A small survey of some recent contributions to the subdifferential of the supremum function

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1. Introduction

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- To provide a general formula for the optimal set of a relaxed minimization problem in terms of the approximate minima of the data function.
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- To derive explicit characterizations for the subdifferential mapping of the supremum function of an arbitrarily indexed family of convex functions, exclusively in terms of the data functions.
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- To provide a general formula for the optimal set of a relaxed minimization problem in terms of the approximate minima of the data function.
- To derive explicit characterizations for the subdifferential mapping of the supremum function of an arbitrarily indexed family of convex functions, exclusively in terms of the data functions.
- To present alternative approaches and applications to subdifferential calculus.
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2. Notation

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$\theta$: zero in all the involved spaces.

$\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. 
Associated with $A \neq \emptyset$ we consider the sets

$$A^\circ := \{ x^* \in X^* \mid \langle x, x^* \rangle \geq -1 \ \forall x \in A \},$$
$$A^- := - (\text{cone } A)^\circ = \{ x^* \in X^* \mid \langle x, x^* \rangle \leq 0 \ \forall x \in A \},$$
$$A^\perp := (-A^-) \cap A^- = \{ x^* \in X^* \mid \langle x, x^* \rangle = 0 \ \forall x \in A \},$$

i.e. the (one-sided) polar, the negative dual cone, and the orthogonal subspace (or annihilator) of $A$, respectively.

If $A \subset X$ and $x \in X$, we define the indicator function of $A$ as

$$I_A(x) := \begin{cases} 0, & \text{if } x \in A, \\ +\infty, & \text{if } x \in X \setminus A, \end{cases}$$

and the normal cone to $A$ at $x$ as

$$N_A(x) := \begin{cases} (A - x)^- & \text{if } x \in A, \\ \emptyset & \text{if } x \in X \setminus A. \end{cases}$$

$A_\infty$ represents its recession cone.
3. Optimal set for the relaxed problem

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The optimal values of both problems coincide:

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\inf_{X} h = \inf_{X} h^{**} =: m \in \bar{\mathbb{R}}.
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Our purpose here is to obtain the *optimal set of* \((P')\), i.e. \( \arg\min_{X} h^{**} \), in terms of the approximate solutions of \((P)\), i.e. \( \varepsilon - \arg\min_{X} h \).
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Our purpose here is to obtain the *optimal set of \((\mathcal{P}')\)*, i.e. \( \text{argmin } h^{**} \), in terms of the approximate solutions of \((\mathcal{P})\), i.e. \( \varepsilon - \text{argmin } h \).

We set \( \varepsilon - \text{argmin } h = \emptyset \) for all \( \varepsilon \geq 0 \) whenever \( m \notin \mathbb{R} \).
Next we establish the \textbf{main result in this section}. 
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**Theorem 1 ([LoVo’10-Th.3.3])**

For any function $h : X \to \overline{\mathbb{R}}$ such that $\text{dom } h^* \neq \emptyset$, one has

$$\text{argmin } h^{**} = \bigcap_{\varepsilon > 0} \overline{\text{co}} \left( (\varepsilon - \text{argmin } h) + \{x^*\}^- \right).$$

If $\text{cone}(\text{dom } h^*)$ is $w^*$-closed or $\text{ri}(\text{cone}(\text{dom } h^*)) \neq \emptyset$, then

$$\text{argmin } h^{**} = \bigcap_{\varepsilon > 0} \overline{\text{co}} \left( (\varepsilon - \text{argmin } h) + (\text{dom } h^*)^- \right).$$

In particular, if $\text{cone}(\text{dom } h^*)) = X^*$, then

$$\text{argmin } h^{**} = \bigcap_{\varepsilon > 0} \overline{\text{co}} (\varepsilon - \text{argmin } h).$$
Given $h : X \to \overline{\mathbb{R}}$ and $\varepsilon \geq 0$, consider the mapping
$M_\varepsilon h : X^* \rightrightarrows X$ defined by $M_\varepsilon h = (\partial_\varepsilon h)^{-1}$. 
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**Theorem 2 ([LoVo’10-Th.3.6])**

If $\text{dom } h^* \neq \emptyset$, one has for all $x^* \in X^*$,

$$\partial h^*(x^*) = \bigcap_{\varepsilon > 0} \overline{\text{co}} \left( (M_\varepsilon h)(x^*) + \{u^* - x^*\}^- \right).$$

