DESCRIPTION OF THE COMMUTANT OF COMPOSITIONS OF DUNKL OPERATORS

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Abstract. In this paper the commutant of a composition $\widetilde{D} = D_1 D_2 \dots D_n$ of Dunkl operators $D_j f(z) = \dfrac{df(z)}{dz} + k_j \dfrac{f(z) - f(-z)}{z}$ with parameters $k_j \geq 0,\ j = 1, 2, \dots, n$, is described using power series in the space A_R of the analytic functions in the disk $D_R = \{z \in \mathbb{C} : |z| < R\}$.

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1 Introduction

The Dunkl operator is a differential-difference operator defined in [3] in 1989 and since then many mathematicians have studied its properties and applications.

Let A_R be the space of the analytic functions in the disk $D_R = \{z \in \mathbb{C} : |z| < R\}$.

Definition 1. *For $f \in A_R$, the operator $D_k : A_R \to A_R$ defined by*

$$D_k f(z) = \frac{df(z)}{dz} + k \frac{f(z) - f(-z)}{z} \tag{1}$$

is called the Dunkl operator with parameter $k \geq 0$.

Definition 2. *It is said that a continuous linear operator M commutes with a fixed operator L, if $ML = LM$. The set of all operators commuting with L is called the commutant of L and will be denoted by* $\mathrm{Comm}(L)$.

M.S. Hristova describes in [5] the commutant $\mathrm{Comm}(D_k)$ of the Dunkl operator D_k and in [6] the commutant $\mathrm{Comm}(D_k^n)$ of arbitrary power n of D_k. Here our goal is to extend this description to the case of a composition $\widetilde{D} = D_1 D_2 \ldots D_n$ of Dunkl operators $D_j = D_{k_j}$ with arbitrary parameters $k_j \geq 0$, $j = 1, 2, \ldots, n$.

2 Representation of the commutant

Theorem 3. *Let f be an analytic function from A_R with a Taylor series $f(z) = \sum_{m=0}^{\infty} a_m z^m$. Then every continuous linear operator $M : A_R \to A_R$ commutes with the composition $\widetilde{D} = D_1 D_2 \ldots D_n$ of Dunkl operators $D_j = D_{k_j}$, $k_j \geq 0$, $j = 1, 2, \ldots, n$, i.e. $M \in \mathrm{Comm}(\widetilde{D})$, if and only if it can be represented in a power series form as*

$$\begin{aligned} Mf(z) &= \sum_{\mu=0}^{n-1} \sum_{m=0}^{\infty} a_m d_{m,\mu} z^{\mu} + \\ &+ \sum_{\mu=n}^{\infty} \sum_{m=[\frac{\mu}{n}]n}^{\infty} a_m \left(\prod_{\nu=1}^{[\frac{\mu}{n}]} \prod_{j=1}^{n} \frac{c_{j,m-\nu n+j}}{c_{j,\mu-\nu n+j}} \right) d_{m-[\frac{\mu}{n}]n,\mu-[\frac{\mu}{n}]n} \cdot z^{\mu}, \end{aligned} \tag{2}$$

where

$$c_{j,m} = m + k_j(1 - (-1)^m), \quad 1 \leq j \leq n, \ m \geq 0, \tag{3}$$

$d_{m,\mu}$, $0 \leq \mu \leq n-1$, $m = 0, 1, 2, \ldots$, are arbitrary complex numbers with the only restriction the series in the representation (2) *to be convergent, and $[A]$ denotes the integer part of the number A.*

Proof. First, let us consider the action of the Dunkl operator $D_j = D_{k_j}$ on a single power z^m of the variable $z \in \mathbb{C}$. If the power is even, i.e. $m = 2s$, then

$$D_j z^{2s} = \begin{cases} \dfrac{dz^{2s}}{dz} + k_j \dfrac{z^{2s} - (-z)^{2s}}{z} = 2s z^{2s-1} & \text{for } s \geq 1, \\ 0 & \text{for } s = 0. \end{cases}$$

If the power is odd, i.e. $m = 2s+1$, then

$$D_j z^{2s+1} = \frac{dz^{2s+1}}{dz} + k_j \frac{z^{2s+1} - (-z)^{2s+1}}{z} = (2s+1) z^{2s} + 2k_j z^{2s} = (2s+1+2k_j) z^{2s}.$$

