On the small-time local attainability [1]

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Abstract. The concept of small-time local attainability (STLA) is discussed. A sufficient geometrical condition for small-time local attainability of a control system with respect to a closed set is proved.

Keywords. nonlinear control system, small-time local attainability

1 Small-time local attainability of a set.

The problem of small-time local attainability of a control system with respect to a closed set has been partially studied using only zero order and one order approach ([2], [19], [23], [24], [7] etc.). We would like to point out that the small-time local attainability of a control system with respect to a given set is not reduced to the problem of small-time local attainability at every point of the set. So, it needs a specific study.

To present a general sufficient STLA condition of zero-order of a control system with respect to a set, we shall follow the notations from [7]: Let S be a compact subset of R^n. We set

$$S_r \ := \ \{y \in R^n | \ \mathrm{d}_S(y, S) \ \leq r\},$$

where

$$\mathrm{d}_S(y) \ := \ \inf \{\|y - s\| \mid s \in S\}.$$

Let the control system is determined by the following differential inclusion

$$\dot{x}(t) \ \in \ F(x(t)), \ \text{ for a. e. } t \in [0, T], \ x(0) \ = \ x_0, \tag{1}$$

in which $F : R^n \Rightarrow R^n$ is a multifunction with compact and convex values. It is assumed that F is continuous of modulus ω near S, i.e.

$$\Delta(F(x), F(y)) \ \leq \ \omega(\|x - y\|), \ \text{ for all } x, y \ \text{ near } S,$$

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where Δ denotes the Hausdorff metric and $\omega : [0, \infty) \to [0, \infty)$ is a nondecreasing continuous function with $\omega(0) = 0$. Let y belongs to the boundary ∂S of the set S. A vector $\xi \in R^n$ is called a proximal normal to S at y provided there exists $r > 0$ so that the point $y + r\xi$ has closest point y in S. The set of all proximal normals at a point y is a cone, and is denoted by $N_S^p(y)$ (for a detailed treatment of proximal analysis and its applications, see [8]).

Definition 1.1 *It is said that a closed set $S \subset R^n$ is small-time locally attainable (STLA) with respect to a control system iff for any $T > 0$ there exists a neighborhood Ω of S such that for every point $x \in \Omega$ there exists a solution of the control system starting from the point x and reaching the set S in time not greater than T.*

Then the results in [23], [24] and [7] can be formulated as follows:

Theorem 1.1 *Suppose that S is nonempty and compact subset of R^n, and $F : R^n \Rightarrow R^n$ is a continuous multifunction of modulus ω with compact convex values. Suppose that there exists $\delta > 0$ so that, whenever $y \in S$ and $\xi \in N_S^p(y)$, there exists $v \in F(y)$ for which*

$$< \xi, v > \leq \; -\delta \|\xi\|. \tag{2}$$

Then (1) is STLA with respect to S.

Using some of the ideas related to small-time local controllability at a point (cf. [10], [12], [25], a class of high order variations is defined in [18] and a new sufficient STLA condition of a control system with respect to a closed set is proved. In this paper we generalize this result.

Let $\mathcal{Q}$ be a subset of $R^n \times R$ such that for every point $x \in R^n$ there exists a positive number ε_x such that

$$\{(x,t) \mid t \in (-\varepsilon_x, \varepsilon_x)\} \subset \mathcal{Q}.$$

We consider a generalized control system, namely a set valued map

$$\mathcal{R} : \mathcal{Q} \Rightarrow R^n$$

with closed nonempty values which is continuous in t and satisfies the following semigroup property: Let x and y be arbitrary points from R^n and $\mathcal{R}(x,t)$, respectively, and let $\mathcal{R}$ be defined at (y,s). Then the relation $z \in \mathcal{R}(y,s)$ implies that $z \in \mathcal{R}(x, t+s)$. A trajectory of the generalized control system R defined on $[0,T]$, $T > 0$, and starting from the point $x \in R^n$ is called any absolutely continuous function $z(\cdot)$ defined on $[0,T]$ and such that

$$z(0) = x \text{ and } z(t) \in \mathcal{R}(x,t), \quad \text{for all } t \in [0,T].$$

