Stability in Linear Optimization: applications

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Some definitions

Two applications to duality theory:
- Stability of the primal-dual partition in linear semi-infinite programming. (Qualitative perspective)
- Lipschitz-like property of the stability of the feasible sets of the primal and the dual problem in Banach spaces. (Quantitative perspective)

An application to convex geometry:
- Stability of the Voronoi cells. (Qualitative perspective)
Definitions

Let $Z$ and $Y$ be topological spaces. Consider a set-valued mapping $M : Z \rightrightarrows Y$. The mapping $M$ is

- **lower semicontinuous** (lsc, in brief) at $z_0 \in Z$ if, for each open set $W \subset Y$ such that $W \cap M(z_0) \neq \emptyset$, there exists an open set $V \subset Z$, containing $z_0$, such that $W \cap M(z) \neq \emptyset$ for each $z \in V$.

- **upper semicontinuous** (usc, in brief) at $z_0 \in Z$ if, for each open set $W \subset Y$ such that $M(z_0) \subset W$, there exists an open set $V \subset Z$, containing $z_0$, such that $M(z) \subset W$ for each $z \in V$.

- **closed** at $z_0 \in \text{dom } M$ if for all nets $z_r \in Z$ and $x_r \in Y$ satisfying $x_r \in M(z_r)$ for all $z_r$ in some index set, $z_r \to z_0$ and $x_r \to x_0$, one has $x_0 \in M(z_0)$.
Definitions

$Z$ and $Y$ topological spaces.
$\mathcal{M} : Z \rightrightarrows Y$ a set-valued mapping

- $\mathcal{M}$ is \textit{lower semicontinuous} at $z_0 \in Z$ (lsc, in brief) if, for each open set $W \subset Y$ such that $W \cap \mathcal{M}(z_0) \neq \emptyset$, there exists an open set $V \subset Z$, containing $z_0$, such that $W \cap \mathcal{M}(z) \neq \emptyset$ for each $z \in V$.

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- $\mathcal{M}$ is **closed** at $z_0 \in \text{dom}\mathcal{M}$ if for all nets $\{z_r\} \subset Z$ and $\{x_r\} \subset Y^n$ satisfying $x_r \in \mathcal{M}(z_r)$ for all $r$ in some index set, $z_r \rightarrow z_0$ and $x_r \rightarrow x_0$, one has $x_0 \in \mathcal{M}(z_0)$.
$\mathcal{M} : Z \nrightarrow Y$ a set-valued mapping

We consider the lower (inner) and upper (outer) limits of set-valued mappings

$$\liminf_{z \to z_0} \mathcal{M}(z) \quad \text{and} \quad \limsup_{z \to z_0} \mathcal{M}(z)$$

According to Rockafellar and Wets (1998), $\mathcal{M}$ closed and lsc at $z_0$ is equivalent to

$$\liminf_{z \to z_0} \mathcal{M}(z) = \limsup_{z \to z_0} \mathcal{M}(z) = \mathcal{M}(z_0).$$

In this case we say that $\lim_{z \to z_0} \mathcal{M}(z) = \mathcal{M}(z_0)$ and call it the Painlevé-Kuratowski limit.
Lipschitz properties

$Z$ and $Y$ normed spaces
$\mathcal{M} : Z \rightrightarrows Y$ a set-valued function

- $\mathcal{M}$ is Lipschitz-like around $(\hat{z}, \hat{y}) \in \text{gph} \mathcal{M}$ if there exist $\ell \geq 0$, and neighborhood $U$ of $\hat{z}$ and $V$ of $\hat{y}$ such that

$$\mathcal{M}(z) \cap V \subset \mathcal{M}(u) + \ell \|z - u\| \mathcal{B}, \quad \text{for all } z, u \in U.$$
Lipschitz properties

\( \mathcal{Z} \) and \( \mathcal{Y} \) normed spaces
\( \mathcal{M} : \mathcal{Z} \rightrightarrows \mathcal{Y} \) a set-valued function

- \( \mathcal{M} \) is Lipschitz-like around \((\hat{z}, \hat{y}) \in \text{gph} \mathcal{M}\) if there exist \( \ell \geq 0 \), and neighborhood \( U \) of \( \hat{z} \) and \( V \) of \( \hat{y} \) such that

\[
\mathcal{M}(z) \cap V \subset \mathcal{M}(u) + \ell \| z - u \| B, \quad \text{for all } z, u \in U.
\]

- The infimum of such moduli \( \ell' \)s over all possible combinations \( \{\ell, U, V\} \) satisfying this is the exact Lipschitzian bound of \( \mathcal{M} \) around \((\hat{z}, \hat{y})\)

\[
\text{lip } \mathcal{M} (\hat{z}, \hat{y}) = \limsup_{(z,y) \to (\hat{z},\hat{y})} \frac{\text{dist}(y, \mathcal{M}(z))}{\text{dist}(z, \mathcal{M}^{-1}(y))}.
\]
This Lipschitz property is one of the various regularity concepts used in variational analysis.

Mordukhovich, Variational Analysis and Generalized Differentiation (2005)
Rockafellar & Wets, Variational Analysis (1998)
Bonnans, Borwein, Cánovas, Dontchev, Ioffe, Klatte, Lewis, López, Parra, Sekiguchi, Shapiro, Zhu (in the 2000’s), etc.

- the metric regularity property
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- the metric regularity property
- the open mapping property of the inverse mapping $\mathcal{M}^{-1} : Y \Rightarrow Z$. 
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- the metric regularity property
- the open mapping property of the inverse mapping $M^{-1} : Y \Rightarrow Z$.

\[
\text{lip } M (\hat{z}, \hat{y}) = (\text{sur } M^{-1}(\hat{y}, \hat{z}))^{-1} \\
= \liminf_{(y,z,\lambda) \rightarrow (\hat{y}, \hat{z}, 0^+)} \frac{1}{\lambda} \sup \{ r \geq 0 : z + r \mathbb{B}_Z \subset M^{-1}(y + \lambda \mathbb{B}_Y) \}
\]

$sur \ M^{-1}(\hat{y}, \hat{z})$ is the rate of surjection (openness) of $M^{-1}$ around $(\hat{y}, \hat{z})$. 
Part I: Stability of the primal-dual partition in LSIP


Primal problem:

\[ P : \inf c'x, \]
\[ \text{s.t. } a'_t x \geq b_t, \text{ for all } t \in T, \]

- \( T \) is an arbitrary index set
- \( c \) and \( x \) are elements of \( \mathbb{R}^n \)
- \( a : T \to \mathbb{R}^n \) and \( b : T \to \mathbb{R} \) are arbitrary functions.

Haar’s dual problem of \( P \):

\[ D : \sup \sum_{t \in T} \lambda_t b_t, \]
\[ \text{s.t } \sum_{t \in T} \lambda_t a_t = c, \lambda \in \mathbb{R}_{+}^{(T)}, \]

\( P \) and \( D \) involve the same data \( a, b \) and \( c \).
Parameter space

\[ \Pi = (\mathbb{R}^n)^T \times \mathbb{R}^T \times \mathbb{R}^n \text{ (general case)} \]

The Banach space \( \Pi_C = C(T)^n \times C(T) \times \mathbb{R}^n \) (continuous case)

- Extended distance \( d : \Pi \times \Pi \to [0, +\infty] \),

\[
  d(\pi^1, \pi^2) := \max \left\{ \| c^1 - c^2 \|_\infty, \sup_{t \in T} \| (a^1_t, b^1_t) - (a^2_t, b^2_t) \|_\infty \right\},
\]

\[ \pi^i = (a^i, b^i, c^i) \in \Pi, \ i = 1, 2. \]
Differences with the continuous case in $\Pi_3 = \Pi_{IC}^P \cap \Pi_{UB}^D$ and $\Pi_4 = \Pi_{IC}^P \cap \Pi_{IC}^D$. 

