \| = \sup\{|f(x)|; x \in X\}$.

For any function $\psi : X \longrightarrow \mathbb{R}^\infty$ and $Y \subseteq X$ we put $\inf_Y(\psi) = \inf\{\psi(x) : x \in Y\}$, $m_Y(\psi) = \{x \in Y : \psi(x) = \inf_Y(\psi)\}$ and $\mathrm{dom}\,(\psi) = \{x \in X : \psi(x) < +\infty\}$.

The function ψ is called *proper* if hers domain $\mathrm{dom}\,(\psi)$ is non-empty.

A minimization problem (X, ψ) is called:

– Tychonoff well-posed if $m_X(\psi)$ is a singleton and every minimizing sequence $\{x_n \in X : n \in \mathbb{N}\}$ of the function ψ is convergent to a point from $m_X(\psi)$;

– almost-well-posed if every minimizing sequence $\{x_n \in X : n \in \mathbb{N}\}$ of the function ψ has a cluster point in X;

– weakly Tychonoff well-posed if $m_X(\psi)$ is a compact set and every minimizing sequence $\{x_n \in X : n \in \mathbb{N}\}$ of the function ψ has an accumulation point.

A function $f : X \longrightarrow \mathbb{R}^\infty$ is called *lower semi-continuous* (respectively, *upper semi-continuous*) if and only if the set $\{x \in X : f(x) > t\}$ (respectively, $\{x \in X : f(x) < t\}$) is an open set for every $t \in \mathbb{R}$.

Let B be a Banach space and $\Phi : B \longrightarrow C(X)$ be a continuous linear operator. For a proper bounded from below lower semi-continuous function $\psi : X \longrightarrow \mathbb{R}^\infty$ on a space X consider the following sets of continuous perturbations:

– $SM(\psi, B, \Phi) = \{b \in B : m_X(\Phi(b) + \psi)$ *is a singleton*$\}$;

– $TWP(\psi, B, \Phi) = \{b \in B :$ *the minimization problem* $(X, \Phi(b) + \psi)$ *is Tychonoff well-posed*$\}$;

– $aWP(\psi, B, \Phi) = \{b \in B :$ *the minimization problem* $(X, \Phi(b)+\psi)$ *is almost-well-posed*$\}$;

– $wTWP(\psi, B, \Phi) = \{b \in B :$ *the minimization problem* $(X, \Phi(b) + \psi)$ *is weakly Tychonoff well-posed*$\}$.

We present conditions under which the set of continuous perturbations of a given lower semi-continuous function attains minimum on a subset with concrete properties is "big" in a topological sense. These problems are typical for the distinct variational principles in optimization. Some optimization problems in topological spaces were studied in [1, 3, 4, 5].

References

[1] I. Ekeland, *On the variational principle*, J. Math. Anal Appl. 74 (1974), 324-353.

[2] R. Engelking, *General Topology*, PWN. Warszawa, 1977.

[3] P. S. Kenderov and J. Revalski, *Dense existence of solutions of perturbed optimization problems and topological games*, Com. rendus Acad. Bulgare Sci. 63:7 (2010) 937–942.

[4] R. Deville, G. Godefroy and V. Zizler, *A smooth variational principle with applications to Hamilton-Jacobi equations in nfinite dimensions*, J. Functional Analysis 111 (1993), 197–212.

[5] R. Lucchetti and J. Revalski (eds), *Recent Developments in Well-Posed Variational Problems*, Kluwer Academic Publishers. Dordrecht-Boston-London, 1995.

Approximations, attractors and singular perturbations of evolution systems

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In the talk we consider discrete approximations of non-convex valued evolution inclusions in a evolution triple $\mathfrak{X} \subset H \subset \mathfrak{X}^*$, where H is a Hilbert space and $\mathfrak{X}$ a reflexive and separable Banach space embedded densely in H with embedding map continuous and compact.

with the following right–hand side:

$$\dot{x}(t) + Ax \in F(x), \; x(0) = x_0 \in H, \; t \in I = [0, 1]. \tag{4}$$

where $A : \mathfrak{X} \to \mathfrak{X}^*$ and $F : H \rightrightarrows H$.

