Open problems session

13\textsuperscript{th} Workshop on Well-Posedness
of Optimization Problems and Related Topics
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This file contains detailed descriptions of the problems that were posed by Roberto Cominetti, Alexander Ioffe, Roberto Lucchetti and Tullio Zolezzi at the 13\textsuperscript{th} WOP workshop. The contents of this file are kindly provided by the authors; the order follows the one in which the problems were presented.

Alexander Ioffe

Find a function $f$ on a Hilbert space such that:

(a) $f$ is defined and Lipschitz in a neighborhood of zero;

(b) $\emptyset \neq \partial F f(0) \neq \partial DH f(0)$.

Here $\partial F$ and $\partial DH$ stand for the Frechet and Dini-Hadamard subdifferentials respectively. The first consists of $x^*$ such that

$$\liminf_{\|h\| \to 0} \|h\|^{-1}(f(h) - f(0) - \langle x^*, h \rangle) \geq 0.$$ 

The second is the collection of $x^*$ such that

$$f_H'(0; h) \geq \langle x^*, h \rangle, \quad \forall h \in X,$$

where

$$f_H'(0, h) = \liminf_{(t,u) \to (0,h)} t^{-1}(f(tu) - f(0)).$$
Tullio Zolezzi

An open problem: to find a reasonable definition of the sensitivity of a convex programming problem in the form, to minimize the objective function \( f(x) \) subject to the constraints

\[
g_1(x) \leq 0, g_2(x) \leq 0, \ldots, g_k(x) \leq 0,
\]

where \( x \in \mathbb{R}^N \) and the data with respect to which we consider sensitivity are the convex functions \( f, g_1, \ldots, g_k \).

The definition should allow to prove distance theorems (condition number theorems) in order to be able to deduce error estimates for the optimal solutions, and information about the computational complexity of the optimization problem. A comparison with the many results known e.g. in linear programming should be possible.

Roberto Cominetti

Consider the map \( f : [0,1]^n \to \mathbb{R} \) defined by

\[
f(x) \triangleq \mathbb{E} [\Phi(X_1 + \cdots + X_n)]
\]

where \( \Phi : \mathbb{N} \to \mathbb{R} \) is a (discrete) convex function and the \( X_i \)'s are independent Bernoulli random variables with \( \mathbb{P}(X_i = 1) = x_i \). Observe that \( f(x) \) is a polynomial of degree \( n \), affine with respect to each variable \( x_i \) separately. Consider the increasing sequence \( c_u = \Phi(u) - \Phi(u - 1) \) and let

\[
d = \max_{u=2,\ldots,n-1} [c_{u+1} - c_u] = \max_{u=2,\ldots,n-1} [\Phi(u + 1) - 2\Phi(u) + \Phi(u - 1)].
\]

**Problem 1:** Find necessary/sufficient conditions on \( \Phi \) for \( f \) to be convex.

**Problem 2:** Prove or disprove: the entropically perturbed map

\[
\tilde{f}(x) \triangleq f(x) + \sum_{i=1}^n x_i \ln x_i + (1-x_i) \ln(1-x_i)
\]

is convex as long as \( \delta \leq 2 \).

**Comments:** These questions help to characterize the rest points of dynamics in repeated games studied in “A payoff-based learning procedure and its application to traffic games”, Games and Economic Behavior 70 (2010) 71–83. The following is known (see Props. 12 and 13)

a) If \( \Phi(u) = a + bu \) then \( f(x) = a + b \sum_{i=1}^n x_i \) is linear, hence convex.

b) If \( \Phi(u) = a + bu + cu^2 \) then \( \tilde{f} \) is known to be convex for \( \delta \leq 2 \).

c) In general \( \tilde{f} \) is known to be convex for \( \delta \leq 1 \).
Roberto Lucchetti