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<tr>
<th>$D \setminus P$</th>
<th>IC</th>
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<th>UB</th>
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<td>IC</td>
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Some important sets:

\[ C := \text{conv} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix} : t \in T \right\}. \]

The first and the second moment cones of \( \pi \):

\[ M := \text{cone} \left\{ a_t : t \in T \right\} \quad \text{and} \quad N := \text{cone} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix} : t \in T \right\}, \]

The characteristic cone \( K \):

\[ K := N + \text{cone} \left\{ \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\}. \]
\[ \pi \in \text{int} \left( \Pi_B^P \cap \Pi_B^D \right) \iff \begin{array}{l}
\bullet \ 0_{n+1} \notin \text{cl} \ C \\
\bullet \ c \in \text{int} \ M
\end{array} \]

\[ \pi \in \text{int} \left( \Pi_{UB}^P \cap \Pi_{IC}^D \right) \iff \begin{array}{l}
\bullet \ 0_n \notin \text{cl} \ \text{conv} \ \{a_t : t \in T\} \\
\bullet \ (0_n; 1)' \notin \text{cl} \ N \\
\bullet \ c \notin \text{cl} \ M
\end{array} \]

\text{int} \ (\Pi_B^P \cap \Pi_B^D) \text{ is dense in } \Pi_B^P \cap \Pi_B^D, \text{ if } |T| \geq n.

\text{int} \ (\Pi_{UB}^P \cap \Pi_{IC}^D) \text{ is dense in } \Pi_{UB}^P \cap \Pi_{IC}^D, \text{ for any } T \text{ and any } n.
\[ \pi \in \text{int} \left( \Pi_{IC}^P \cap \Pi_{UB}^D \right), \ M = \mathbb{R}^n \quad \iff \quad \bullet \ (0_n, 1)^t \in \text{int} \ N \]

\[ \pi \in \text{int} \left( \Pi_{IC}^P \cap \Pi_{UB}^D \right), \ M \neq \mathbb{R}^n \quad \iff \quad \bullet \ 0_n \notin \text{int} \ \text{conv}\{a_t : t \in T\} \]
\[ \text{int} \left( \Pi_{IC}^P \cap \Pi_{UB}^D \right) \text{ is dense in } \Pi_{IC}^P \cap \Pi_{UB}^D, \text{ if } |T| \geq n + 1. \]
Stability of the primal-dual partition in LSIP

\[ \pi \in \text{int} \left( \Pi_{IC}^P \cap \Pi_{IC}^D \right) \iff \begin{align*}
\bullet & \quad 0_n \not\in \text{cl conv} \{ a_t : t \in T \} \\
\bullet & \quad (0_n, 1)' \in O^+ (\text{cl } C) \\
\bullet & \quad c \not\in \text{cl } M
\end{align*} \]

\[ \text{int} \left( \Pi_B^P \cap \Pi_{IC}^D \right) = \emptyset. \]

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\[ \Pi_B^P \cap \Pi_{IC}^D \subset \text{bd} \left( \Pi_B^P \cap \Pi_B^D \right) \cap \text{bd} \left( \Pi_{UB}^P \cap \Pi_{IC}^D \right) \]

\[ \Pi_{IC}^P \cap \Pi_B^D \subset \text{bd} \left( \Pi_{IC}^P \cap \Pi_{IC}^D \right). \]
Part II: On coderivatives and Lipschitzian properties of the dual pair in optimization

Characterizations of Lipschitzian stability of feasible solutions for the primal and the dual problem in an infinite-dimensional setting.

López, Ridolfi & V. (2011) (primal and dual problem with a conic constraint)
Duffin (1956), Kretschmer (1961)
Anderson & Nash (1987) *Linear Programming and Infinite-Dimensional Spaces.*
Dinh, Goberna & López (2010)
An infinite-dimensional linear optimization problem

\[ P : \text{Sup} \quad \langle \bar{c}^*, x \rangle \]
\[ \text{s.t.} \quad \langle a_t^*, x \rangle \leq \bar{b}_t, \ t \in T, \]
\[ x \in Q, \]

- \( T \) an arbitrary index set, possibly infinite
- \( X \) real Banach space (not necessarily reflexive)
- \( Q \) closed convex cone in \( X \)
- \( \bar{c}^* \) in \( X^* \)
- \( \{a_t^*, \ t \in T\} \subset X^* \) bounded
- \( A : X \to \ell_\infty(T) \) defined as \( Ax := \langle a^*_t, x \rangle \)
- \( \bar{b}_t, \ t \in T, \) real numbers.
$P$ can be reformulated

\[
P: \quad \text{Sup} \quad \langle \overline{c}^*, x \rangle \\
\text{s.t.} \quad Ax \leq \overline{b}, \\
\quad x \in Q.
\]

$\overline{b} = (\overline{b}_t)_{t \in T}$.

The associated dual problem $D$:

\[
D: \quad \text{Inf} \quad \langle \mu, \overline{b} \rangle \\
\text{s.t.} \quad A^* \mu \in \overline{c}^* - Q^\circ, \\
\quad \mu \geq 0,
\]

$\mu \in \ell_\infty(T)^* = \{ \mu : 2^T \to \mathbb{R} : \mu \text{ is bounded and additive} \}$

$A^* : \ell_\infty(T)^* \to X^*$ adjoint operator of $A$

$Q^\circ$ the dual cone of $Q$
Tools from variational analysis:

- coderivative and its norm
Tools from variational analysis:

- coderivative and its norm
- their relationship with the exact bound of the Lipschitzian moduli.
$\mathcal{M} : Z \Rightarrow Y$

$(\hat{z}, \hat{y}) \in gph\mathcal{M}$, the \textit{coderivative} of $\mathcal{M}$ at $(\hat{z}, \hat{y})$ is

$$D^*\mathcal{M} (\hat{z}, \hat{y}) : Y^* \Rightarrow Z^*$$

$$D^*\mathcal{M} (\hat{z}, \hat{y}) (y^*) := \{ z^* \in Z^* : (z^*, -y^*) \in N((\hat{z}, \hat{y}) ; gph\mathcal{M}) \},$$

$y^* \in Y^*$.