Remark 1.1 *We do not need to fix the form of the control system, because we present a general approach for studying the reachable set of control systems. It is based on the* *notion of local variation which is defined below. The STLA property of some particular cases of the control systems, determined by differential equations, differential inclusions and etc., is studied in [18]. Thus, one can think that $\mathcal{R}(x,t)$ is just the reachable set of a control system starting from the point x at the moment of time t.*

Our main assumption on $\mathcal{R}$ is the following:

C. Let $\{x_k(.)\}_{k=1}^{\infty}$ be a sequence of trajectories of $\mathcal{R}$ defined on $[0, T_k]$, respectively. Let $\{T_k\}_{k=1}^{\infty} \to T$. Then there exists a subsequence $\{x_{k_j}(.)\}_{k=1}^{\infty}$ of the sequence $\{x_k(.)\}_{k=1}^{\infty}$ and a trajectory $x(.)$ of $\mathcal{R}$ defined on $[0, T]$ such that $x_{k_j}(t) \to x(t)$ uniformly over $[0, \min\{T_{k_j}, T\}]$.

REMARK 1.2 *The assumption C means compactness of the set of all trajectories of the considered control system $\mathcal{R}$. The property compactness of the set of all trajectories is typical for a sufficiently general class of control systems. For example, results ensuring compactness of the set of trajectories of differential inclusions are proved in [6], [8], etc.*

Throughout the paper, we shall use some notations and definitions we introduce now.

By $\mathcal{P}$ we shall denote the set of all function $p(t), t \in R$, of the following type: $p(t) = \sum_{i=1}^{k} p_i t^{q_i}$, where $1 \leq q_1 < q_2 < ... < q_k$, and $0 < p_i, i = 1, ...k$. Further we shall also use the notations $o(t), o_1(t), ...$, to indicate any family of vector functions $o(t) = o(t,.), ...$, defined on R^n (parameterized on t), continuous in (t,x) and such that for some $r > 0$ and $\omega > 1$ the ratio $o(t,x)/t^{\omega}$ is bounded uniformly with respect to $x \in S_r$. Let $\mathcal{V}_S^0$ be the set of all families of smooth vector functions $a(t) = a(t,.)$ defined on R^n (parameterized on t), continuous in (t,x) and such that for every element $a(t)$ from $\mathcal{V}_S^0$ there exist some positive reals r, θ and M such that $\|a(t,x)\| \leq M.t^{\theta}.\mathrm{d}_S(x)$ for all $x \in S_r$.

DEFINITION 1.2 *Let S be an arbitrary closed subset of R^n and A be a smooth function defined on R^n. It is said that A belongs to the set $\mathcal{S}_S^{\alpha}$ iff there exist positive real numbers r and T, an element $p \in \mathcal{P}$, two families of vector functions $o(t)$ and $a(t) \in \mathcal{V}_S^0$ such that for every point x from $S_r \setminus S$ and each $t \in [0,T]$*

$$x + t^{\alpha} A(x) + a(t,x) + o(t^{\alpha}, x) \in \mathcal{R}(x, p(t)). \tag{3}$$

REMARK 1.3 *A natural question is to know how to construct elements of the set $\mathcal{S}_S^{\alpha}$ when a control system is given in the form of differential equation, differential inclusion and etc. A possible answer is given in [18] where different particular cases of control systems are studied. For example, let us consider again the following control system:*

$$\dot{x}(t) = f_0(x(t)) + u(t) f_1(x(t)), \; u \in [-1, 1], \tag{4}$$

where the state variable x belongs to R^n and f_0 and f_1 are smooth real vector fields defined on some neighbourhood of the origin, and such that $f(0) = 0$ and $f_1(0) \neq 0$. Assuming that $f \in \mathcal{V}_S^0$, it is proved in [18] that there exists $\alpha > 0$, such that $f_1 \in \mathcal{S}_S^{\alpha}$ and $\pm \left(ad^k \; f_1, \; f_0 \right) \in \mathcal{S}_S^{\alpha}, k = 1, 2, ..., n$.