If $\text{cone} \left( (\text{dom } h^*) - x^* \right)$ is $w^*$-closed or $\text{ri(cone((\text{dom } h^*) - x^*))} \neq \emptyset$, then

$$\partial h^*(x^*) = \bigcap_{\varepsilon > 0} \overline{\text{co}} \left( (M_\varepsilon h)(x^*) + N_{\text{dom } h^*}(x^*) \right).$$
**Theorem 3 ([LoVo’10-Th.4.1])**

Given a family $\emptyset \neq \{f_t, \; t \in T\} \subset \overline{\mathbb{R}}^X$, consider the supremum function $f := \sup_{t \in T} f_t$, and assume that $\text{dom } f \neq \emptyset$. If

$$f^{**} \equiv \left(\sup_{t \in T} f_t \right)^{**} = \sup_{t \in T} f_t^{**},$$

the subdifferential of $f$ at any point $x \in X$ is given by

$$\partial f(x) = \bigcap_{\varepsilon > 0, \; z \in \text{dom } f} \overline{\text{co}} \left( \bigcup_{t \in T_{\varepsilon}(x)} \partial_{\varepsilon} f_t(x) + \{z - x\}^- \right),$$

where $T_{\varepsilon}(x) := \{t \in T : f_t(x) \geq f(x) - \varepsilon\}$ if $f(x) \in \mathbb{R}$ and $T_{\varepsilon}(x) = \emptyset$ if $f(x) \notin \mathbb{R}$. 
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Theorem 3

Moreover, if either $\text{cone co}(\text{dom } f - x)$ is closed or $\text{ri}(\text{cone co}(\text{dom } f - x)) \neq \emptyset$, then

$$
\partial f(x) = \bigcap_{\epsilon > 0} \overline{\text{co}} \left( \bigcup_{t \in T_\epsilon(x)} \partial \epsilon f_t(x) + N_{\text{dom } f}(x) \right).
$$
The following theorem is an extension of Theorem 1:
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**Theorem 4 ([LoVo’10-Th.4.8])**

For \( h : X \to \mathbb{R} \) and any family \( \{ C_i, \ i \in I \} \) of convex sets of \( X^* \) satisfying

\[
\text{dom} \ h^* \subseteq \bigcup_{i \in I} C_i,
\]

and

\[
\text{ri} \ (\text{cone}(C_i \cap \text{dom} \ h^*)) \neq \emptyset, \text{ for all } i \in I,
\]

one has

\[
\text{argmin} \ h^{**} = \bigcap_{\varepsilon > 0, \ i \in I} \overline{\text{co}} \left( (\varepsilon - \text{argmin} \ h) + (C_i \cap \text{dom} \ h^*)^- \right).
\]

If \( \{ C_i, \ i \in I \} = \{ \{ x^* \}, \ x^* \in \text{dom} \ h^* \} \), we get Theorem 1.
Corollary ([LoVo’10-Cor.4.9])

For any function $h : X \to \overline{\mathbb{R}}$ with $\text{dom } h^* \neq \emptyset$, if

$$\mathcal{F}_{x^*} := \left\{ L \subset X^* \mid \text{L is a finite-dimensional linear subspace such that } x^* \in L \right\},$$

one has for all $x^* \in X^*$,

$$\partial h^*(x^*) = \bigcap_{\varepsilon > 0, L \in \mathcal{F}_{x^*}} \overline{\text{co}} \left( (M_\varepsilon h)(x^*) + N_{L \cap \text{dom } h^*}(x^*) \right).$$

We applied that $(M_\varepsilon h)(x^*) = \varepsilon - \text{argmin} \left( h(\cdot) - \langle \cdot, x^* \rangle \right)$. 
4. More on the subdifferential of the supremum function

The last corollary is applied to obtain an extension of Theorem 4 in [HLZ08]:

**Theorem ([LoVo’10-Cor.4.11])**

Given \( \emptyset \neq \{f_t, \ t \in T\} \subset \mathbb{R}^X \), and the supremum function \( f := \sup_{t \in T} f_t \), assume that \( \text{dom} f \neq \emptyset \) and \( f^{**} = \sup_{t \in T} f_t^{**} \). Then

\[
\partial f(x) = \bigcap_{\varepsilon > 0, L \in \mathcal{F}_x} \text{cl} \left( \text{co} \left( \bigcup_{t \in T \varepsilon(x)} \partial \varepsilon f_t(x) \right) + N_{L \cap \text{dom} f(x)} \right).
\]
Different characterizations of $N_{\text{dom} f}(x)$ yield:

Theorem ([HLZ’08])