The nonempty closed bounded subset $\mathfrak{A}$ of H is said to be invariant attractor of (4), when it is minimal invariant set such that $\lim\limits_{t\to\infty} Ex(Reach(t,(4)),\mathfrak{A}) = 0$ for any initial condition x_0. Here $Reach(t,(4))$ is the reachable set of (4) at the time t and $Ex(A,B) = \sup\limits_{a\in A} \inf\limits_{b\in B} \|a-b\|$.

We show existence of attractor in two cases – asymptotic compactness and one sided Lipschitz condition with negative constant.

Examples of partial differential equations control systems are provided.

Afterwards we discuss the connections between existence of attractors and existence of exact upper limit of the solution and reachable set of singularly perturbed control systems.

Generalized Nash equilibrium problems under relaxed assumptions

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The generalized Nash equilibrium problems have increasing popularity because of their applications in many applied areas. We consider a class of generalized Ky Fan inequalities (quasi-variational inequalities) in which the involved multi-valued mapping is lower semi-continuous. We present a relaxed version of the generalized Nash equilibrium problem involving strategy maps, which are only lower semi-continuous. This relaxed version may have no exact Nash equilibrium, but has ε-Nash equilibrium for every $\varepsilon > 0$. The proof involves parametric variational principles. We give positive answers to two questions (in the compact case) raised in a recent paper of Cubiotti and Yao.

We consider several definitions of semi-continuity for multifunctions, combining the topological and the ordered structure of a Banach space induced by a closed convex cone. Several types of Nash equilibrium theorems for multifunctions are presented. As corollaries we obtain saddle point theorems for convex-concave multifunctions, which can be considered as generalization to the vector-valued set-valued case of the classical minimax theorems.

Variational Approach to Second-Order Optimality Conditions for Control Problems with Pure State Constraints

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We consider the following optimal control problem of the Bolza form:

$$(P) \qquad \text{Minimize} \int_0^1 l(t, x(t), u(t))dt$$

over measurable u and solutions x of the control system

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)) \qquad u(t) \in U(t) \quad a.e. \\ x(0) = x_0 \\ x(t) \in K, \quad \forall t \in [0,1] \end{cases}$$

where the maps $f\colon [0,1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $l\colon [0,1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, the set-valued map $U\colon [0,1] \rightsquigarrow \mathbb{R}^m$, the subset $K \subset \mathbb{R}^n$ and the initial state $x_0 \in \mathbb{R}^n$ are given.

Second-order optimality conditions for optimal control problems have been studied for almost half a century but it remains a challenge to formulate them in a very general context. In some recent works (see [1, 4]), second-order necessary optimality conditions for problems with pure state constraints were obtained by using an abstract infinite dimensional optimization problem. On the other hand, in the absence of pure state constraints, second-order necessary optimality conditions can be obtained by using a variational approach (see for instance [2, 3]). Inspired by these variational techniques, we use a variational approach to deduce new second-order necessary optimality conditions for the problem (P).

In addition to the usual second-order derivative of the Hamiltonian, our second-order necessary conditions contain extra terms involving second-order tangents to the set of admissible trajectory-control pairs at the extremal process under consideration.

This approach allows a direct proof in which we use a new second-order variational equation. Further, it allows to separate the proofs of the first- and second-order necessary conditions. In particular, our result applies to any first-order necessary optimality conditions in the form of the constrained maximum principle. Another important advantage is that the presence of pure state constraints does not lead to any restrictions on the control constraints, i.e. the result is applicable for any measurable set-valued map U. Finally, we do not assume any regularity of the optimal control other than measurability and also the dynamics of the control system are allowed to be merely measurable in the time variable t.

To prove our results we consider a new second-order Mangasarian-Fromovitz like qualification condition. This raises up new open problems concerning its verification and stability.

References

[1] J. F. Bonnans and A. Hermant. Second-order analysis for optimal control problems with pure state constraints and mixed control-state constraints. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26 (2009), 561–598.

[2] A. A. Milyutin and N. P. Osmolovskii. *Calculus of Variations and Optimal Control.* American Mathematical Society, 1998.

[3] N. P. Osmolovskii. Quadratic Extremality Conditions for broken Extremals in the General Problem of the Calculus of Variations. *J. Math. Sci. (N. Y.)*, 123 (2004), 3987–4122.

