On MIC(0) Preconditioning of Red-Black Ordered FEM Systems

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CONTENTS

1. Introduction

The Galerkin variational formulation of the problem reads:
given $f \in L^2(\Omega)$ **find** $u \in H^1_D(\Omega)$**, satisfying**

$$\mathcal{A}(u,v) \equiv \int_\Omega a(x)\nabla u(x) \cdot \nabla v(x) dx = (f,v) \qquad \forall v \in H^1_D(\Omega).$$

Linear FEs on triangles are implemented. Then the FEM formulation is: find $u_h \in V_h$**, satisfying**

$$\mathcal{A}_h(u_h,v_h) \equiv \int_\Omega \sum_{e\in\mathcal{T}} a(e)\nabla u_h \cdot \nabla v_h dx = (f,v_h) \qquad \forall v_h \in V_h,$$

$$a(e) = \frac{1}{|e|}\int_e a(x)dx.$$

The standard FEM procedure leads to the linear system of equations $A\mathbf{u} = \mathbf{f}$.

2. MIC(0) factorization

Let $A = D - L - L^t$, where D is the diagonal and $(-L)$ is the strictly lower triangular part. The MIC(0) preconditioner of A has the form

$$\mathcal{C}_{MIC(0)}(A) = (X - L)X^{-1}(X - L)^t$$

where $X = diag(x_1, \cdots, x_N)$ is determined by the condition $\mathcal{C}\underline{e} = A\underline{e}$.

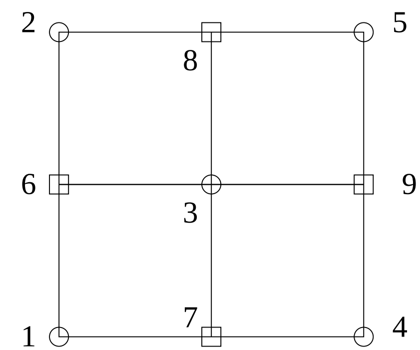
Theorem 1 ***Let us assume that: (a) $L \geq 0$, (b) $A\underline{e} \geq 0$, (c) $A\underline{e} + L^t\underline{e} > 0$. Then the relation***

$$x_i = a_{ii} - \sum_{k=1}^{i-1} \frac{a_{ik}}{x_k} \sum_{j=k+1}^{N} a_{kj}$$

gives the positive values x_i and the diagonal matrix $X = diag(x_1, \cdots, x_N)$ defines stable $MIC(0)$ factorization of A.

3. Red-black ordering

$$A = \frac{1}{2}\begin{bmatrix}
2 & & & & & -1 & -1 & & \\
 & 2 & & & & -1 & & -1 & \\
 & & 8 & & & -2 & -2 & -2 & -2 \\
 & & & 2 & & & -1 & & -1 \\
 & & & & 2 & & & -1 & -1 \\
-1 & -1 & -2 & & & 4 & & & \\
-1 & & -2 & -1 & & & 4 & & \\
 & -1 & -2 & & -1 & & & 4 & \\
 & & -2 & -1 & -1 & & & & 4
\end{bmatrix}$$



Red-black ordering

The idea of this ordering is to get a stiffness matrix in a block form where the diagonal blocks are diagonal matrices. This allows for efficient 1D parallel decomposition.

4. Numerical tests

The presented below numerical tests illustrate the PCG convergence rate of the studied MIC(0) preconditioner when the mesh-size parameter is varied. Here, $-\Delta u = [2/pi^2 y(1-y)]\sin \pi x$, $\Omega = (0,1)^2$, and the exact solution is $u(x,y) = y(1-y)\sin \pi x$.

The observation is that
$N_{it} = O(n) = O(N^{1/2})$.

PCG **iterations and FEM accuracy.**

n	N	N_{it}	$\|\|e_h\|\|_C$
2	25	4	6.77E-3
4	81	11	1.65E-3
8	289	22	4.12E-4
16	1029	36	1.03E-4
32	4225	60	2.57E-5
64	16641	103	6.39E-6
128	66049	185	1.57E-6
256	263169	350	3.70E-7

5. Concluding remarks

- In the case of red-black ordering, MIC(0) preconditioner is applicable only after a diagonal perturbation of the original stiffness matrix.
- The observed convergence of such a MIC(0) preconditioner is too slow.
- The potential of the related algorithm for parallel implementation is not strong enough for real-life large-scale applications.
- Red-black ordering could be applicable as a smoother in the sense of MG/ML preconditioning methods.
- The direct application of MIC(0) preconditioner has better chances in the case of skewed meshes or rotated quadrilateral FEM discretizations.