$N((\hat{z}, \hat{y}) ; gph\mathcal{M})$ the limiting normal cone to $gph\mathcal{M}$ at $(\hat{z}, \hat{y})$ (the normal cone when $gph\mathcal{M}$ is convex)

The \textit{norm} of this coderivative is

$$\| D^*\mathcal{M} (\hat{z}, \hat{y}) \| := \sup \{ \| z^* \| : z^* \in D^*\mathcal{M} (\hat{z}, \hat{y}) (y^*) , \| y^* \| \leq 1 \} .$$
For a $C^1$ function $f : \mathbb{R} \to \mathbb{R}$ we have $D^* f (z, f(z)) (y) = f'(z) y$

Let $Z = \mathbb{R}^n$, $Y = \mathbb{R}^m$. For a $C^1$ function $f : \mathbb{R}^n \to \mathbb{R}^m$, the coderivative $D^* f (\hat{z}, \hat{y}) : \mathbb{R}^m \to \mathbb{R}^n$ is the adjoint operator of the differential map $df (\hat{z}, \hat{y}) : \mathbb{R}^n \to \mathbb{R}^m$.

$$D^* f (\hat{z}, \hat{y}) (y) = \nabla f (\hat{z})^* y$$
Ioffe and Sekiguchi (2009) defined a convex set-valued mapping $M^{-1}$ as being \textit{perfectly regular} at $(\hat{y}, \hat{z}) \in \text{gph } M^{-1}$ if and only if

\[
\text{sur } M^{-1}(\hat{y}, \hat{z}) = \inf \left\{ \|y^*\| : y^* \in D^* M^{-1}(\hat{y}, \hat{z})(z^*), \|z^*\| = 1 \right\}.
\]

If $M^{-1}$ is perfectly regular at $(\hat{y}, \hat{z}) \in \text{gph } M^{-1}$, the following equality holds:

\[
\text{lip } M(\hat{z}, \hat{y}) = \|D^* M(\hat{z}, \hat{y})\|.
\]
We allow for perturbations $c^* \in X^*$ and $b \in \ell_\infty(T)$ of the fixed $\bar{c}^*$ and $\bar{b}$.

**perturbed primal problem**

\[
P(b, c^*): \quad \text{Sup} \quad \langle \bar{c}^* + c^*, x \rangle \\
\langle a^*_t, x \rangle \leq \bar{b}_t + b_t, \quad t \in T, \\
\text{s.t.} \quad x \in Q,
\]

**perturbed dual problem**

\[
D(b, c^*): \quad \text{Inf} \quad \langle \mu, \bar{b} + b \rangle \\
A^* \mu \in \bar{c}^* + c^* - Q^\circ, \\
\text{s.t.} \quad \mu \geq 0.
\]
Feasible mappings

\[ \mathcal{F}_P : \ell_\infty(T) \Rightarrow X \]

\[ \mathcal{F}_P(b) := \{x \in X : Ax \leq \bar{b} + b \text{ and } x \in Q\} \]

\[ \mathcal{F}_D : X^* \Rightarrow \ell_\infty(T)^* \]

\[ \mathcal{F}_D(c^*) := \{\mu \in \ell_\infty(T)^* : A^*\mu \in \bar{c}^* + c^* - Q^\circ \text{ and } \mu \geq 0\} \]
The corresponding inverse mappings

\[
\mathcal{F}_P^{-1}(x) := \begin{cases} 
A x - \bar{b} + \ell_{\infty}(T)_+, & \text{if } x \in Q, \\
\emptyset, & \text{if } x \notin Q,
\end{cases}
\]

and

\[
\mathcal{F}_D^{-1}(\mu) := \begin{cases} 
A^* \mu - \bar{c}^* + Q^\circ, & \text{if } \mu \geq 0, \\
\emptyset, & \text{otherwise}.
\end{cases}
\]
Reformulations to apply known results to study their stability properties

Choose a convenient closed bounded set \( \tilde{Q} \) not containing the null vector and spanning the cone \( Q \).

Write

\[
P(b, c^*) : \quad \text{Sup} \quad \langle \overline{c}^*, x \rangle + \langle c^*, x \rangle \\
\quad \text{s.t.} \quad \langle a^*_t, x \rangle \leq \overline{b}_t + b_t, \quad t \in T, \\
\quad \langle q^*, x \rangle \leq 1, \quad q^* \in Q^\circ,
\]

\[
D(b, c^*) : \quad \text{Inf} \quad \langle \mu, \overline{b} \rangle + \langle \mu, b \rangle \\
\quad \text{s.t.} \quad \langle \mu, Aq \rangle \geq \langle \overline{c}^*, q \rangle + \langle c^*, q \rangle, \quad q \in \tilde{Q}, \\
\quad \langle \mu, p \rangle \geq -1, \quad p \in \ell_\infty(T)_+.
\]
\( \mathcal{F}_P \) satisfies the **strong Slater condition** at \( b \in \ell_\infty(T) \) if there is some \( \hat{x} \in Q \) such that

\[
\sup_{t \in T} \left\{ \langle a_t^*, \hat{x} \rangle - \bar{b}_t - b_t \right\} < 0.
\]

\( \mathcal{F}_D \) satisfies the **strong Slater condition** at \( c^* \in X^* \) if there is some \( \hat{\mu} \in \ell_\infty(T)^* \), \( \hat{\mu} \geq 0 \), such that

\[
\inf_{q \in \tilde{Q}} \left\{ \langle \hat{\mu}, Aq \rangle - \langle \bar{c}^* + c^*, q \rangle \right\} > 0.
\]
Q has a \textit{compact base} \( \tilde{Q} \) if there is some \( \bar{x}^* \in X^*, \|\bar{x}^*\| = 1 \), such that the set

\[
\tilde{Q} = \{ q \in Q : \langle \bar{x}^*, q \rangle = 1 \},
\]

is weakly compact and spans \( Q \).

\begin{quote}
\textbf{Proposition}

If \( v_P \) is finite, \( Q \) has a compact base, and there exists a strong Slater point \( \hat{\mu} \) of \( \mathcal{F}_D \) at 0, then there is no-duality gap (i.e. \( v_P = v_D \)) and \( P \) is solvable.
\end{quote}
Theorem

Let $\mathcal{F}_P(b) \neq \emptyset$. If $\text{int} \ Q \neq \emptyset$, then $\mathcal{F}_P$ is Lipschitz-like around $(b, x)$ for all $x \in \mathcal{F}_P(b)$ if and only if there exists some $\overline{x} \in X$ such that $(b, \overline{x}) \in \text{int} (\text{gph} \mathcal{F}_P)$.

Theorem

Let $\widehat{x} \in \mathcal{F}_P(0)$, then $\mathcal{F}_P$ is Lipschitz-like around $(0, \widehat{x})$ if and only if

$$D^* \mathcal{F}_P(0, \widehat{x})(0) = \{0\}.$$ 

Theorem

Let $\widehat{x} \in \mathcal{F}_P(0)$, then

$$\text{lip} \mathcal{F}_P(0, \widehat{x}) = \|D^* \mathcal{F}_P(0, \widehat{x})\|.$$
Characterization of the coderivative for the primal problem

\[ D^* \mathcal{F}_P (0, \hat{x}) \]

**Theorem**

Let \( \hat{x} \in \mathcal{F}_P (0) \), \( \mu \in \ell_\infty (T)^* \), and \( x^* \in X^* \). Then

\[ \mu \in D^* \mathcal{F}_P (0, \hat{x}) (x^*) \]

if and only if

\[ (\mu, -x^*, - \langle x^*, \hat{x} \rangle) \in \text{cl}^* (\text{cone} \{( -\delta_t, a_t^*, b_t \), \ t \in T \} + \{0\} \times Q^\circ \times \{0\}) \]
Theorem

Let $\hat{x} \in \mathcal{F}_P(0)$. Then,

(i) If $\hat{x}$ is a strong Slater point of $\mathcal{F}_P$ at $b = 0$, then $\| D^* \mathcal{F}_P(0, \hat{x}) \| = 0$. 