[4] Z. Páles and V. Zeidan. Optimal control problems with set-valued control and state constraints. *SIAM J. Optim.*, 14 (2003), 334–358.

Subdifferential estimate of the directional derivative of lower semicontinuous functions

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Let X be a Banach space and let $f : X \to \mathbb{R}\cup\{\infty\}$ be a function. The (radial) *directional derivative of f at point $\bar{x} \in \operatorname{dom} f$ in direction $d \in X$* is defined by

$$f'(\bar{x}; d) := \liminf_{t \searrow 0} \frac{f(\bar{x} + td) - f(\bar{x})}{t} \in \overline{\mathbb{R}} .$$

For a subdifferential ∂ on the class of lower semicontinuous functions on X, we discuss the following Controlled subdifferential estimate of the Directional Derivative, which extends the known formula for convex functions:

(CDD) *For any lower semicontinuous $f : X \to \mathbb{R} \cup \{\infty\}$, $\bar{x} \in \operatorname{dom} f$ and $d \in X$, there exists a sequence $\{(\bar{x}_n, \bar{x}_n^*)\} \subset \partial f$ such that*

(i) $x_n \to \bar{x}$ *and* $f(x_n) \to f(\bar{x})$;

(ii) $f'(\bar{x}; d) \leq \liminf_{n\to\infty} \langle x_n^*, d\rangle$;

(iii) $\limsup_{n\to\infty} \langle x_n^*, x_n - \bar{x}\rangle \leq 0$.

We show that this property is equivalent to other subdifferential properties, such as controlled dense subdifferentiability, optimality criterion, mean value inequality and separation principles. As an application, we obtain a first-order sufficient condition for optimality, which extends the known condition for differentiable functions in finite-dimensional spaces. Such a condition can be regarded as expressing a maximality property of the subdifferential operator with respect to monotone operators, namely:

Assume that (CDD) holds. Then, for every lower semicontinuous $f : X \to \mathbb{R} \cup \{\infty\}$, the operator $\partial f : X \rightrightarrows X^*$ is not properly contained in any monotone operator. In particular, if f is convex, then ∂f is maximal monotone.

*Joint work with Florence Jules.

Approximating fixed points of Bregman nonexpansive type mappings in Banach spaces

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Diverse notions of nonexpansive type mappings have been generalized to the more general framework of Bregman distances in reflexive Banach spaces. We study these classes of mappings, where the existence and approximation of fixed points and asymptotic fixed points is concerned. Then the asymptotic behavior of Picard and Mann type iterations is discussed for quasi-Bregman nonexpansive mappings. Likewise we provide parallel algorithms for approximating common fixed points of a finite family of Bregman strongly nonexpansive mappings by means of a block operator which preserves the Bregman strong nonexpansivity. In particular, all the results hold for the smaller class of Bregman firmly nonexpansive mappings, a class which contains the generalized resolvent with respect to the Bregman distance of monotone operators. This talk is based on a joint work with Prof. Simeon Reich and Shoham Sabach.

References

[1] Bauschke, H. H., Borwein, J. M. and Combettes, P. L.: Bregman monotone optimization algorithms, *SIAM J. Control Optim.* **42** (2003), 596–636.

[2] Bregman, L. M.: The method of successive projection for finding a common point of convex sets, *Soviet Math. Dokl.* **6** (1965), 688–692.

[3] Butnariu, D. and Resmerita, E.: Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces, *Abstr. Appl. Anal.* **2006**, Art. ID 84919, 1–39.

[4] Reich, S. and Sabach, S.: Existence and approximation of fixed points of Bregman firmly nonexpansive mappings in reflexive Banach spaces, *Fixed-Point Algorithms for Inverse Problems in Science and Engineering, Springer*, Springer, New York, 299–314, 2011.