Let $\emptyset \neq \{f_t, \ t \in T\} \subset \mathbb{R}^X$ and $f := \sup_{t \in T} f_t$. Then, for every $x \in X$ we have

$$
\partial f(x) = \bigcap_{L \in \mathcal{F}_x, \varepsilon > 0} \overline{\text{co}} \left( A_L + \bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x) \right)
= \bigcap_{L \in \mathcal{F}_x, \varepsilon > 0} \overline{\text{co}} \left( B_L + \bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x) \right),
$$

where

$$
x^* \in A_L \iff (x^*, \langle x^*, x \rangle) \in \left[ \overline{\text{co}} \left( (L^\perp \times \mathbb{R}^+) \cup \bigcup_{t \in T} \text{epi} f_t^* \right) \right]_\infty,
$$

$$
x^* \in B_L \iff (x^*, \langle x^*, x \rangle) \in \left[ \overline{\text{co}} \left( L^\perp \times \{0\} \cup \bigcup_{t \in T} \text{gph} f_t^* \right) \right]_\infty.
$$
When the functions $f_t, t \in T,$ are **affine** our formula becomes:

**Corollary ([HLZ'08])**

Assume that $T \neq \emptyset$ and $f(x) := \sup \{ \langle a_t^*, x \rangle - \beta_t \mid t \in T \},$ with $a_t^* \in X^*$ and $\beta_t \in \mathbb{R}.$ Then, for every $x \in X$ we have

$$
\partial f(x) = \bigcap_{L \in \mathcal{F}_x, \varepsilon > 0} \text{cl} \left( \text{co} \{ a_t^* \mid t \in T_{\varepsilon}(x) \} + B_L \right), \text{ where }
$$

$$
x^* \in B_L \iff (x^*, \langle x^*, x \rangle) \in \left[ \overline{\text{co}} \left( (L^\perp \times \{0\}) \cup \{(a_t^*, \beta_t), t \in T\} \right) \right]_{\infty}.
$$
Corollary (Volle’93, for normed spaces)

Let \( \emptyset \neq \{f_t : X \to \overline{\mathbb{R}} \mid t \in T \} \) be a family of convex functions, and set \( f := \sup_{t \in T} f_t \). If \( f \) is finite and continuous at \( z \in X \), then

\[
\partial f(z) = \bigcap_{\varepsilon > 0} \co \left( \bigcup_{t \in T_\varepsilon(z)} \partial_\varepsilon f_t(z) \right).
\]

Corollary (Brøndsted’72)

Consider convex functions \( f_i : X \to \overline{\mathbb{R}} \) for \( i = 1, \ldots, k \), and \( f := \max\{f_1, \ldots, f_k\} \), and assume that (CC) holds. Given \( z \in X \) such that \( (\cl f)(z) = (\cl f_i)(z) \) for \( i = 1, \ldots, k \), we have

\[
\partial f(z) = \bigcap_{\varepsilon > 0} \co \left( \bigcup_{i=1}^{k} \partial_\varepsilon f_i(z) \right).
\]
Theorem (CoHaLo’11)

Let $f_t : X \rightarrow \mathbb{R}$, $t \in T$, be convex, $f = \sup_{t \in T} f_t$, and $C \subset X$ such that $\emptyset \neq C \cap \text{dom } f$ is convex. If

$$\text{cl}(f + I_C)(x) = \sup_{t \in T} (\text{cl} f_t)(x), \quad \forall x \in C \cap \text{dom } f,$$

then, $\forall x \in X$ and $i = 1, 2$, we have

$$\partial (f + I_C)(x) = \bigcap_{\varepsilon > 0} \overline{\text{co}} \left\{ \bigcup_{L \in \mathcal{G}^i_x} \partial_{\varepsilon f_t}(x) + N_{L \cap C \cap \text{dom } f}(x) \right\},$$

where

$$\mathcal{G}^1_x = \left\{ L \subset X \text{ convex} \mid x \in L \text{ and } \text{ri} (L \cap C \cap \text{dom } f) \neq \emptyset \right\},$$

$$\mathcal{G}^2_x = \left\{ L \subset X \text{ convex} \mid \text{cone } \left\{ (L \cap C \cap \text{dom } f) - x \right\} \text{ is closed and } x \in L \right\}.$$
5. An alternative approach

Theorem (Ioffe’11)

Given $\emptyset \neq \{f_t, t \in T\} \subset \mathbb{R}^X$, and $f := \sup_{t \in T} f_t$. Assume that $x \in \text{dom } f$ and that (CC) is satisfied. Let $\{C_i, i \in I\}$ be a family of convex sets such that

$x \in C_i, \forall i \in I, \text{ and } \text{dom } f \subset \bigcup_{i \in I} C_i \subset \text{dom } f^{**}.$