(ii) If $\hat{x}$ is not a strong Slater point of $\mathcal{F}_P$ at $b = 0$, then $\| D^* \mathcal{F}_P(0, \hat{x}) \| > 0$ and it can be calculated as

$$\| D^* \mathcal{F}_P(0, \hat{x}) \| = \sup \left\{ \| x^* \|^{-1} : (x^*, \langle x^*, \hat{x} \rangle) \in \text{cl}^* \mathcal{C}_P(0) \right\}.$$

The characteristic set of $\mathcal{F}_P(b)$ is

$$\mathcal{C}_P(b) := \text{conv} \left( \left\{ (a^*_t, \bar{b}_t + b_t) : t \in T \right\} \cup (Q^\circ \times \{1\}) \right).$$
Consistency of the dual problem

$c^* \in X^*$ and the linear system in $\ell_\infty(T)^*$

$$\sigma_D(c^*) := \left\{ \begin{array}{l}
\langle \mu, Aq \rangle \geq \langle \bar{c}^* + c^*, q \rangle, \quad q \in \bar{Q}, \\
\langle \mu, p \rangle \geq -1, \quad p \in \ell_\infty(T)_+
\end{array} \right\},$$

**Theorem**

$\sigma_D(c^*)$ is consistent $\iff (0, 1) \notin \text{cl} \ H(c^*)$,

where

$$H(c^*) = \{(Ax, \langle \bar{c}^* + c^*, x \rangle) : x \in Q\} + \ell_\infty(T)_+ \times (-\mathbb{R}_+).$$
Characterization of stably consistent dual problems

Definition

A dual problem $D(b, c^*)$ is stably consistent if and only if $c^* \in \text{int}(\text{dom} F_D)$.

This condition is equivalent to $F_D$ being Lipschitz-like around $(c^*, \mu)$ for all $\mu \in F_D(c^*)$, because the graph of $F_D^{-1} : \ell_{\infty}(T)^* \rightarrow X^*$ is closed and convex, and $\ell_{\infty}(T)^*$ and $X^*$ are Banach spaces.
Proposition

Let \( c^* \in \text{dom} \, \mathcal{F}_D \). Suppose that \( 0 \notin \text{cl} \, \text{conv} \, \tilde{Q} \), then the following statements are equivalent:

(i) There is some \( \hat{\mu} \geq 0 \) that is strong Slater for \( \mathcal{F}_D \) at \( c^* \).

(ii) \((0,0) \notin \text{cl}^* \, C_D (c^*) \).

(iii) \( c^* \in \text{int} (\text{dom} \, \mathcal{F}_D) \)

The characteristic set of \( \mathcal{F}_D(c^*) \), relative to \( \tilde{Q} \), is the following subset of \( \ell_\infty (T) \times \mathbb{R} \):

\[
C_D (c^*) := \text{conv} \left( \left\{ (Aq, \langle \bar{c}^* + c^*, q \rangle) : q \in \tilde{Q} \right\} \cup \left\{ (p, -1) : p \in \ell_\infty (T)_+ \right\} \right).
\]
Remark

- The hypothesis $0 \notin \text{cl conv } \tilde{Q}$ is only needed for the implication $(iii) \Rightarrow (i)$.

- The existence of a strong Slater point implies the condition $0 \notin \text{cl conv } \tilde{Q}$.

- $(ii) \ (0,0) \notin \text{cl}^* C_D (c^*)$ is equivalent to $(ii)' \ (0,0) \notin \text{cl} C_D (c^*)$, (we always consider the $w^*$—topology on $\ell_\infty (T)^{**}$)
Characterization of the normal cone for the dual problem

\[ N \left( (\widehat{c^*}, \widehat{\mu}) ; \text{gph} \mathcal{F}_D \right) \]

Let \((\widehat{c^*}, \widehat{\mu}) \in \text{gph} \mathcal{F}_D\) and let \((x^{**}, b^{**}) \in X^{**} \times \ell_{\infty}(T)^{**}\). Then

\[(x^{**}, b^{**}) \in N \left( (\widehat{c^*}, \widehat{\mu}) ; \text{gph} \mathcal{F}_D \right) \]

if and only if \(- (x^{**}, b^{**}, \langle \widehat{c^*}, x^{**} \rangle + \langle \widehat{\mu}, b^{**} \rangle)\) belongs to

\[ \text{cl}^* \left\{ \{(-q, Aq, \langle \widehat{c^*}, q \rangle) : q \in Q \} + \{0\} \times \ell_{\infty}(T)_+ \times (-\mathbb{R}_+) \right\} . \]
Lemma

Let $\hat{\mu} \in {\mathcal{F}}_D(0)$, $x \in X^{**}$ and $b^{**} \in \ell_\infty(T)^{**}$.
If $x^{**} \in D^* {\mathcal{F}}_D(0, \hat{\mu})(b^{**})$ then there exists a net

$$\{(q_v, p_v)\}_{v \in \mathcal{N}} \subset Q \times \ell_\infty(T)_+$$

such that

$$x^{**} = w^* - \lim_{v \in \mathcal{N}} q_v,$$

$$b^{**} = w^* - \lim_{v \in \mathcal{N}} (Aq_v + p_v),$$

$$\langle \hat{\mu}, b^{**} \rangle = \lim_{v \in \mathcal{N}} \langle \overline{c}^*, q_v \rangle.$$

Moreover, if $\hat{\mu}$ is strong Slater for ${\mathcal{F}}_D$ at 0, then $x^{**} = 0$. 
Characterization of the coderivative of the dual problem

**Theorem**

Let $\hat{\mu} \in \mathcal{F}_D(0)$. If $x^{**} \in X^{**}$ and $b^{**} \in \ell_\infty(T)^{**}$, then $x^{**} \in D^* \mathcal{F}_D(0, \hat{\mu})(b^{**})$ if and only if $(x^{**}, b^{**}, \langle \hat{\mu}, b^{**} \rangle)$ belongs to

$$\text{cl}^* \left\{ \{(q, Aq, \langle \bar{c}^*, q \rangle) : q \in Q\} + \{0\} \times \ell_\infty(T)_+ \times \{0\} \right\}.$$
Estimate of the norm of the coderivative

(A): \( \tilde{Q} \) is a closed spanning subset of \( Q \) such that there are two positive real numbers \( r \) and \( R \), and some \( \bar{x}^* \in X^* \), \( \| \bar{x}^* \| = 1 \), satisfying

\[
0 < r \leq \langle \bar{x}^*, q \rangle \leq \| q \| \leq R
\]

for all \( q \in \tilde{Q} \).

Theorem

Suppose that \( Q \) satisfies condition (A) and that \( \hat{\mu} \in \mathcal{F}_D(0) \). Then,

(i) If \( \hat{\mu} \) is a strong Slater point of \( \mathcal{F}_D \) at 0, then \( \| D^* \mathcal{F}_D(0, \hat{\mu}) \| = 0 \).

(ii) If \( \hat{\mu} \) is not a strong Slater point of \( \mathcal{F}_D \) at 0, then \( \| D^* \mathcal{F}_D(0, \hat{\mu}) \| > 0 \) and

\[
r \Delta \leq \| D^* \mathcal{F}_D(0, \hat{\mu}) \| \leq R \Delta
\]

where

\[
\Delta := \sup \left\{ \| b^{**} \|^{-1} : (b^{**}, \langle \hat{\mu}, b^{**} \rangle) \in \text{cl}^* C_D(0) \right\}.
\]
Example

Constants \( r = R = 1 \). \( T = \mathbb{N} \), \( X = \ell_1 \)

\[
Q = \{(x_n) \in \ell_1 : x_n \geq 0 \text{ for all } n \in \mathbb{N}\}.
\]

\[
\overline{c}^* = 1^* := (1_n^*) \in \ell_\infty = X^* ,
\]

\[
a_n^* := -e_n^* \in \ell_\infty, \quad n = 1, 2, \ldots
\]

\[
\tilde{Q} := \{q \in Q : \|q\|_1 = 1\}
\]

Condition (A) holds for \( \tilde{Q} \) with \( r = R = 1 \).

