Branching Models Applications and Statistical Inferences

Maroussia Slavtchova-Bojkova
Sofia University and *Bulgarian Academy of Sciences*

1. Introduction

GOAL: to explain some phenomena arising in the biological treatment of wastewater by cells feeding on a substrate in a bio-reactor

- "One of the very points of branching processes theory is the establishment of the paradoxical possibility of frequent extinction in spite of general growth. Thus the extinct species population could well have been supercritical, but just suffered from bad luck."

- P. Haccou, P. Jagers and V. A. Vatutin (2005). **Branching Processes: Variation, Growth, and Extinction of Populations, Cambridge University Press, Cambridge**;

- How long does take the final establishment of bacterial cultures in wastewater laboratory experiments?

  As there is always a positive probability of extinction, it is possible to have several unsuccessful trials before the bacterial cultures start to grow irreversibly;

- What conclusions one can draw from an early extinction of a bacterial culture in different types of wastewater? Does it imply that the offspring mean in these environments is low?

- Similarly, our study might help decision-makers to take a choice based on comparative laboratory results in similar wastewater cultivated with different bacterial strains.

2. Background

- We present an age-dependent branching model with immigration and the main idea is to point out that the duality between super-critical and sub-critical branching processes given extinction can make decision-makers take the wrong decision.

  –in the **discrete-time case**, the problem concerning the total progeny was investigated in:

- P. Jagers (1975): **Branching Processes with Biological Applications.** London: John Wiley and Sons.

- T. Harris (1989): **The Theory of Branching Processes.** Dover Publications Inc., New York.

- S. Karlin, S. Tavaré (1982): Detecting particular genotypes in populations under nonrandom mating. Math. Biosci., v. 59, 57-75. – the asymptotic behavior of the probabilities of hitting the absorbing states, the times needed to hit these states, and the conditional distributions of the number of particles (for models allowing catastrophes).

- F. T. Bruss, and Slavtchova-Bojkova, M.(1999): On waiting times to populate an environment and a question of statistical inference. J. Appl. Probab., v. 36, 261-267. – the problem of inference from expected waiting times and expected progeny on fertility rates was treated for the first time.

  The simple case, in which all newly introduced populations are supposed to behave like independent identically distributed (i.i.d.) Bienaymé-Galton-

Watson branching processes was studied. As an example population experiments with trout were provided.

- Slavtchova-Bojkova, M.(2000): Computation of waiting time to successful experiment using age-dependent branching model. Ecological Modelling , v. 133, 125-130. – the above results were generalized in **continuous time**, i.e. when the newly introduced populations are supposed to be Bellman-Harris branching processes, and the results of inference remain valid. It is interesting to see how the duality is applied in a concrete application where it appears in a natural way.

3. Model Formulation

Consider a population branching process $(Z(t) : t \geq 0)$ having a state-space the non-negative integers with zero as an absorbing state.

$$Z_0 = 1, \quad T_0 = 0, \quad T = \inf\{t : Z(t) = 0\} \leq \infty$$

$\{(Z_t(n)) : n = 1, 2, \ldots\}$ be i.i.d. copies of $Z(t)$
$Z_0(n) = 1$, $T_n = \inf\{t : Z_t(n) = 0\}$, $H_0 = 0$ and

$$H_n = \sum_{0 \leq j \leq n} T_j, (n \geq 1).$$

$\{T_n\}$ are independent identically distributed random variables (i.i.d.r.v.).

Thus H_n is the time of the $n-$th extinction event, provided it is finite, and the convention $H_0 = 0$ implies the entire process begins with an extinction at $t = 0^-$.

Hence $N = \sup\{n : T_n = \infty\}$ is the number of extinction events, including the event at $t = 0^-$.

Then, the process we are interested in is

$$\widetilde{Z}_t = Z_t(n) \quad \text{if} \quad H_{n-1} \leq t < H_n \quad (n = 1, \ldots, N).$$

Life cycles

For the branching process $\{\widetilde{Z}_t\}_{t\geq 0}$ we shall call life cycles (periods) the intervals (H_{n-1}, H_n), $n = 1, 2, \ldots, N-1$ on which
$\inf_{H_{n-1}<t<H_n} \widetilde{Z}_t > 0$.

Thus $\{\widetilde{Z}_t\}$ may have several life periods, the last one always being infinite, provided the process is super-critical. If the process is sub-critical it will have a.s. infinitely many life periods.

Lifetime of the process $\{\widetilde{Z}_t\}_{t\geq 0}$ before escaping from extinction.