(a) If all $C_i, \forall i \in I$, are closed, then

$$\partial f(x) = \bigcap_{\varepsilon > 0} \text{co} \left( \bigcup_{t \in T_{\varepsilon}(x)} \partial_{\varepsilon f_t(x)} + N_{C_i}^\varepsilon(x) \right),$$

where $N_{C_i}^\varepsilon(x) := \{x^* \in X^* \mid \langle x^*, z - x \rangle \leq \varepsilon, \forall z \in C_i\}.$
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\[ x \in C_i, \, \forall i \in I, \, \text{and} \, \text{dom} f \subset \bigcup_{i \in I} C_i \subset \text{dom} f^{**}. \]

(a) If all \( C_i, \, \forall i \in I, \) are closed, then

\[ \partial f(x) = \bigcap_{\varepsilon > 0} \overline{\text{co}} \left( \bigcup_{t \in T_{\varepsilon}(x)} \partial_{\varepsilon f_t}(x) + N_{C_i}^\varepsilon(x) \right), \]

where \( N_{C_i}^\varepsilon(x) := \{x^* \in X^* \mid \langle x^*, z - x \rangle \leq \varepsilon, \forall z \in C_i\} \).
Theorem

(b) If either $\text{cone}(C_i - x)$ is closed or $\text{ri} C_i \neq \emptyset$, $\forall i \in I$, then

$$\partial f(x) = \bigcap_{\varepsilon > 0} \overline{\text{co}} \left( \bigcup_{t \in T_\varepsilon(x)} \partial_{\varepsilon f_t}(x) + N_{C_i}(x) \right).$$

If we take $I := \text{dom} f$, $C_z := \text{co}\{x, z\}$, $\forall z \in \text{dom} f$, it is obvious that $x \in C_z$, $\forall z \in \text{dom} f$, $\text{dom} f \subset \bigcup_{i \in I} C_i \subset \text{dom} f^{**}$, and $\text{cone}(C_i - x)$ is a closed ray. Applying the last theorem one gets

$$\partial f(x) = \bigcap_{\varepsilon > 0, z \in \text{dom} f} \overline{\text{co}} \left( \bigcup_{t \in T_\varepsilon(x)} \partial_{\varepsilon f_t}(x) + \{z - x\}^- \right),$$
Theorem (CoHaLo’11)

Let $f_t : X \to \overline{\mathbb{R}}$, $t \in T$, be convex functions, define $f = \sup_{t \in T} f_t$, and let $C \subset X$ be so that $C \cap \text{dom} f$ is a nonempty convex set. Assume that, for every $x \in C \cap \text{dom} f$,

$$\text{cl}(f + I_C)(x) = \sup_{t \in T} (\text{cl} f_t)(x).$$

Then, for every $x \in X$ we have that

$$\partial (f + I_C)(x) = \bigcap_{\varepsilon > 0} \text{co} \left\{ \bigcup_{F \in \mathcal{H}_x} \partial_{\varepsilon f_t} (x) + N^\varepsilon_{F \cap C \cap \text{dom} f} (x) \right\},$$

where

$$\mathcal{H}_x := \{ F \subset X \mid x \in F \text{ and } F \cap C \cap \text{dom} f \text{ is closed and convex} \}.$$
6. Other calculus rules
Given an arbitrary set $T$, we call space of the generalized sequences to
\[ \mathbb{R}^{(T)} := \{ \gamma \in \mathbb{R}^T \mid \text{only finitely many } \gamma_t \text{ are different from 0} \}. \]
Its non-negative cone is $\mathbb{R}^{(T)}_+ = \{ \gamma \in \mathbb{R}^{(T)} \mid \gamma_t \geq 0, \forall t \in T \}$. Given $\gamma \in \mathbb{R}^{(T)}$, we define
\[ \text{supp } \gamma := \{ t \in T : \gamma_t \neq 0 \}. \]
We also use the notation
\[ S_T := \left\{ \gamma = (\gamma_t) \in \mathbb{R}^{(T)}_+ : \sum_{t \in T} \gamma_t = 1 \right\}. \]
When $T = \{1, 2, ..., n\}$,
\[ S_T := \left\{ \lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n_+ : \sum_{i=1}^{n} \lambda_i = 1 \right\}. \]
Given \( f = \sup_{t \in T} f_t, x \in X \), and \( \varepsilon > 0 \) we define

\[
S_T(x, \varepsilon) := \left\{ \gamma \in S_T : \sum_{t \in \text{supp} \gamma} \gamma_t f_t(x) \geq f(x) - \varepsilon \right\},
\]

and

\[
D_1 := \bigcap_{t \in T} \text{dom} f_t \supset \text{dom} f =: D_2.
\]