A bimatrix \((A, B) = (a_{ij}, b_{ij})\), \(i = 1, \ldots, n\), \(j = 1, \ldots, m\) is a finite game in strategic form (\(a_{ij}\), \(b_{ij}\) utilities of the players). (In the zero sum case \(b_{ij} = -a_{ij}\)). Let \(I = \{1, \ldots, n\}\), \(J = \{1, \ldots, m\}\) and \(X = I \times J\). Let \(\Delta_k\) denote the standard simplex in \(\mathbb{R}^k\). A Nash equilibrium is a pair \((\bar{p}, \bar{q})\), \((\bar{p} \in \Delta_n), (\bar{q} \in \Delta_m)\) such that

\[
\sum_{i,j} \bar{p}_i \bar{q}_j a_{ij} \geq \sum_{i,j} p_i \bar{q}_j a_{ij} \\
\sum_{i,j} \bar{p}_i \bar{q}_j b_{ij} \geq \sum_{i,j} \bar{p}_i q_j b_{ij}
\]

for all \(p \in \Delta_n, q \in \Delta_m\). If both \(p, q\) are extreme points of the simplexes: pure Nash equilibria, if inequalities are strict for \(p \neq \bar{p}\) and \(q \neq \bar{q}\): strict Nash equilibria.

A correlated equilibrium is a probability distribution \(p = (p_{ij})\) on \(X\) such that, for all \(i \in I\),

\[
\sum_{j=1}^{m} p_{ij} a_{ij} \geq \sum_{j=1}^{m} p_{ij} a_{ij} \quad \forall i \in I,
\]

and such that, for all \(j \in J\)

\[
\sum_{i=1}^{n} p_{ij} b_{ij} \geq \sum_{i=1}^{n} p_{ij} b_{ij} \quad \forall j \in J.
\]

**Example 1**

\[
\begin{pmatrix}
(6, 6) & (2, 7) \\
(7, 2) & (0, 0)
\end{pmatrix}.
\]

Nash outcomes: \((2, 7), (7, 2)\) (pure), also a mixed providing \(\frac{14}{3}\) to both.

\[
\begin{pmatrix}
\frac{1}{3} & \frac{1}{3} \\
\frac{2}{3} & 0
\end{pmatrix}.
\]

is the nice correlated equilibrium maximizing the total sum of the payoffs (among correlated equilibria).

The problem of finding correlated equilibria is of finding, in the appropriate simplex, solutions \(p\) to the system of linear inequalities \(C(A, B)p \geq 0\), where \(C\) is called the matrix of
incentive constraints and looks like:

\[
C(A, B) = \begin{pmatrix}
    a_{11} - a_{21} & \cdots & a_{1m} - a_{2m} & \cdots & \cdots & 0 & 0 & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    a_{11} - a_{n1} & \cdots & a_{1m} - a_{nm} & \cdots & \cdots & 0 & 0 & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    0 & 0 & 0 & \cdots & \cdots & a_{n1} - a_{11} & \cdots & a_{nm} - a_{1m} \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    b_{11} - b_{12} & \cdots & 0 & \cdots & \cdots & b_{n1} - b_{n2} & \cdots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    b_{11} - b_{1m} & \cdots & 0 & \cdots & \cdots & b_{n1} - b_{nm} & \cdots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    0 & b_{1m} - b_{11} & \cdots & 0 & \cdots & 0 & b_{nm} - b_{n1} \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    0 & b_{1m} - b_{1(m-1)} & 0 & 0 & \cdots & 0 & 0 & b_{nm} - b_{n(m-1)}
\end{pmatrix}
\]

**Definition 1** We say that a correlated equilibrium \( \bar{p} \) for \((A, B)\) is lower stable if for any sequence \((A_n, B_n)\) of games converging to \((A, B)\), there exists \(p_n\) correlated equilibrium for \((A_n, B_n)\) converging to \(p\).

The following are results on the topic.

**Proposition 1 (the two by two case)** In a two by two bimatrix game the following happens:

- If there are no dominated strategies, then all correlated equilibria are lower stable
- in presence of dominated strategies, only pure strict Nash equilibria are lower stable

**Proposition 2** A correlated equilibrium of a zero-sum game is stable within the space of zero-sum games if and only if it is the unique correlated equilibrium of the game.

**Open question** Generalize the above result to more general classes of games