Lifetime of the process $\{\widetilde{Z}_t\}_{t\geq 0}$ before escaping from extinction will be defined as the last instant
$M =: H_{n-1}$ of immigration, i. e. the "birth time" of that process $Z_n(t)$, which finally survives or the last exit time from the state zero.

We shall also study the distribution and expectation of the lifetime M. Finally, we shall analyze the total progeny during the lifetime of the process $\{\widetilde{Z}_t\}_{t\geq 0}$ and shall obtain its expectation and variance.

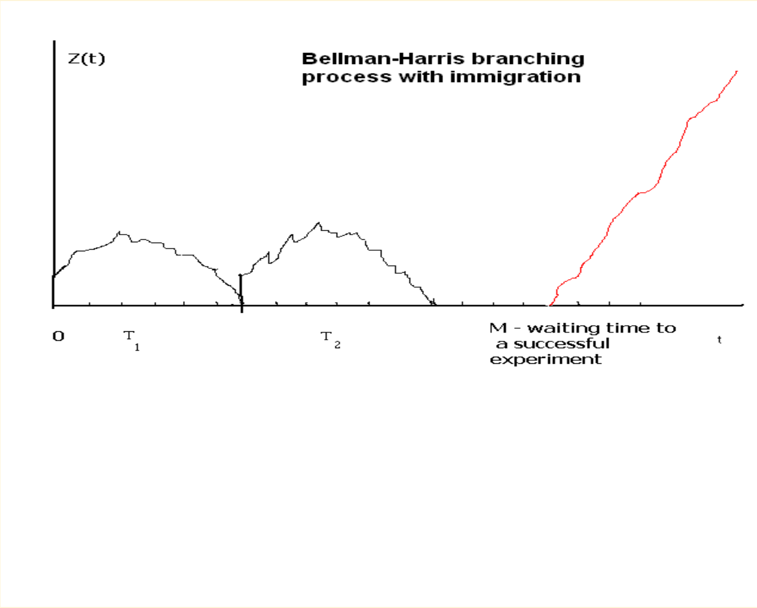
particle's life-length τ distribution $G(t) = P(\tau \leq t)$, $G(0^+) = 0$.

particle's reproduction distribution, (p_k), $\sum_{k=0}^{\infty} p_k = 1$.

p.g.f. of the number of ξ offspring is

$$f(s) = \sum_{k=0}^{\infty} p_k s^k, \quad |s| \leq 1, p_k = P(\xi = k).$$

where $p_k = P(\xi = k)$,



4. Theoretical Results

Suppose $q = P(T < \infty) < 1$, so that $\tau(\theta) = E(e^{-\theta T})$ satisfies $\tau(0) = q$.

Thus M = 0 if the initial population is immortal, i.e. $T_1 = \infty$.

Let
$$F(t) = P(T \leq t) = \int_0^t f(F(t-u))dG(u).$$

Theorem 1.

For the Laplace transform $\lambda(\theta) := E(e^{-\theta M})$ of the total waiting time until the initiation of that cycle whose path goes to infinity, it is held:

$$\lambda(\theta) := \frac{1-q}{1-\tau(\theta)}.$$

The total progeny of a life cycle

We are interested in the total progeny V of a life cycle and its expectation, in order to make some inferences on the fertility rates of the particles. So, let $g(s;t) = E(s^V; T \leq t)$.

Theorem 2.

The expected total progeny V of a life cycle satisfies the following equation:

$$\nu(t) := E(V; T \leq t) = F(t) + \int_0^t \nu(t-y) f'(F(t-y)) dG(y).$$

In the case $\tilde{m} \neq 1$ (i. e. non-critical cases) for the estimated total progeny of a life cycle it is held:

$$E(V_\infty; T < \infty) = \frac{1}{1 - f'(q)}.$$

5. Numerical Method

- M. E. Jacobson,(1985): Computation of Extinction Probabilities for the Bellman-Harris Branching Processes. Math. Biosci., v. 77, 173 - 177. – a numerical investigation of the rate of convergence of the extinction probability for a discrete-time age-dependent branching process was carried out.

  The most significant result illustrated by his simulations was the long tail of the distribution of life period for the case where the mean number of offspring was 1, as opposed to the quick convergence for the other runs.

- E. D. Powell,(1955): Some features of the generation times of individual bacteria. Biometrika, v. 42, 16-44. – found that the life-period of bacteria follows a gamma distribution, and reproduction at death is characteristic of bacteria-like organisms. That is why a discretized gamma density was used for all computations and simulations.