On the Validity of the Euler-Lagrange Equation

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Consider an open subset Ω of $\mathbb{R}^n$, a Carathéodory map $L(x, u, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, convex with respect to ξ, and the problem of minimizing the integral functional

$$\int_\Omega L(x, u(x), \nabla u(x))\ \mathrm{d}x \tag{1}$$

on $u \in u_0 + W_0^{1,1}(\Omega)$. Under some standard assumptions on the Lagrangian L (i.e. regularity properties, structure and growth conditions with respect to the variable ξ), it is a classical result that the Euler-Lagrange equation holds: if $\overline{u}$ is a minimizer of (1), then

$$\mathrm{div}_x \nabla_\xi L(x, \overline{u}(x), \nabla\overline{u}(x)) = L_u(x, \overline{u}(x), \nabla\overline{u}(x)). \tag{2}$$

We investigate the validity of (2) for larger classes of functionals. For L satisfying an upper growth condition of exponential type, but without assuming differentiability, we prove that a weak form of the Euler-Lagrange equation holds. This condition turns out to be equivalent to the normal version of the Pontryagin maximum principle for multidimensional domains. Afterward, we obtain the validity of the Euler-Lagrange equation for functionals of the kind

$$\int_\Omega \left[F\left(\nabla u\left(x\right)\right) + g\left(x, u\left(x\right)\right)\right]\ \mathrm{d}x,$$

where F is convex and g is a Carathéodory function such that the map $u \mapsto g\left(x, u\right)$ is concave and satisfies some growth assumptions. The main feature here is that no growth assumptions on F are needed. Finally we consider again general functionals of the form (1) and we prove that it is possible to overcome the exponential growth condition by using higher integrability properties of minimizers.

These are joint works with Giovanni Bonfanti and Arrigo Cellina.

Numerical Optimization in Support to Graph-based Scenario Modelling

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The study considers a practical approach for using the liner-quadratic [1] and heuristic fuzzy optimization [2] in support to an ad-hoc discrete clusterisation task of a weighted graph representation regarding a fictitious scenario model. The main idea is to enhance the utilization of the experts' opinions transformation into a complex dynamic system model represented by a weighted labeled graph. As a result of this, a graph nodes classification is performed. The optimization is used for repositioning a certain (desired) node amongst the clusterisation classes, defining boundary conditions for the rest of the nodes. A practical direct implementation of the achieved results is visible as a supporting solution to the new 21st century security challenges meeting [3].

References

[1] Paul Jensen, Jonathan Bard, *Operations Research Models and Methods*, John Wiley and Sons, 2003.

[2] Zlatogor Minchev, *Intelligent Scenario Development for CAX*, In Proceedings of NATO ARW: "Scientific Support for the Decision Making in the Security Sector" (Eds O. Kounchev, R. Willems, V. Shalamanov and Ts. Tsachev), Velingrad, Bulgaria, October 21–25, 2006, Published by IOS Press Amsterdam, NATO Science for Peace and Security, Series D: Information and Communication Security, vol. 12, 16–24, 2007.

[3] Zlatogor Minchev, Velizar Shalamanov, *Scenario Generation and Assessment Framework Solution in Support of the Comprehensive Approach*, In Proceedings of SAS-081 Symposium on "Analytical Support to Defence Transformation", RTO-MP-SAS-081 AC/323 (SAS-081)TP/329, Sofia, Boyana, April 26–28, 22-1–22-16, 2010.

Approximate values for mathematical programs with variational inequality constraints

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In general the infimal value of a mathematical program with variational inequality constraints $(MPVI)$ is not stable under perturbations in the sense that, even in presence of nice data, it may occur that the sequence of infimal values for the perturbed programs may not converge to the infimal value of the original problem. Thus, we present for these programs different types of values which approximate, under or without perturbations, the exact value from below or/and from above.

Ergodic convergence of the Forward-Backward algorithm to a zero of the extended sum of two maximal monotone operators

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A wide range of problems in physics, economics, etc. can be formulated as a solution of the inclusion $0 \in T(x)$, where T is a set-valued mapping on a Hilbert space $\mathcal{H}$. When T is maximal monotone, a classical tool to solve those problems is the *proximal point algorithm* of Rockafellar [3].