DEFINITION 1.3 *Let $r > 0$ and $T > 0$. It is said that $\mathcal{A}$ is a regular subset of the set $\mathcal{S}_S^\alpha$ on the set $S_r \times [0,T]$ provided that there exist positive constants $\omega > \alpha$, θ, L, M, N, C and P such that for every $A \in \mathcal{A}$ with corresponding $p(.)$, $a(t,x)$ and $o(t,x)$ (according to definition 2.1), the following relations hold true:*
i) $\|A(x) - A(y)\| \ \leq \ L\|x-y\|$ for all x, y from S_r;
ii) $\|A(x)\| \ \leq \ C$ for all x from S_r;
iii) $\|a(x,t))\| \ \leq \ M.t^\theta.d_S(x)$ for all x from S_r and all $t \in [0,T]$;
iv) $\|o(x,t^\alpha))\| \ \leq \ N.t^\omega$ for all x from S_r and all $t \in [0,T]$;
v) $|p(t)| \ \leq \ P.t$ for all $t \in [0,T]$;
vi) $x + t^\alpha A(x) + a(t,x) + o(t^\alpha, x) \ \in \ \ \mathcal{R}(x, p(t))$ for all x from S_r and all $t \in [0,T]$.

Now we can formulate the main result

THEOREM 1.2 *Let S be a closed subset of R^n, $r_0 > 0$, $T_0 > 0$, $\delta > 0$ and $\mathcal{A}$ be a regular subset of the set $\mathcal{S}_S^\alpha$ on $S_{r_0} \times [0, T_0]$. Let us assume that whenever y belongs to the boundary ∂S of the set S and $\xi \ \in \ N_S^p(y)$ there exists $A \in \mathcal{A}$ for which*

$$< \xi, A(y) > \leq \ -\delta.\|\xi\|. \tag{5}$$

Then the control system $\mathcal{R}$ is STLA with respect to S.

Let $\Theta(x)$ be the minimal time of steering the set S from the point x by means of a trajectory of the general system $\mathcal{R}$, i.e.

$$\Theta(x) \ := \ \inf\{t \geq 0, \text{ such that } z(0) = x, \ z(t) \in S \text{ for some trajectory } z(.) \text{ of } \mathcal{R}\}.$$

The map $\Theta(.)$ is called time optimal map of reaching the set S. Theorem 2.1 implies directly the following corollary:

COROLLARY 1.1 *Suppose that the assumptions of theorem 2.1. hold true. Then there exists a constant $C > 0$ such that*

$$\Theta(x) \leq C.d_S(x)^{1/\alpha} \tag{6}$$

for every x from some neighborhood U of S.

2 Numerical results.

The idea of the proof of the main result is to move towards the set S using suitable high-order control variations defined on suitable intervals of time. For an illustrative example, we consider the following three-dimensional control system

$$\begin{array}{lll} \dot{x}_1 = u, & x_1(0) = 0, & u \in [-1,1], \\ \dot{x}_2 = v, & x_2(0) = 0, & v \in [-1,1], \\ \dot{x}_3 = vx_1 - ux_2, & x_3(0) = 0, & \end{array}$$

which is small-time locally controllable at the origin but does not satisfy the Brockett necessary condition (cf. [2]), and hence can not be asymptotic stabilized by means of a continuous feedback law. Let $x = (x_1, x_2, x_3)^T$, $B(x) = (1, 0, -x_2)^T$ and $C(x) = (0, 1, x_1)^T$. Then the Lie bracket $[B, C](\cdot)$ of the vector fields $B(\cdot)$ and $C(\cdot)$ is

$$[B, C](x) := \partial C(x)B(x) - \partial B(x)C(x) = (0, 0, 2)^T,$$

where by $\partial B(\cdot)$ and $\partial C(\cdot)$ are denoted the Jacobians of $B(\cdot)$ and $C(\cdot)$, respectively.