6. Preliminaries

- the probability q of eventual extinction of Bellman-Harris process $(\tilde{Z}(t))$ is the smallest non-negative root of the equation $f(s) = s$.

  $q = 1 \iff m = f'(1) \leq 1.$

- m is called the reproduction mean, and the super-critical, critical and sub-critical cases correspond to the relations $m > 1$, $m = 1$ and $m < 1$, respectively.

- l be the maximum number of offspring an individual can have,

- r be the greatest age an individual can live to,

- $g(.)$ be the mortality density.

$F(t)$ denotes the probability of extinction by time t, M is the last instant of immigration in the process $Z(t)$

If $t < r$, $F(t) = p_0 G(t) + \sum_{s=1}^{t-1} \sum_{k=1}^{l} p_k F^k(t-s) g(s)$,

If $t > r$, $F(t) = p_0 + \sum_{s=1}^{r} \sum_{k=1}^{l} p_k F^k(t-s) g(s)$

$v(t) = P(T \leq t)$.

$P(T \leq t | T < \infty) = \frac{v(t)}{q}$,

$E(T|T < \infty) = \frac{1}{q} \int_0^\infty (q - v(t)) dt$;

$E(M) = \frac{q}{(1-q)} E(T|T < \infty)$.

7. Numerical results

To study the implications of the above equations for extinction probabilities we use Mapple 6. Computations were then made with extinction probabilities up to 150 generations.

- we compute the conditional distribution of a life cycle T of an age-dependent branching process with immigration

- the conditional distribution of the total waiting time M when adopting as a probability density function (p.d.f.) for cell generation times the $\Gamma(\alpha, \beta)$ form for this distribution with density,

$$h(x) = \frac{e^{-x\beta}\beta^{\alpha}x^{\alpha-1}}{\Gamma(\alpha)}$$

  for $x > 0$,

- We consider two cases for the offspring distribution.

  First we suppose the offspring distribution to belong to the family of p.g.f. $h_p(s) = p + 0.4s + (0.6 - p)s^2$, parameterized by

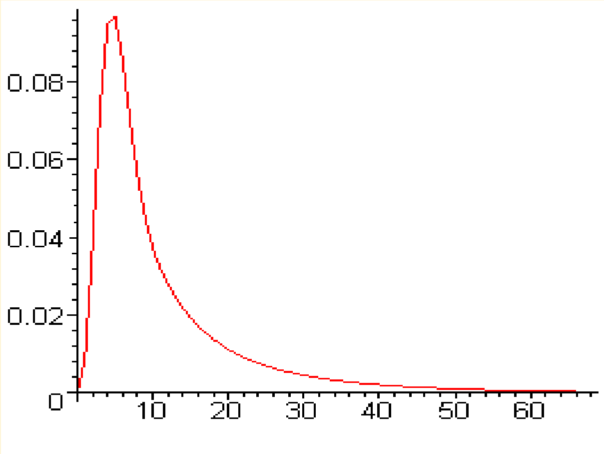
$$p = P\{\text{the initial progenitor dies without any offspring}\}.$$

  The computational results for $p = 0.05$ (which corresponds to a supercritical case with $m = 1.5$ and $q = 0.09$)

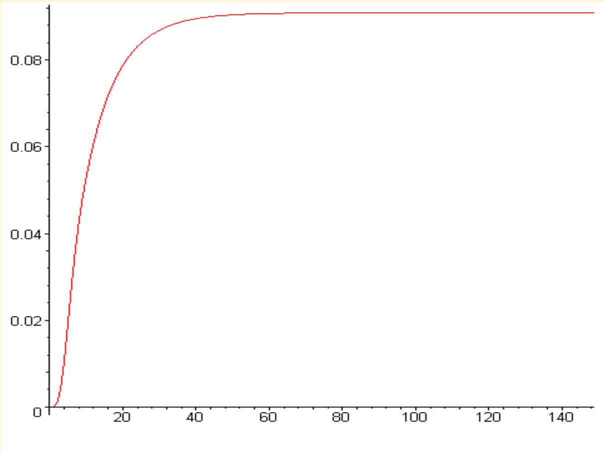
In addition, we obtained for the conditional expected value of T on the event of certain extinction, i.e. $E(T|T < \infty) = 11.92$ and for the unconditional $E(M) = 1.17$ in this case.

- Secondly, we implemented the computation when the offspring distribution is geometrical one with $p = 2/5$ (which corresponds to a supercritical case with $m = 1.5$ and $q = 0.66$).