The two representations can be combined in one formula:

$$D_j z^m = \begin{cases} c_{j,m} z^{m-1}, \ c_{j,m} = m + k_j[1 - (-1)^m] & \text{for } m \geq 1, \\ 0 & \text{for } m = 0. \end{cases} \tag{4}$$

Next, if the composition $\widetilde{D} = D_1 D_2 \ldots D_n$ is considered, its action on an arbitrary power m of the variable z can be expressed as

$$\widetilde{D} z^m = \begin{cases} c_{n,m} c_{n-1,m-1} \ldots c_{1,m-n+1} z^{m-n} = \left(\prod_{j=1}^{n} c_{j,m-n+j} \right) z^{m-n} & \text{for } m \geq n, \\ 0 & \text{for } 0 \leq m \leq n-1. \end{cases} \tag{5}$$

Now consider an arbitrary operator M from the commutant $\mathrm{Comm}(\widetilde{D})$. Let us represent its action again on an arbitrarily fixed power z^m by the power series

$$Mz^m = \sum_{\mu=0}^{\infty} d_{m,\mu} z^{\mu}, \quad m = 0, 1, 2, \dots \tag{6}$$

Here the coefficients $d_{m,\mu}$ are unknown, but they will be determined below.

In order to analyze the commutation $M\widetilde{D} = \widetilde{D}M$, we start by expressing $M\widetilde{D}z^m$ and $\widetilde{D}Mz^m$ for arbitrarily fixed power z^m:

$$M\widetilde{D}z^m = \begin{cases} Mc_{n,m}\dots c_{1,m-n+1}z^{m-n} = \sum_{\mu=0}^{\infty} c_{n,m}\dots c_{1,m-n+1}d_{m-n,\mu}z^{\mu} & \text{for } m \geq n, \\ 0 & \text{for } 0 \leq m \leq n-1. \end{cases} \tag{7}$$

$$\begin{aligned} \widetilde{D}Mz^m &= \widetilde{D}\sum_{\mu=0}^{\infty} d_{m,\mu}z^{\mu} = \sum_{\mu=0}^{\infty} d_{m,\mu}\widetilde{D}z^{\mu} \\ &= \sum_{\mu=n}^{\infty} d_{m,\mu}c_{n,\mu}\dots c_{1,\mu-n+1}z^{\mu-n} = \sum_{\mu=0}^{\infty} d_{m,\mu+n}c_{n,\mu+n}\dots c_{1,\mu+1}z^{\mu}. \end{aligned} \tag{8}$$

In the last formula $\mu - n$ was replaced by a single letter μ for convenience.

We want to have $M\widetilde{D}f(z) = \widetilde{D}Mf(z)$ for every $f \in A(R)$. By the uniqueness theorem for analytic functions this will be true if and only if for every $m \geq 0$ one has $M\widetilde{D}z^m = \widetilde{D}Mz^m$, i.e. if the expressions in (7) and (8) coincide.

Let us consider first the case $0 \leq m \leq n-1$. Then one must have

$$0 = \sum_{\mu=0}^{\infty} d_{m,\mu+n}c_{n,\mu+n}\dots c_{1,\mu+1}z^{\mu}.$$

By the uniqueness theorem the power series on the right is zero if and only if all its coefficients are equal to zero, i.e. $d_{m,\mu+n}c_{n,\mu+n}\dots c_{1,\mu+1} = 0$ for every $\mu = 0, 1, 2, \dots$. But all $c_{j,\mu+j}$, $1 \leq j \leq n$, are different from zero and hence it is necessary to have

$$d_{m,\mu+n} = 0, \quad 0 \leq m \leq n-1,\ \mu = 0, 1, 2, \dots.$$

This can be written in a better way if $\mu + n$ is replaced by a single index μ:

$$d_{m,\mu} = 0, \quad 0 \leq m \leq n-1,\ \mu \geq n. \tag{9}$$

Now a recurrent formula for arbitrary $m \geq n$ will be found.