\( \overline{\mu} = 0 \) is a strong Slater point of \( \mathcal{F}_D \) at 0,

the Dirac measure \( \hat{\mu} = \delta_1 \in \mathcal{F}_D(0) \) is not.

The equality

\[
\| D^* \mathcal{F}_D(0, \hat{\mu}) \| = \sup \left\{ \| b^{**} \|^{-1} : (b^{**}, \langle \hat{\mu}, b^{**} \rangle) \in \text{cl}^* C_D (0) \right\}
\]

takes place at \((0, \hat{\mu}) = (0, \delta_1)\).
Example

Let $X = \mathbb{R}^2$

$T = \mathbb{N}$,

$\bar{c}^* = (0, 1)$,

$a_n := (1, 2)$ for all $n$,

$\hat{\mu} = \delta_1$,

$$Q = \{ (x, y) \in \mathbb{R}^2 : y \geq |x| \} ,$$

Take

$$\tilde{Q}_1 = \{ (x, y) \in Q : |x| + |y| = 1 \} .$$

The largest $r$ and smallest $R$ we can choose for $\tilde{Q}_1$ are $r_1 = \frac{1}{2}$ and $R_1 = 1$.

Another possibility is to take

$$\tilde{Q}_2 = \{ (x, y) \in Q : |x| \leq 2, y = 2 \} .$$

Now $r_2 = 2 (\langle (0, 1), (x, y) \rangle = y \geq 2)$ and $R_2 = 2\sqrt{2}$.
We have, for $i = 1, 2$, \[ C_i^D(0, 0) := \text{conv} \left( \left\{ (x + 2y)_{n \in \mathbb{N}}, y \right\} : (x, y) \in \tilde{Q}_i \right) \cup \left\{ (p, -1) : p \in \ell_\infty(\mathbb{N}_+) \right\}. \]

In particular,
\[ C_2^D(0, 0) = \text{conv} \left( \left\{ (x + 4)_{n \in \mathbb{N}}, 2 \right\} : |x| \leq 2 \right) \cup \left\{ (p, -1) : p \in \ell_\infty(\mathbb{N}_+) \right\}. \]

Then:
(i) \[ \sup \left\{ \|b^*\|^{-1}_\infty : (b^*, \langle \hat{\mu}, b^* \rangle) \in \text{cl } C_1^D(0, 0) \right\} = 2, \]
(ii) \[ \sup \left\{ \|b^*\|^{-1}_\infty : (b^*, \langle \hat{\mu}, b^* \rangle) \in \text{cl } C_2^D(0, 0) \right\} = \frac{1}{2} \text{ and } R_2 \sup \left\{ \|b^*\|^{-1}_\infty : (b^*, \langle \hat{\mu}, b^* \rangle) \in \text{cl } C_2^D(0, 0) \right\} = \sqrt{2}, \]
(iii) \[ \|D^*F_D(0, \hat{\mu})\| = \sqrt{2}, \]
so \[ \|D^*F_D(0, \hat{\mu})\| = R_2 \sup \left\{ \|b^*\|^{-1}_\infty : (b^*, \langle \hat{\mu}, b^* \rangle) \in \text{cl } C_2^D(0, 0) \right\} < R_1 \sup \left\{ \|b^*\|^{-1}_\infty : (b^*, \langle \hat{\mu}, b^* \rangle) \in \text{cl } C_1^D(0, 0) \right\}. \]
Theorem

Let $\widehat{\mu} \in \mathcal{F}_D(0)$ and suppose that $0 \notin \text{cl conv } \tilde{Q}$. Then, $\mathcal{F}_D$ is Lipschitz-like around $(0, \widehat{\mu})$ if and only if

$$D^* \mathcal{F}_D(0, \widehat{\mu})(0) = \{0\}.$$ 

Remark

If the condition $0 \notin \text{cl conv } \tilde{Q}$ does not hold, we may have $D^* \mathcal{F}_D(0, \widehat{\mu})(0) = \{0\}$ and $\mathcal{F}_D$ not Lipschitz-like around $(0, \widehat{\mu})$. 

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Example

\[ T = \{ t_0 \} \]
\[ X = c_0 \text{ the Banach space of bounded real sequences converging to 0, with the supremum norm.} \]
\[ X^* = \ell_1 = \ell_1 (\mathbb{N}) \text{ and } X^{**} = \ell_\infty = \ell_\infty (\mathbb{N}) . \]

\[ Q = \{ q \in c_0 : q_n \geq 0, \ n \in \mathbb{N} \} \quad \text{and} \quad \widetilde{Q} = \{ q \in Q : \| q \|_\infty = 1 \} . \]

\[ a^*_t = \bar{c}^* = \left( \frac{1}{n!} \right)_{n=1}^\infty \in \ell_1 \]

Observing that each \( e^n = (0, \ldots, 0, 1, 0, \ldots) \), is in \( \widetilde{Q} \), it is easy to see that

\[ \hat{\mu} = 1 \in \mathcal{F}_D (0) \]

\[ D^* \mathcal{F}_D (0, \hat{\mu})(0) = \{ 0 \} \]

\[ c^* = 0 \notin \text{int (dom } \mathcal{F}_D ) \]

\( \mathcal{F}_D \) is not Lipschitz-like around \((0, 1)\).
Estimate of the exact Lipschitzian bound for the dual problem

\[ \text{lip } \mathcal{F}_D(0, \hat{\mu}) = \limsup_{(c^*, \mu) \to (0, \bar{\mu})} \frac{\text{dist}(\mu, \mathcal{F}_D(c^*))}{\text{dist}(c^*, \mathcal{F}_D^{-1}(\mu))}. \]

Extended Ascoli distance formula

\[ \text{dist}(\mu, \mathcal{F}_D(c^*)) = \sup_{(b^{**}, \alpha) \in \text{cl}^* \mathcal{C}_D(c^*)} \frac{[\alpha - \langle b^{**}, \mu \rangle]_+}{\|b^{**}\|} \]

holds true when \( \mathcal{F}_D \) satisfies the strong Slater condition at \( c^* \).

If \( \|q\| \leq R \) for all \( q \in \tilde{Q} \). Let \( c^* \in X^* \) and \( \mu \in \ell_\infty(T)^* \) be such that \( (c^*, \mu) \notin \text{gph } \mathcal{F}_D \) and \( \mathcal{F}_D^{-1}(\mu) \neq \emptyset \). Then

\[ \text{dist}(c^*, \mathcal{F}_D^{-1}(\mu)) \geq R^{-1} \sup_{q \in \tilde{Q}} \left[ -\langle \mu, Aq \rangle + \langle \bar{c}^* + c^*, q \rangle \right]_+ > 0. \]
Theorem

Assume condition (A) and let $\hat{\mu} \in F_D(0)$. Then:

(i) If $\hat{\mu}$ is a strong Slater point of $F_D$ at 0, then $\text{lip} F_D(0, \hat{\mu}) = 0$.