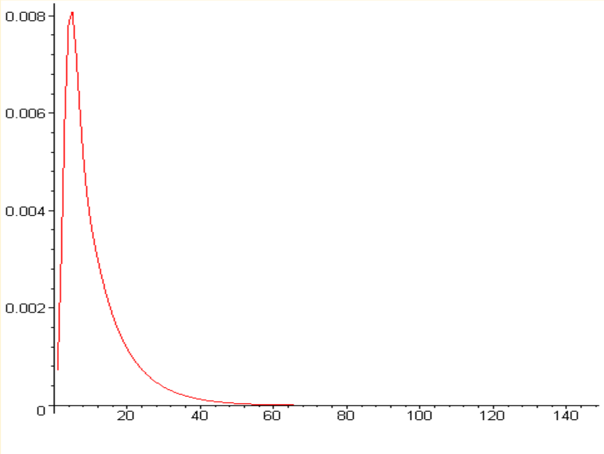
  We obtain in this case, that $E(T|T < \infty) = 12.46$ and $E(M) = 24.2$.



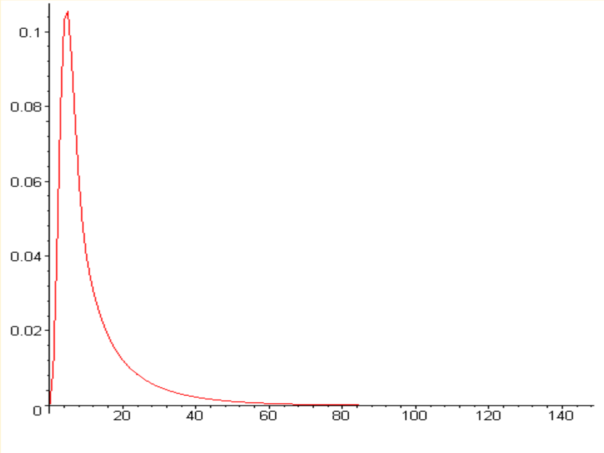
Conditional density function of life-period T, parametric reproduction



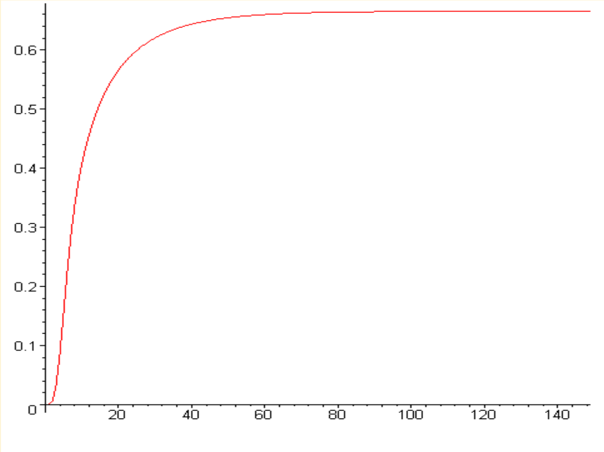
Probability of extinction by time t, parametric reproduction



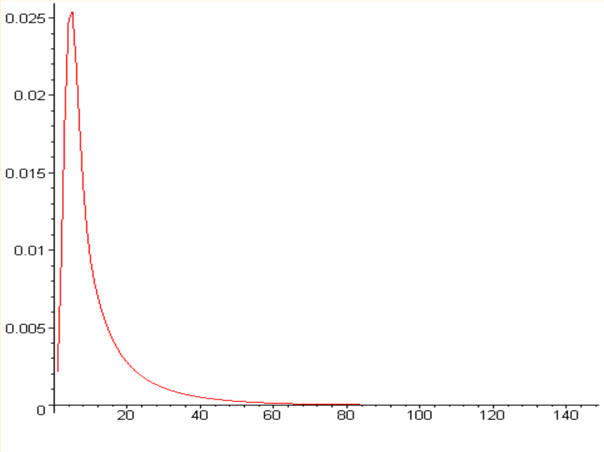
Conditional density function of the total progeny M, parametric reproduction