**Theorem (LoVo’11-Th.1)**

Assume that \( f^{**} \equiv (\sup_{t \in T} f_t)^{**} = \sup_{t \in T} f^{**} \) and it is proper. Then, for \( D \subset X \) such that \( D_2 \subset D \subset D_1 \), and \( \forall x \in X \),

\[
\partial f(x) = \bigcap_{\varepsilon > 0} \text{cl} \bigcup_{\gamma \in S_T(x, \varepsilon)} \partial_{\varepsilon} \left( \sum_{t \in \text{supp} \gamma} \gamma_t f_t + i_D \right)(x).
\]
In order to prove Theorem 11 the following function associated with the set $D$ satisfying $D_2 \subset D \subset D_1$ plays a crucial role:

$$
\varphi := \inf_{\gamma \in \mathcal{S}_T} \left( \sum_{t \in \text{supp } \gamma} \gamma_t f_t + i_D \right)^*.
$$

(3)

**Theorem**

Assume that $f^{**} \equiv (\sup_{t \in T} f_t)^{**} = \sup_{t \in T} f^{**}$ and it is proper. Then $\varphi \in (\mathbb{R} \cup \{+\infty\})^{X^*}$ is convex and proper. Moreover

$$
\text{cl}^* \varphi = f^*.
$$

(4)
The use of epigraphs
Associated with the function

\[ \varphi := \inf_{\gamma \in \mathcal{S}_T} \left( \sum_{t \in \text{supp } \gamma} \gamma_t f_t + i_D \right)^*, \]

we introduce the set

\[ E := \bigcup_{\gamma \in \mathcal{S}_T} \text{epi} \left( \sum_{t \in \text{supp } \gamma} \gamma_t f_t + i_D \right)^*. \]

One can prove

\[ \text{epi}(\text{cl}^* \varphi) = \text{cl}^* E. \] (5)

Moreover, if \( E \) is \( w^* \)-closed, then

\[ \text{epi } \varphi = E. \]
Under this closedness criterion the formula in Theorem 11, i.e.

$$\partial f(x) = \bigcap_{\varepsilon > 0} \cl \bigcup_{\gamma \in S_T(x, \varepsilon)} \partial_\varepsilon \left( \sum_{t \in \supp \gamma} \gamma_t f_t + i_D \right)(x),$$

can be significantly simplified.

**Theorem (LoVo’11-Th.2)**

Assume that $f^{**} \equiv (\sup_{t \in T} f_t)^{**} = \sup_{t \in T} f_t^{**}$ and it is proper, and that for $D \subset X$ such that $D_2 \subset D \subset D_1$, the set $E$ is $w^*-$closed. Then,

$$\partial f(x) = \bigcup_{\gamma \in S_T(x)} \partial \left( \sum_{t \in \supp \gamma} \gamma_t f_t + i_D \right)(x), \ \forall x \in X, \quad (6)$$

where $S_T(x) := \{ \gamma \in S_T : f_t(x) = f(x), \ \forall t \in \supp \gamma \}$. 
Finitely many functions

Consider the case of finitely many functions $f_1, f_2, ..., f_n$, i.e. $T = \{1, 2, ..., n\}$. The 'closedness condition' for the supremum function $f = \max_{1 \leq i \leq n} f_i$ now is

$$f^{**} = \max_{1 \leq i \leq n} f_i^{**}$$

and it is proper. \hspace{1cm} (7)

Only a domain is involved:

$$D := \text{dom} f = \bigcap_{1 \leq i \leq n} \text{dom} f_i.$$  

According to the rule $0 \times (+\infty) = +\infty$ one easily checks that

$$\sum_{i \in \text{supp} \lambda} \lambda_i f_i + i_D = \sum_{1 \leq i \leq n} \lambda_i f_i, \forall (\lambda_i) \in S_T.$$ \hspace{1cm} (8)
Now

\[ \varphi := \inf_{(\lambda_i) \in S_T} \left( \sum_{1 \leq i \leq n} \lambda_i f_i \right)^*, \]

and if \( f^{**} = \max_{1 \leq i \leq n} f_i^{**} \) and it is proper, then \( \varphi \in \Gamma(X^*) \) and so, \( \varphi = f^* \).

The theorem below extends [Za’02, Cor. 2.8.11].

**Theorem**

*Assume that (7) holds. Then*

\[ \partial f(x) = \bigcup_{(\lambda_i) \in S(x)} \partial \left( \sum_{1 \leq i \leq n} \lambda_i f_i \right)(x), \ \forall x \in X, \quad (9) \]

*where \( S(x) := \{(\lambda_i) \in S_T : f_i(x) = f(x) \text{ if } \lambda_i \neq 0\} \).*