(ii) If $\hat{\mu}$ is not a strong Slater point of $F_D$ at 0, then

$$r\Delta \leq \text{lip} F_D(0, \hat{\mu}) \leq R\Delta$$

where

$$\Delta := \sup \left\{ \|b^*\|^{-1} : (b^*, \langle \hat{\mu}, b^* \rangle) \in \text{cl}^* C_D(0) \right\}$$
Example

Let $X = \mathbb{R}^2$

$T = \mathbb{N}$,

$\overline{c}^* = (0, 1)$,

$a_n := (1, 2)$ for all $n$,

$\hat{\mu} = \delta_1$,

$$Q = \left\{ (x, y) \in \mathbb{R}^2 : y \geq |x| \right\},$$

$$\tilde{Q} = \tilde{Q}_2 = \left\{ (x, y) \in Q : |x| \leq 2, y = 2 \right\}.$$ 

Now $r_2 = 2$ and $R_2 = 2\sqrt{2}$, gives $\text{lip} \mathcal{F}_D(0, \hat{\mu}) = \sqrt{2}$ because

$$\sqrt{2} = \| D^* \mathcal{F}_D (0, \hat{\mu}) \| \leq \text{lip} \mathcal{F}_D (0, \hat{\mu})$$

$$\leq R_2 \max \left\{ \| b^{**} \|_\infty^{-1} : (b^{**}, \langle \hat{\mu}, b^{**} \rangle) \in \text{cl} C_D^2 \right\}$$

$$= \sqrt{2}. $$
Theorem

In relation to the dual feasible set mapping $F_D$ the following two statements hold:

(i) Assume condition (A) and suppose that $F_D$ satisfies the strong Slater condition at $c^* = 0$. Let $\hat{\mu} \in F_D(0)$, then there are constants $r$ and $R$, $0 < r \leq R$, which only depends on the cone $Q$, such that

$$0 \leq \text{lip} F_D(0, \hat{\mu}) - \| D^* F_D(0, \hat{\mu}) \| \leq (R - r) \max \left\{ \| b^{**} \|^{-1} : (b^{**}, \langle \hat{\mu}, b^{**} \rangle) \in \text{cl}^* C_D (0) \right\}.$$

(ii) If $F_D^{-1}(l_\infty(T)_+) \text{ has nonempty interior for the norm topology in } X^*$ and

$$\{ q \in Q : \| q \| = 1 \}$$

is $w^*$-closed in $X^{**}$, then

$$\text{lip} F_D(0, \hat{\mu}) = \| D^* F_D(0, \hat{\mu}) \|.$$
The equality \( \text{lip} \mathcal{F}_D(0, \hat{\mu}) = \| D^* \mathcal{F}_D(0, \hat{\mu}) \| \) may hold even when \( r \neq R \) and \( \{ q \in Q : \| q \| = 1 \} \) is not \( w^* \)-closed in \( X^{**} \).

**Example**

\( X = c_0 \) the Banach space of bounded real sequences converging to 0, with the supremum norm.

\( X^* = \ell_1 = \ell_1(\mathbb{N}) \) and \( X^{**} = \ell_{\infty} = \ell_{\infty}(\mathbb{N}) \).

\( Q = \{ q \in c_0 : 2q_1 \geq q_n \geq 0, \ n \in \mathbb{N} \} \) and \( \tilde{Q} = \{ q \in Q : \| q \|_{\infty} = 1 \} ; \)

\( 1 = (1)_{n=1}^\infty \in \text{cl}^* \tilde{Q} \), so this set \( \tilde{Q} \) is not \( w^* \)-closed in \( \ell_\infty \).

\( T = \{ t_0 \} \) and put \( a^*_0 := a^* = \left( \frac{-1}{2^{n-1}} \right)_{n=1}^\infty \in \ell_1 \), also fix \( \overline{c}^* = a^* \).

\( r = \frac{1}{2} \) is the largest \( r \) we can choose to satisfy \( \langle z^*, q \rangle \geq r \) for some \( z^* \in \ell_1, \| z^* \|_1 = 1 \), and for all \( q \in \tilde{Q} \). Here \( R = 1 \).

Then \( \mu = 0 \) is a strong Slater point for \( \mathcal{F}_D \) at \( c^* = 0 \), while \( \hat{\mu} = 1 \in \mathcal{F}_D(0) \) is not strong Slater.

\[
C_D(0) = \text{conv} \left( \left\{ \langle a^*, q \rangle, \langle a^*, q \rangle \right\} : q \in \tilde{Q} \right) \cup \left\{ (p, -1) : p \in \mathbb{R}, \ p \geq 0 \right\} \subset \mathbb{R}^2 ;
\]
\[ \| D^* \mathcal{F}_D(0, \hat{\mu}) \| \leq \sup \left\{ \| b^* \|^{-1} : (b^*, \langle \hat{\mu}, b^* \rangle) = (b^*, b^*) \in \text{cl}^* C_D(0) \right\} \]

Also \( \text{lip} \mathcal{F}_D(0, \hat{\mu}) \leq 2 \)
On the other hand, for each \( k \in \mathbb{N}, \; k \neq 1, \; \bar{q}^k \) defined by
\[
\bar{q}_1^k = 1 - \frac{1}{2^{k-1}}, \; \bar{q}_{k+1}^k = 2\bar{q}_1^k, \; \text{and} \; q_n^k = 0 \; \text{otherwise},
\]
belongs to \( D^* \mathcal{F}_D(0, \hat{\mu}) (b^k) \), where \( b^k := \langle a^*, \bar{q}^k \rangle \).
Since \( |b^k| = |\langle a^*, \bar{q}^k \rangle| = 1 - \frac{1}{2^{k(k-2)}} < 1 \), we obtain that
\[
\| D^* \mathcal{F}_D(0, \hat{\mu}) \| = \sup \left\{ \| z^* \| : z^* \in D^* \mathcal{F}_D(0, \hat{\mu}) (b^* \rangle, \; \| b^* \| \leq 1 \right\} \\
\geq \| \bar{q}^k \| = \| \bar{q}^k \|_\infty = 2 \left( 1 - \frac{1}{2^{k-1}} \right).
\]
By letting \( k \to \infty \) \( \| D^* \mathcal{F}_D(0, \hat{\mu}) \| \geq 2 \).
It follows that \( \| D^* \mathcal{F}_D(0, \hat{\mu}) \| = 2 \).
Finally, since \( \| D^* \mathcal{F}_D(0, \hat{\mu}) \| \leq \text{lip} \mathcal{F}_D(0, \hat{\mu}) \leq 2 \), we obtain the equalities
\[
\| D^* \mathcal{F}_D(0, \hat{\mu}) \| = \text{lip} \mathcal{F}_D(0, \hat{\mu}) = 2.
\]
Goberna & V. (2011)

Nature provides beautiful examples:
The world’s largest salt flats: Salar de Uyuni (Bolivia)
A salty Voronoi cell
Voronoi Cells

Let $T \subset \mathbb{R}^n$ be the Voronoi sites. ($T$ with more than one element)

The **Voronoi cell** of $s \in T$ is the set

$$V_T(s) = \{ x \in \mathbb{R}^n : d(x, s) \leq d(x, T \setminus \{s\}) \}.$$

Descartes (1644), Dirichlet (1850), Voronoi (1908)

$$V_T(s) = \left\{ x \in \mathbb{R}^n : (t - s)' x \leq \frac{||t||^2 - ||s||^2}{2}, t \in T \right\}$$

Voronoi Diagram

\[ \text{Vor} (T) := \{ V_T (s) : s \in T \} \]

(Wikipedia)
An unbounded Voronoi cell.

$V_T(s)$ is bounded if and only if $s \in \text{int conv } T$.