Conditional density function of life-period T, geometric reproduction



Probability of extinction by time t, geometric reproduction



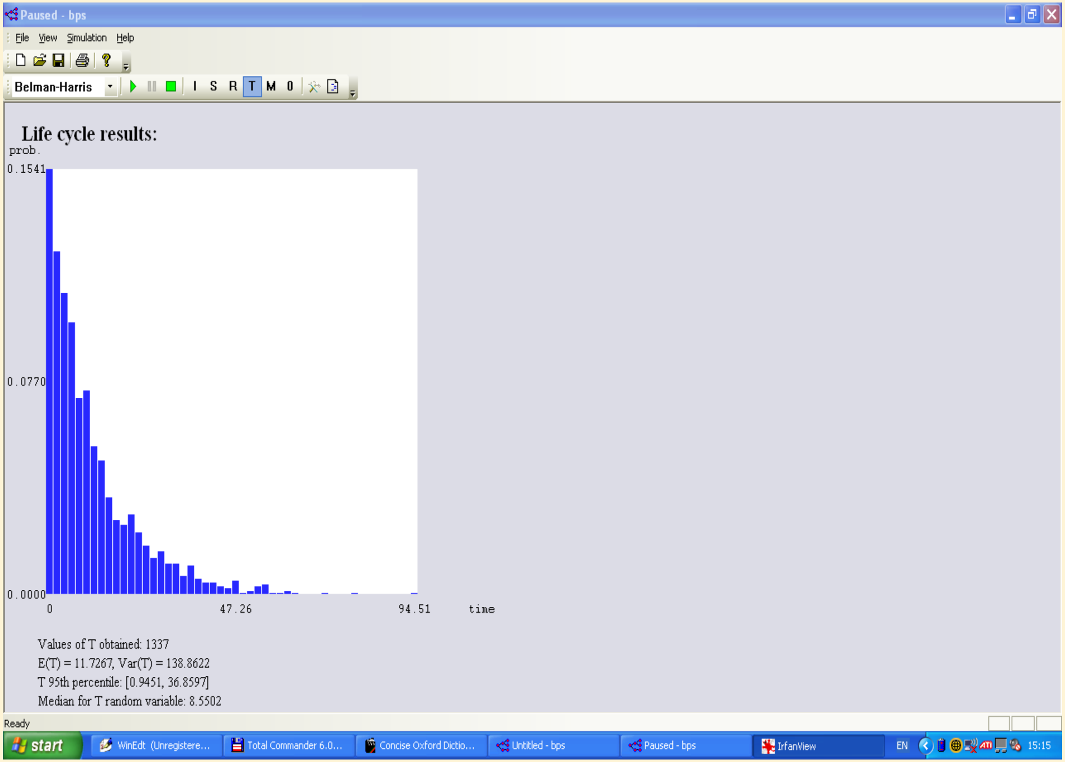
Conditional density function of the total progeny M, geometric reproduction

8. Simulation experiments

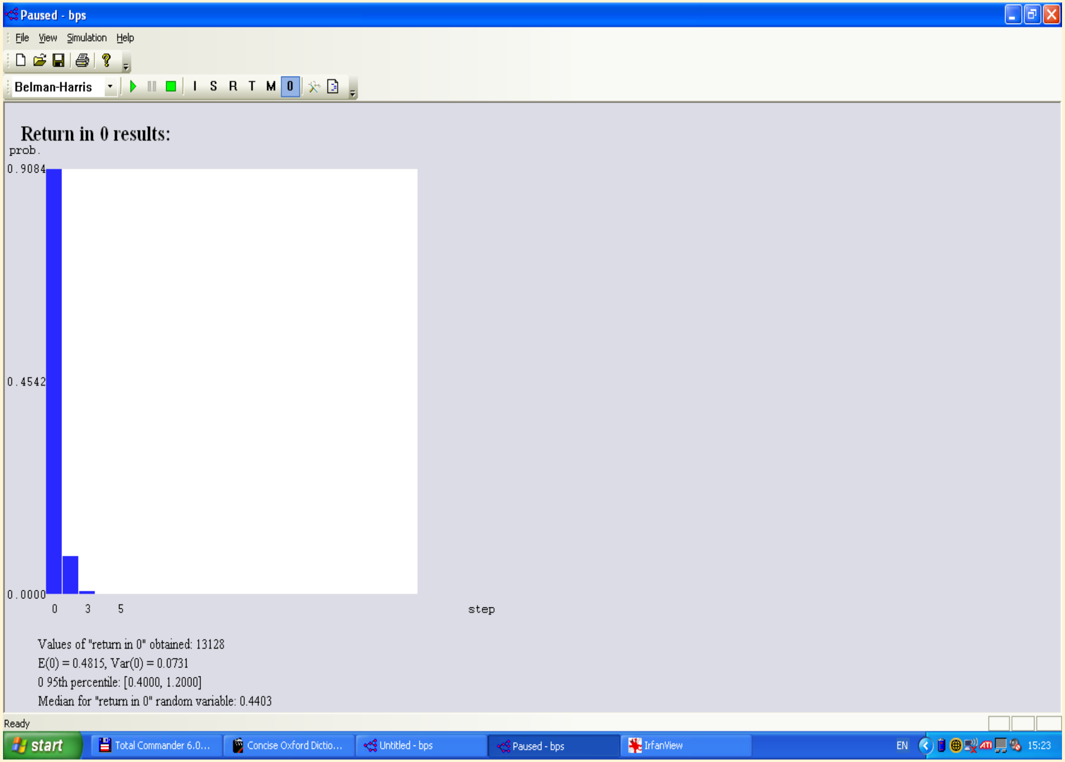
A code for simulation system for branching processes is developed. No programming is necessary and all input data can be entered in user-friendly dialog boxes and graphics and (numerical) results can be easily and quickly obtained.