Splitting methods as *forward-backward*, *backward-backward* and *proximal barycentric* algorithms can be used in the investigation of zeros of monotone operators which admit a decomposition $T = A + B$, where A and B are maximal monotone operators on a Hilbert space. However, the pointwise sum of two maximal monotone operators is not necessarily maximal monotone. The *extended sum* of two maximal monotone operators is an extension of $A + B$ in the sense of graphs inclusion, and it can be maximal monotone in some cases where the pointwise sum is not (see [2, Theorem 4.4]). Consequently, it is interesting to study how we can obtain a zero of this sum, that is, to solve the inclusion:

$$0 \in \left(A \underset{\text{ext}}{+} B\right)(x).$$

Moudafi and Théra in [1] have proposed two splitting methods: the *backward-backward* and the *barycentric proximal point* algorithms in order to solve this problem. In the same goal, we propose to use the forward-backward algorithm given by:

$$x_{n+1} = J_{\lambda_n A}(x_n - \lambda_n y_n) \quad \text{with} \quad y_n \in B(x_n), \quad \forall n \in \mathbb{N}, \tag{3}$$

and with $\mathrm{D}(A) \subset \mathrm{D}(B)$ in order to well define the iteration.

References

[1] Abdellatif Moudafi and Michel Théra. Ergodic convergence to a zero of the extended sum of two maximal monotone operators. 8:369–379, 2001.

[2] Julian P. Revalski and Michel Théra. Enlargements and sums of monotone operators. *Nonlinear Anal.*, 48(4, Ser. A: Theory Methods):505–519, 2002.

[3] R. Tyrrell Rockafellar. Monotone operators and the proximal point algorithm. *SIAM J. Control Optimization*, 14(5):877–898, 1976.

Regularization of nonsmooth functions

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We discuss some classes of smoothing approximations of nonsmooth functions arising from complementarity, variational and hemivariational inequality problems. For details we refer e.g. to [1], Vol. II, and the references given there to the numerous works of Chen, Qi, Sun, Zang et al.
All regularizations are based in a direct or indirect way on convolution. To overcome the calculation of multivariate integrals that appears inevitable in a general approach of constructing smoothing functions via convolution, we focus our attention only to some specific instances like maximum, minimum or nested maxmin functions. These functions still describe some practice-oriented cases as many problems arising in mechanics and engineering show. We point out that smoothing functions are the basis of the smoothing Newton methods and the regularization method used to regularize a non-differentiable functional by a sequences of differentiable ones. As a model problem we consider an obstacle problem with a nonmonotone unilateral contact without friction [2], which gives rise to hemivariational inequality defined on a boundary. We also give some relations to Moreau-Yosida and Lasry-Lions regularizations [3].

References

[1] Francisco Facchinei, Jong-Shi Pang, *Finite-Dimensional Variational inequalities and Complementarity Problems*, Vol. I, Vol. II, Springer-Verlag, 2003.

[2] J. Haslinger, M. Miettinen, P. D. Panagiotopoulos, *Finite element methods for hemivariational inequalities*, 1999.

[3] N. Ovcharova, *Second-Order Analysis of the Moreau-Yosida and the Lasry-Lions Regularizations*, Optimization Methods and Software, 25, No. 1, 109–116, 2010.

Evolutionary Stable Strategies and Well Posedness Property

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In this paper I study Evolutionary stable strategies as defined by Maynard Smith and I introduce approximate evolutionary stable equilibria and a new concept of well posedness wich characterizes some problems with only one evolutionary stable equilibria.

Key-words: non cooperative games, evolutionary stable strategies, approximate equilibria, well posedness

References

[1] Aumann, R. J. *Almost strictly competitive games*, J. Soc. Industr. Appl. Math, **9**, 544–550, (1961).

[2] Bazzani A., Buiatti M., Freguglia P. *Metodi matematici per la Teoria dell'Evoluzione* (with specific contributes of C. Bocci, V. Capasso, L. Pusillo, E. Rogora), Springer Verlag, (2011).

[3] Binmore K. *Fun and Games. A Text on Game Theory*, D. C. Haealth and Company ed. Lexington, Massachusetts, Toronto, (1992).

[4] Cavazzuti, E. Cobwebs and something else in "Decision Process in Economics" (Ricci G. ed.), Proc. Conf. Modena, Italy, Springer, Berlin, 34–43, (1989).

[5] Gabay, D.,Moulin, H. *On the uniqueness and stability of Nash equilibria in noncooperative games*, Applied Stochastic Control in Econometrics and Management Science, edited by A. Bensoussan et al., North-Holland Publishing Company, (1980).