$T = \{0_2\} \cup (\{-1\} \times \mathbb{Z})$, \quad s = 0_2
Let $T = \{(k - 1, 1 - \frac{1}{k}), (-k - 1, -1 + \frac{1}{k}), k \in \mathbb{N}\}$
$k = 2, 10$
Perturbations of the sites

In many real applications, the position of some elements of $T$ is uncertain.

What are the effect on the Voronoi cell of small changes in $s$ or in a given non empty set $P \subset T \setminus \{s\}$?

Different types of perturbations can be considered:

- only the point $s$ is allowed to be perturbed.
Perturbations of the sites

In many real applications, the position of some elements of $T$ is uncertain.

What are the effect on the Voronoi cell of small changes in $s$ or in a given non empty set $P \subset T \setminus \{s\}$?

Different types of perturbations can be considered:

- only the point $s$ is allowed to be perturbed.
- global perturbations of the subset of sites $P$. 

V. Vera (UNCuyo)
Perturbations of the sites

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Different types of perturbations can be considered:

- only the point $s$ is allowed to be perturbed.
- global perturbations of the subset of sites $P$.
- pointwise perturbations of $P$. 

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Perturbations of the sites

In many real applications, the position of some elements of $T$ is uncertain.

What are the effect on the Voronoi cell of small changes in $s$ or in a given non empty set $P \subset T \setminus \{s\}$?

Different types of perturbations can be considered:

- only the point $s$ is allowed to be perturbed.
- global perturbations of the subset of sites $P$.
- pointwise perturbations of $P$.
- simultaneous perturbations in $s$ and $P$. 

Stability of Voronoi cells

Strong Slater condition:

Given a set $P$ with $\emptyset \neq P \subset T \setminus \{s\}$, an element $\bar{x}$ of the set

$$X_P^s := \left\{ x \in \mathbb{R}^n : (t - s)'x \leq \frac{\|t\|^2 - \|s\|^2}{2}, \ t \in T \setminus P \right\}$$

is called a **strong Slater point** for the pair $(P, s)$ with associated scalar $\delta > 0$ whenever

$$(t - s)' \bar{x} \leq \frac{\|t\|^2 - \|s\|^2}{2} - \delta, \ \text{for all} \ t \in P.$$  

When there exists a strong Slater point for $(P, s)$ we say that $P$ satisfies the **strong Slater condition for $s$**.
Proposition

Given a set $P$ such that $\emptyset \neq P \subset T \setminus \{s\}$, the following statements are equivalent:

(i) $P$ satisfies the strong Slater condition for $s$.

(ii) $s \notin \text{cl } Q$, i.e., $d(s, P) > 0$.

(iii) $s$ is a strong Slater point for $(P, s)$.

Proposition

Assume that $P$ is bounded. If $\bar{x}$ is a strong Slater point for $(P, s)$, then there exists $\varepsilon > 0$ such that $\bar{x}$ is a strong Slater point for any pair $(P_1, s)$ such that $P_1 \subset P + \varepsilon B_n$. 
Stability of Voronoi cells  Case 1

Case 1: only $s$ can be perturbed.
Parameter space $\Omega_1 = \mathbb{R}^n$.

$\mathcal{V} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the mapping

$$\mathcal{V}(s_1) = \left\{ x \in \mathbb{R}^n : (t - s_1)' x \leq \frac{\| t \|^2 - \| s_1 \|^2}{2}, \ t \in T \setminus \{ s \} \right\}$$

for any $s_1 \in \mathbb{R}^n$.

Theorem

The following statements are true for any $s \in T$ :

(i) $\mathcal{V}$ is closed at $s$.

(ii) $\mathcal{V}$ is lsc at $s$ if and only if $V_T(s) = \{ s \}$ or $s$ is an isolated point of $T$.

(iii) $\mathcal{V}$ is usc at $s$ if $V_T(s)$ is bounded.
Stability of Voronoi cells  Case 2

**Case 2:** $P$ is a given **nonempty compact** subset of sites, $s \notin P$.

Global perturbations of $P$ that are also nonempty compact sets not containing $s$ are allowed.

$s$ is kept fixed.

Parameter space $\Omega$: the family of nonempty compact subsets of $\mathbb{R}^n$ equipped with the Hausdorff distance $d_H$.

The set-valued mapping to be analyzed is $\mathcal{V} : \Omega \rightrightarrows \mathbb{R}^n$

$$\mathcal{V} (P_1) = \left\{ x \in X : (t - s)' x \leq \frac{\|t\|^2 - \|s\|^2}{2}, \ t \in P_1 \right\},$$

for $P_1 \in \Omega$, where

$$X := X_P^s = \left\{ x \in \mathbb{R}^n : (t - s)' x \leq \frac{\|t\|^2 - \|s\|^2}{2}, \ t \in T \setminus P \right\}$$

will remain fixed.

$$\mathcal{V} (P) = V_T (s)$$
Theorem

The following statements are true for any $P \subset T$, $P \in \Omega$:

(i) $\mathcal{V}$ is closed at $P$.

(ii) $\mathcal{V}$ is lsc at $P$.

(iii) $\mathcal{V}$ is usc at $P$ if $V_T(s)$ is bounded.
Case 3: \( P \) is a given nonempty bounded subset of sites, \( s \notin P \).

Pointwise perturbations on \( P \), \( s \notin P \) remains fixed.

**Pointwise perturbations** of \( P \): The result of perturbing \( P \) is represented by the (bounded) range, \( f(P) \), of certain \( f \in l^\infty(P) \).

The set \( P \) is represented by the identity mapping on it, say \( i_P \).

The parameter space is the Banach space \( l^\infty(P) \) of all bounded functions from \( P \subset \mathbb{R}^n \) to \( \mathbb{R}^n \) equipped with the norm

\[
\| f \|_\infty := \sup_{t \in P} \| f(t) \|_\infty, \text{ for all } f \in l^\infty(P).
\]
Stability of Voronoi cells  Case 3

The set-valued mapping to be analyzed is \( \mathcal{V} : l^n_\infty (P) \rightarrow \mathbb{R}^n \) defined by

\[
\mathcal{V}(f) = \left\{ x \in X : (f(t) - s)'x \leq \frac{\|f(t)\|^2 - \|s\|^2}{2}, \ t \in P \right\},
\]

for all \( f \in l^n_\infty (P) \), where \( X := X_P^s \)

\[
\mathcal{V}(i_P) = V_T(s)
\]

In this framework we have:

- If \( T \) is finite, then \( \mathcal{V}(f) = V_{(T\setminus P) \cup f(P)} (s) \) is always a polyhedral convex set because \( (T\setminus P) \cup f(P) \) is finite.
- If \( T \) is discrete, \( (T\setminus P) \cup f(P) \) is discrete too because it is the union of discrete sets, so that \( \mathcal{V}(f) \) is quasipolyhedral for all \( f \in l^n_\infty (P) \).
- The boundedness of \( V_T(s) \) is maintained in some neighborhood of \( i_P \).
The number of extreme points (or facets) of a Voronoi cell can change even for arbitrarily small pointwise perturbations.

Let \( T = \{(0, 0), (1, 0), (2, 0)\} \), \( P = \{(2, 0)\} \), and \( s = 0 \)
\( l_n^\infty (P) = \mathbb{R}^2 \) and \( \mathcal{V}(i_P) = \{ x \in \mathbb{R}^2 : x_1 \leq \frac{1}{2} \} \) has a unique facet and no extreme point.
If \( f(2, 0) = (a, b) \in \mathbb{R}^2 \), we have
\[
\mathcal{V}(f) = \left\{ x \in \mathbb{R}^2 : x_1 \leq \frac{1}{2}, ax_1 + bx_2 \leq \frac{a^2 + b^2}{2} \right\},
\]
and this polyhedral convex set has two facets and one extreme point whenever \( b \neq 0 \).
Theorem

Given $s \in T \setminus P$, the following statements are true:

(i) $\mathcal{V}$ is closed at $i_P$.