The results can be stored in the Database table and may be analyzed easily. The code can be used for actual design, prediction and estimation of the parameters of different classes of branching processes, both in discrete and continuous time.

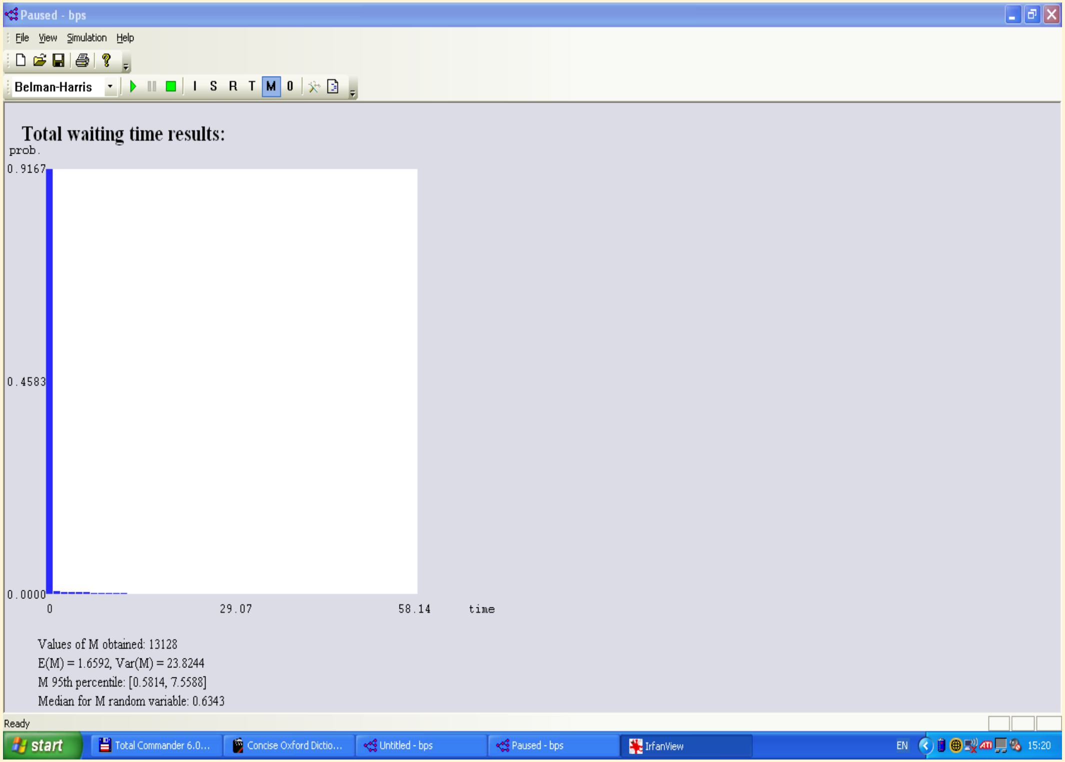
The SIMULATION SYSTEM is a simple professional tool that might be used by biologists, engineers and decision-makers for simulation of the processes which could appear to be suitable for modeling of some real world problems related to population and re-population experiments.