Comparing the first line in (7) with (8), we get by the uniqueness theorem that

$$c_{n,m}\dots c_{1,m-n+1}d_{m-n,\mu} = d_{m,\mu+n}c_{n,\mu+n}\dots c_{1,\mu+1}, \quad m \geq n, \mu \geq 0.$$

Replacing μ by $\mu - n$ we have

$$c_{n,m}\dots c_{1,m-n+1}d_{m-n,\mu-n} = c_{n,\mu}\dots c_{1,\mu-n+1}d_{m,\mu}, \quad m \geq n, \mu \geq n. \tag{10}$$

But all constants $c_{j,\mu-n+j}$, $1 \leq j \leq n$, are different from zero and we obtain the desired recurrent formula

$$d_{m,\mu} = \frac{c_{n,m}\dots c_{1,m-n+1}}{c_{n,\mu}\dots c_{1,\mu-n+1}}\ d_{m-n,\mu-n} = \left(\prod_{j=1}^{n} \frac{c_{j,m-n+j}}{c_{j,\mu-n+j}}\right) d_{m-n,\mu-n},\ m \geq n, \mu \geq n. \tag{11}$$

Now this important recurrent formula (11) will be used for expressing any coefficient $d_{m,\mu}$ with $m \geq n$ and $\mu \geq n$ by a coefficient $d_{p,q}$, where either $0 \leq p \leq n-1$ or $0 \leq q \leq n-1$.

Let us remind that in the sequel $[A]$ will denote the integer part of a number A.

In the case $\left[\frac{m}{n}\right] < \left[\frac{\mu}{n}\right]$ one can apply $\left[\frac{m}{n}\right]$ times the recurrent formula (11) and then

$$\begin{aligned} d_{m,\mu} &= \left(\prod_{j=1}^{n} \frac{c_{j,m-n+j}}{c_{j,\mu-n+j}}\right) d_{m-n,\mu-n} = \left(\prod_{j=1}^{n} \frac{c_{j,m-n+j}}{c_{\mu-n+j}}\right)\left(\prod_{j=1}^{n} \frac{c_{j,m-2n+j}}{c_{\mu-2n+j}}\right) d_{m-2n,\mu-2n} \\ &= \ldots = \left(\prod_{\nu=1}^{\left[\frac{m}{n}\right]}\prod_{j=1}^{n} \frac{c_{j,m-\nu n+j}}{c_{j,\mu-\nu n+j}}\right) d_{m-\left[\frac{m}{n}\right]n,\mu-\left[\frac{m}{n}\right]n}. \end{aligned} \tag{12}$$

Here $m-\left[\frac{m}{n}\right]n \leq n-1$, i.e. the first index is the remainder when m is divided by n, and $\mu-\left[\frac{m}{n}\right]n \geq n$. Then by our first observation (9) the coefficient $d_{m-\left[\frac{m}{n}\right]n,\mu-\left[\frac{m}{n}\right]n}$ must be zero. Therefore (12) gives

$$d_{m,\mu} = 0, \quad \text{for } \left[\frac{m}{n}\right] < \left[\frac{\mu}{n}\right]. \tag{13}$$

In the other case, when $\left[\frac{m}{n}\right] \geq \left[\frac{\mu}{n}\right]$, one can apply $\left[\frac{\mu}{n}\right]$ times the recurrent formula (11) to get

$$\begin{aligned} d_{m,\mu} &= \left(\prod_{j=1}^{n} \frac{c_{j,m-n+j}}{c_{j,\mu-n+j}}\right) d_{m-n,\mu-n} = \left(\prod_{j=1}^{n} \frac{c_{j,m-n+j}}{c_{\mu-n+j}}\right)\left(\prod_{j=1}^{n} \frac{c_{j,m-2n+j}}{c_{\mu-2n+j}}\right) d_{m-2n,\mu-2n} \\ &= \ldots = \left(\prod_{\nu=1}^{\left[\frac{\mu}{n}\right]}\prod_{j=1}^{n} \frac{c_{j,m-\nu n+j}}{c_{j,\mu-\nu n+j}}\right) d_{m-\left[\frac{\mu}{n}\right]n,\mu-\left[\frac{\mu}{n}\right]n}. \end{aligned} \tag{14}$$

Now the second index $\mu-\left[\frac{\mu}{n}\right]n$ is the remainder when μ is divided by n.