The considered control system belongs to the class of the so called "symmetric" control systems (cf. [3]). So, it is natural to assume that all nonvanishing Lie brackets belong to the set of high-order control variations. Indeed, studying the set of admissible trajectories of this control system, one can find the following control variations of second order: Let $\beta = (\beta_1, \beta_2, \beta_3)^T$, where $\beta_i \in [-1, 1]$, $i = 1, 2, 3$. We define the following admissible control:

$$u_\beta(\tau) := \begin{cases} u = \beta_3, \ v = 0, & \text{for } \tau \in [0, t); \\ u = 0, \ v = 1, & \text{for } \tau \in [t, 2t) \\ u = -\beta_3, \ v = 0, & \text{for } \tau \in [2t, 3t); \\ u = 0, \ v = -1, & \text{for } \tau \in [3t, 4t) \\ u = \beta_1, \ v = \beta_2, & \text{for } \tau \in [4t, 4t + t^2); \end{cases}$$

It could be directly verified that the corresponding trajectory $z(x, u_\beta, \cdot)$ is defined on $[0, 4t + t^2]$ and

$$z(x, u_\beta, 4t + t^2) = x + t^2 \left(\beta_1 B(x) + \beta_2 C(x) + \beta_3 [B, C](x)\right).$$

Now we define the following stabilizing feedback $u_{\beta(x)}$ on $\Omega \times [0, 4t_x + t_x^2]$, where

$$\Omega := \left\{x = (x_1, x_2, x_3)^T : \ |x_1| \le 1, \ |x_2| \le 1, \ |x_3| \le 2\right\} \text{ and } t_x := \sqrt{\|x\|}.$$

Since $B(0)$, $C(0)$ and $[B, C](0)$ are linearly independent, every point $x \in \Omega$ can be present as follows

$$x := \alpha_1(x)B(0) + \alpha_2(x)C(0) + \alpha_3(x)[B, C](0), \text{ with } |\alpha_i(x)| \le 1, \ i = 1, 2, 3.$$

We set (similarly to [11])

$$\beta_1(x) := -\frac{\alpha_1(x)}{\|x\|}, \ \beta_2(x) := -\frac{\alpha_2(x)}{\|x\|}, \text{ and } \beta_3(x) := -\frac{|\alpha_3(x)|}{\|x\|}$$

and use the feedback control $u_{\beta(x)}$ on the interval $[0, 4t_x + t_x^2)$. The condition (5) holds true with $\mu = -1$. According to the main result, this control system is STLAS. Moreover, it could be verified directly that the corresponding trajectory $z(x, u_{\beta(x)}, \cdot)$ is defined on $[0, 4t_x + t_x^2]$ and $\bar{z} := z(x, u_{\beta(x)}, 4t_x + t_x^2) = 0$. Choosing different starting points x from Ω, we apply this feedback control. All computer simulations are performed using the computer algebra system Maple V. Some of the corresponding numerical results are presented in the next table:

x_1	x_2	x_3	t_x	$\bar{z}_1$	$\bar{z}_2$	$\bar{z}_3$
$1/3$	$-2/3$	$5/3$	$\sqrt[4]{\frac{10}{3}}$	0	0	-6.10^{-10}
$1/3$	$2/3$	$-5/3$	$\sqrt[4]{\frac{10}{3}}$	0	-3.10^{-10}	-5.10^{-10}
1	1	-2	$\sqrt[4]{6}$	-3.10^{-10}	0	16.10^{-10}
1	-1	2	$\sqrt[4]{6}$	0	0	0

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