[6] Margiocco M., Patrone F., Pusillo L. *Metric characterization of Tihkonov well posedness in value* , Journal of Optimization Theory and Applications, 100, 2, 377–387, (1999).

[7] Karamardian, S. *The nonlinear complementarity problem with applications*, Journal of Optimization Theory and Applications, **4**, n.2, n.3, (1969).

[8] Klein, E., Thompson, A. C. *Theory of Correspondeces: Including Applications to Mathematical Economics.* A Wiley-Interscience Publication, (1984).

[9] Maynard Smith J. *Evolution and the Theory of Games*, Cambridge: Cambridge University Press, (1982).

[10] Mesterton-Gibbons M., Adams S. E. *Animal Contests as Evolutionary Games*, American Scientist, **86**, (1998).

[11] Vega Redondo F. *Evolution Games and Economic Behaviour*, Oxford University Press, (1996).

[12] Vincent T. L., Brown J. S. *Evolutionary Game Theory, Natural Selection and Darwinian Dynamics*, Cambridge University Press, (2005).

[13] Weibull W. J. *Evolutionary Game Theory*, The MIT Press, Cambridge, Massachusetts 02142, (1996).

On sweeping process without convexity

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The classical sweeping process introduced by Moreau (cf. for example [5]) has been extensively studied in the literature. The mathematical formulation of sweeping process is the study under different assumptions of the following constrained differential inclusion

$$\begin{aligned} &\dot{x}(t) \in -N_{C(t)}(x(t)),\\ &x(0) = x_0 \in C(0),\\ &x(t) \in C(t), \end{aligned}$$

where $C(\cdot)$ is a given moving closed set. G. Colombo, V. Goncharov (cf. [2]) and H. Benabdellah (cf. [1]) proved that the sweeping process has a solution if the phase space is R^n, the multifunction $C(\cdot)$ is Lipschitz continuous with respect to the Hausdorff distance and $N_{C(t)}(x(t))$ is the Clarke normal cone to the set $C(t)$ at the point $x(t)$. It is natural to ask the following question (posed by Jourani in 2007): Is it possible to replace the Clarke normal cone by the Morduchovich normal cone in the result of G. Colombo, V. Goncharov and H. Benabdellah? This is a substantially new problem because of the non convexity of the right-hand side of the differential inclusion. We present an example showing that the answer of the so stated problem is "no" even when the set $C(\cdot)$ is moving in the simplest but non trivial possible way, i.e. when $C(t) = K + at$, where K is a fixed closed subset of R^n and $a \in R^n$ is a constant vector. Some positive results on the existence of solution to the sweeping process with Morduchovich normal cone are presented. They are based on the approach proposed in [3]. In particular, a generalization of a classical result of Olech (cf. [6]) and Lojasiewicz (cf. [4]) is obtained.

References

[1] H. Benabdellah, *Existence of solutions to the nonconvex sweeping process*, J. Differential Equations 164 (2000), pp. 286295.

[2] G. Colombo, V.V. Goncharov, *The sweeping process without convexity*, Set-ValuedAnal. 7 (1999), pp. 357374.

[3] M. I. Krastanov, N. K. Ribarska, Ts. Y. Tsachev, *On the Existence of Solutions to Differential Inclusions with Nonconvex Right-Hand Sides*, SIAM Journal on Optimization 18 (2007), pp. 733-751.

[4] St. jun. Lojasiewicz, *Some theorems of Scorza Dragoni type for multifunctions with application to the problem of existence of solutions for differential multivalued equations*, Mathematical control theory, Banach Cent. Publ. 14 (1985), pp. 625-643.

[5] J.J. Moreau, *Evolution problem associated with a moving convex set in a Hilbert space*, J. Differential Equations 26 (1977), pp. 347374.

[6] C. Olech, *Existence of solutions of non-convex orientor fields*, Boll. Un. Mat. Ital. 11 (1975), No. 4, pp. 189–197.

Generalized univariate Newton methods motivated by proximal regularization

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We devise new generalized univariate Newton methods for solving nonlinear equations, motivated by the Bregman distances and the proximal regularization of optimization problems. We prove quadratic convergence of the new methods, and illustrate their benefits over the classical Newton's method by means of three test problems. These test problems provide insight as to which generalized method could be chosen for a given nonlinear equation. Finally, we derive a closed-form expression for the asymptotic error constants of the generalized methods and make further comparisons involving these constants.