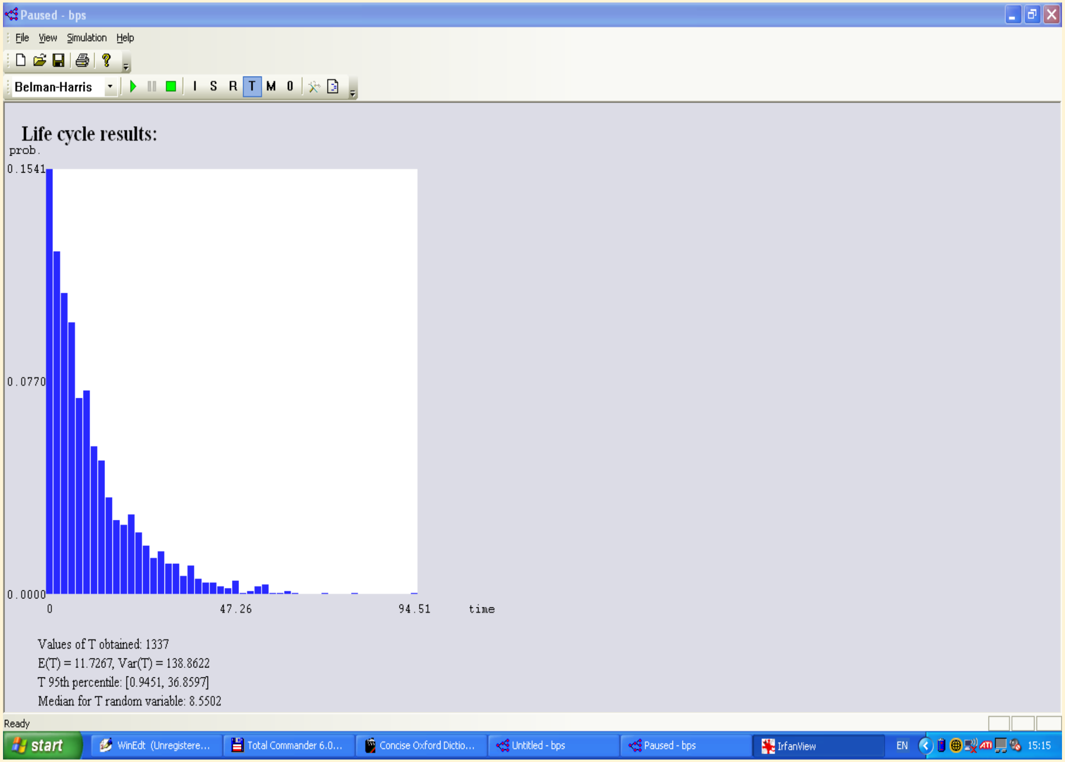
Conditional density function of life-period T, parametric reproduction



Conditional desity function of the return s in zero by t, parametric reproduction



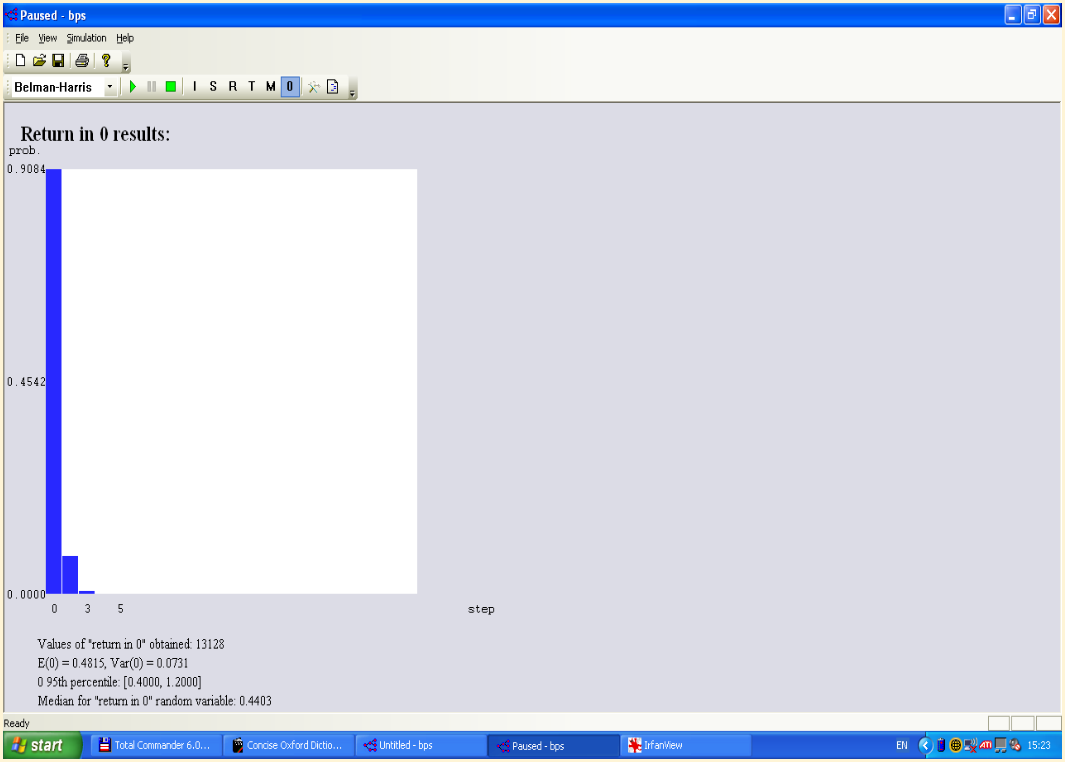
Conditional density function of the total progeny M, parametric reproduction