(ii) $\mathcal{V}$ is lsc at $i_P$ if and only if $V_T(s) = \{s\}$ or $P$ satisfies the strong Slater condition for $s$.

(iii) $\mathcal{V}$ is usc at $i_P$ if $V_T(s)$ is bounded.
Stability of Voronoi cells  Case 3

The assumption $V(i_P) \neq \{s\}$ gives that $s \notin \text{cl } P$ is equivalent to $V$ being lsc at $i_P$.
Since $V$ is closed, the Painlevé-Kuratowski characterization applies:

**Corollary**

If $V(i_P) \neq \{s\}$, then $s \notin \text{cl } P$ if and only if for any sequence
\[ \{f_k\} \subset l_\infty^n(P), \]
\[ \liminf_{k \to \infty} V(f_k) = \limsup_{k \to \infty} V(f_k) = V(i_P). \]

**Corollary**

If $V_T(s) = V(i_P) \neq \{s\}$, then $s \notin \text{cl } P$ if and only if $V_T(s)$ is the set formed by all the possible cluster points of sequences $\{y_k\}$ with $y_k \in V(f_k)$, for all the sequences $\{f_k\} \subset l_\infty^n(P), f_k \to i_P$, and for any $x \in V_T(s)$ there exists a convergent sequence $\{x_k\}$ such that $x_k \in V(f_k)$ and $\lim_{k \to \infty} x_k = x$. 
Stability of Voronoi cells

Simultaneous perturbations in $s$ and $P$:

- the nominal set $P$ will be replaced by a new set $P_1$,
- the nominal site $s$ by a new site $s_1$,
- new set of sites $(T \setminus (P \cup \{s\})) \cup P_1 \cup \{s_1\}$. 

.
**Case 4:** $P$ is a given **nonempty compact** subset of sites, $s \notin P$.

Global perturbations of $P$ that are also nonempty compact sets not containing $s$ are allowed.

The parameter space $\Omega_2$ is the family of pairs $(P_1, s_1)$ such that $P_1$ is a nonempty compact set, $s_1 \in \mathbb{R}^n$, and $s_1 \notin P_1$, equipped with the distance

$$d \left( (P_1, s_1), (P_2, s_2) \right) := \max \{ d_H (P_1, P_2), d (s_1, s_2) \}.$$ 

The corresponding set-valued mapping is $\mathcal{V} : \Omega_2 \rightarrow \mathbb{R}^n$ such that

$$\mathcal{V} (P_1, s_1) = \left\{ x \in X (s_1) : (t - s_1)' x \leq \frac{\| t \|^2 - \| s_1 \|^2}{2}, \ t \in P_1 \right\},$$

where

$$X (s_1) := \left\{ x \in \mathbb{R}^n : (t - s_1)' x \leq \frac{\| t \|^2 - \| s_1 \|^2}{2}, \ t \in T \setminus P, t \neq s, t \neq s_1 \right\}.$$
Theorem

Global perturbations: For $\mathcal{V} : \Omega_2 \Rightarrow \mathbb{R}^n$ and any fixed $(P, s) \in \Omega_2$:
(i) $\mathcal{V}$ is closed at $(P, s)$.
(ii) $\mathcal{V}$ is lsc at $(P, s)$.
(iii) If $\mathcal{V}_T(s)$ is bounded, then $\mathcal{V}$ is locally bounded at $(P, s)$.
(iv) $\mathcal{V}$ is usc at $(P, s)$ if $\mathcal{V}_T(s)$ is bounded.
**Case 5:** $P$ is a given nonempty bounded subset of sites, $s \not\in P$.

**Pointwise perturbations** of $P$: The result of perturbing $P$ is represented by the (bounded) range, $f(P)$, of certain $f \in l^\infty(P)$.

The set $P$ is represented by the identity mapping on it, say $i_P$.

The parameter space $\Omega_3$ is the Banach space $l^n_\infty(P) \times \mathbb{R}^n$ with the product topology.

$\mathcal{V} : \Omega_3 \Rightarrow \mathbb{R}^n$ is defined by

$$\mathcal{V}(f, s_1) = \left\{ x \in X(s_1) : (f(t) - s_1)'x \leq \frac{\|f(t)\|^2 - \|s_1\|^2}{2}, \ t \in P \right\}.$$
Stability of Voronoi cells  Case 5

Theorem

**Pointwise perturbations**: For $\mathcal{V} : \Omega_2 \rightarrow \mathbb{R}^n$, given $s \in T \setminus P$, the following statements are true:

(i) $\mathcal{V}$ is closed at $(i_P, s)$.

(ii) $\mathcal{V}$ is lsc at $(i_P, s)$ if and only if $V_T(s) = \{s\}$ or $s$ is isolated in $T$.

(iii) If $V_T(s)$ is bounded, then $\mathcal{V}$ is locally bounded at $(i_P, s)$.

(iv) $\mathcal{V}$ is usc at $(i_P, s)$ if $V_T(s)$ is bounded.

$T$ finite set of sites: in all the cases $\mathcal{V}$ is closed and lower semicontinuous.

In particular we always have the Painlevé Kuratowski characterization of $V_T(s)$. 
Stability of Voronoi cells  Case 5

The lsc and the usc properties may fail simultaneously in cases where \( P \) is a non closed set with \( s \in \text{bd } P \), and \( s \notin \text{int conv } T \).

Example

Let \( T = \left\{ t \in \mathbb{R}^2 : \| t - e_1 \| = 1 \right\} \), \( s = 0_2 \) and and \( P = T \setminus \{0_2\} \).

(a) For any \( \varepsilon \), \( 0 < \varepsilon < 1 \), let \( f_\varepsilon : P \to \mathbb{R}^2 \) be such that \( f_\varepsilon (t_1, t_2) = (t_1 - \varepsilon, t_2) \). Then, \( \mathcal{V} \) is not lsc at \( (i_P, s) \).

Indeed, \( \mathcal{V} (i_P, s) \) is not bounded and \( \mathcal{V} (f_\varepsilon, s) \) is bounded.
Example

(b) For any \( \varepsilon, 0 < \varepsilon < 1 \), let \( f_\varepsilon : P \to \mathbb{R}^2 \) be such that
\[ f_\varepsilon (t) = (1 - \cos \varepsilon, \sin \varepsilon), \] if \( t_1 < 1 - \cos \varepsilon \), and \( 0 < t_2 < \sin \varepsilon \), and
\[ f_\varepsilon (t) = t, \] otherwise. Then \( \| f_\varepsilon - i_P \|_\infty < \sqrt{2} (1 - \cos \varepsilon) \to 0 \) as \( \varepsilon \downarrow 0 \)
whereas

\[ \mathcal{V} (f_\varepsilon, s) = \left\{ x \in \mathbb{R}^2 : x_1 + \left( \cot \frac{\varepsilon}{2} \right) x_2 \leq 1, x_2 \geq 0 \right\} \]

is not contained in \( \mathcal{V} (i_P, s) + B_2 \) for any \( \varepsilon \). Therefore \( \mathcal{V} \) is not usc at \((i_P, s)\).
References


The end!