This is a joint project with Regina S. Burachik and C. Yalcn Kaya.

Convergence of accelerated proximal methods in the presence of computational errors

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Given a Hilbert space $\mathcal{H}$ we consider the optimization problem

$$\text{(P)} \qquad \min_{x\in\mathcal{H}} F(x) := f(x) + g(x)$$

where $f : \mathcal{H} \to \mathbb{R}$ is convex and differentiable with a Lipschitz continuous gradient, and $g : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semicontinuous. Problem (P) covers a wide range of applications and to solve it, proximal methods - in particular forward-backward splitting - received much attention. More precisely, resorting to the ideas contained in the seminal work of Nesterov [3], recently there has been an active interest in accelerations and modifications of the forward-backward method for solving (P), see e.g. FISTA [1].

The computational effort of accelerated methods is comparable with that of the standard algorithm and mainly lies in the minimization subproblem required to compute the proximal point at each iteration. In fact, very often in the applications, a formula for the proximity operator is not available in closed form. In those cases the proximity operator is usually computed using ad hoc algorithms, and therefore inexactly. For this reason, it is indeed critical to study the convergence of the algorithms under possible perturbations of proximal points. This program has been pursued in the pioneering paper by Rockafellar [4] for what concerns the basic proximal point algorithm, and under different notions of admissible approximations of proximal points. Since then, there has been a growing interest in inexact implementations of proximal methods and many works appeared, treating the problem under different perspectives.

We analyze the convergence of accelerated and inexact versions of the proximal point algorithm and forward-backward splitting. Leveraging on a new concept of admissible errors, convergence of inexact and accelerated schemes is guaranteed, and the rate is the same of the exact ones. Moreover, if $f = 0$, using even a generalization of the type of errors considered in [2], convergence of the inexact and accelerated proximal point algorithm is proved. In both cases conditions on the asymptotic behavior of the errors' magnitude are needed.

References

[1] A. Beck and M. Teboulle. A fast iterative shrinkage-thresholding algorithm for linear inverse problems. *SIAM J. Imaging Sciences*, 2(1):183–202, 2009.

[2] O. Güler. New proximal point algorithms for convex minimization. *SIAM J. on Optimization*, 2(4):649–664, 1992.

[3] Y. Nesterov. A method for unconstrained convex minimization problem with the rate of convergence $o(1/k^2)$. *Doklady AN SSSR*, 269(3):543–547, 1983.

[4] R. T. Rockafellar. Monotone operators and the proximal point algorithm. *SIAM J. Control Optimization*, 14(5):877–898, 1976.

Completely monotone functions and convexity

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We extend a construction by J. Borwein and O. Hijab, by showing how every completely monotone function on $(0, \infty)$ can be extended to a symmetric convex function on $\mathbb{R}^n_{++}$. Since every completely monotone function is a Laplace transform of a measure on $[0, \infty)$ properties of that measure determine properties of the symmetric convex function. We discuss an order between functions g, h on the same domain in $\mathbb{R}$: $g \preceq h$ if for any other function f we have ($g \circ f$ convex $\Rightarrow h \circ f$ is convex). Well-known example is $-1/x \preceq \log(x) \preceq \mathrm{id}(x)$. We show that this extends to every completely monotone function g and its derivative: $-g' \preceq g \preceq \mathrm{id}(x)$.

This is a joint work with Ričardas Zitikis, the University of Western Ontario.

Best approximation problems in a class of Banach spaces

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The best approximation problem to a nonempty closed subset is generalized well-posed, or the set of solutions is either empty or generalized well posed, for the majority of the points in a class of Banach spaces. Under "majority" we understand a set whose complement in the space is sigma-porous or sigma-cone supported. Analogously, to the the case when uniqueness of the best approximation is required, it turns out that the conditions that assure this are certain local uniform, or uniform, properties of the norm of the underlying space.

Perturbation method for variational problems*

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We provide a general method for proving existence of solutions of suitable perturbations of certain variational problems. A novel variational principle enables perturbing only the integrand, thus preserving the form of the problem.

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