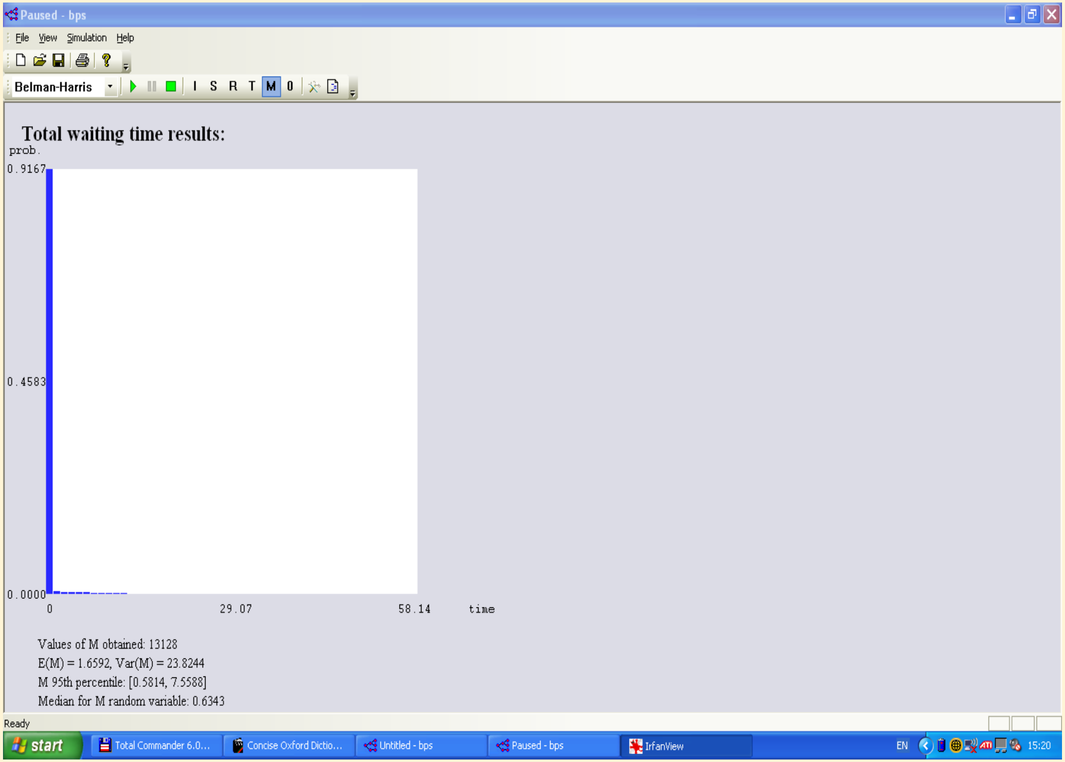
Conditional density function of life-period T, geometric reproduction

Home Page
Title Page
◀◀ ▶▶
◀ ▶

Go Back
Full Screen
Close
Quit



Conditional desity function of the return s in zero by t, geometric reproduction



Conditional density function of the total progeny M, geometric reproduction

9. Discussion

In population experiments it is usually easier to see if a new introduction has been successful than to know whether, and when, extinction has occurred. In many cases statistical data are only provided by interest groups, hunters, photographers, etc. Independent control studies to assess the prior probability of extinction are likely to be environment-biased. On the other hand, it is not always possible to reduce the prior probability of extinction by releasing a large numbers of animals. The point is that extinction involves a very strong bias. The discrete mass in the origin for the density function of M is the consequence of $P(M = 0) = 1 - q$.

THANK YOU VERY MUCH FOR YOUR KIND ATTENTION!