Let us combine (13) and (14) as

$$d_{m,\mu} = \begin{cases} 0 & \text{for } \left[\frac{m}{n}\right] < \left[\frac{\mu}{n}\right], \\ \left(\prod_{\nu=1}^{\left[\frac{\mu}{n}\right]}\prod_{j=1}^{n} \frac{c_{j,m-\nu n+j}}{c_{j,\mu-\nu n+j}}\right) d_{m-\left[\frac{\mu}{n}\right]n,\mu-\left[\frac{\mu}{n}\right]n}. & \text{for } \left[\frac{m}{n}\right] \geq \left[\frac{\mu}{n}\right]. \end{cases} \tag{15}$$

This important formula shows that *all coefficients $d_{m,\mu}$ with $0 \leq \mu \leq n-1$ can be chosen arbitrarily*, and then *all other coefficients $d_{m,\mu}$ with $\mu \geq n$ are either equal to zero or can be expressed by some of the arbitrarily chosen $d_{\nu,\varkappa}$ with $0 \leq \varkappa \leq n-1$.*

The recurrent relation (15) allows a representation of Mz^m as a polynomial of degree at most $\left(\left[\frac{m}{n}\right]+1\right)n-1$:

$$Mz^m = \sum_{\mu=0}^{n-1} d_{m,\mu}z^\mu + \sum_{\mu=n}^{\left(\left[\frac{m}{n}\right]+1\right)n-1} \left(\prod_{\nu=1}^{\left[\frac{\mu}{n}\right]}\prod_{j=1}^{n} \frac{c_{j,m-\nu n+j}}{c_{j,\mu-\nu n+j}}\right) d_{m-\left[\frac{\mu}{n}\right]n,\mu-\left[\frac{\mu}{n}\right]n} \cdot z^\mu. \tag{16}$$

Finally, the action of an operator $M \in \mathrm{Comm}(\widetilde{D})$ on some analytic function $f(z) = \sum_{m=0}^{\infty} a_m z^m$ is

$$\begin{aligned} Mf(z) &= M\sum_{m=0}^{\infty} a_m z^m = \sum_{m=0}^{\infty} a_m M z^m \qquad (17)\\ &= \sum_{m=0}^{\infty} a_m \left(\sum_{\mu=0}^{n-1} d_{m,\mu} z^\mu + \sum_{\mu=n}^{([\frac{m}{n}]+1)n-1} \left(\prod_{\nu=1}^{[\frac{\mu}{n}]} \prod_{j=1}^{n} \frac{c_{j,m-\nu n+j}}{c_{j,\mu-\nu n+j}} \right) d_{m-[\frac{\mu}{n}]n,\mu-[\frac{\mu}{n}]n} \cdot z^\mu \right). \end{aligned}$$

It is natural to interchange the two sums in order to have a standard power series representation of (17):

$$\begin{aligned} Mf(z) &= \sum_{\mu=0}^{n-1} \sum_{m=0}^{\infty} a_m d_{m,\mu} z^\mu \qquad (18)\\ &+ \sum_{\mu=n}^{\infty} \sum_{m=[\frac{\mu}{n}]n}^{\infty} a_m \left(\prod_{\nu=1}^{[\frac{\mu}{n}]} \prod_{j=1}^{n} \frac{c_{j,m-\nu n+j}}{c_{j,\mu-\nu n+j}} \right) d_{m-[\frac{\mu}{n}]n,\mu-[\frac{\mu}{n}]n} \cdot z^\mu. \end{aligned}$$

This is in fact the desired representation (2) and thus, we proved the necessity, i.e. if $M \in \mathrm{Comm}(\widetilde{D})$, then the operator M must be of the form (2).

Now, let us check the *sufficiency*, i.e. if an operator M has the form (2), then it commutes with the composition $\widetilde{D} = D_1 D_2 \dots D_n$ of the Dunkl operators $D_j = D_{k_j}$, $j = 1, 2, \dots, n$, i.e. $M\widetilde{D} = \widetilde{D}M$. It is enough to verify this for all powers z^m, $m = 0, 1, 2, \dots$, since they form a basis of the space of the analytic functions A_R. In fact, for arbitrarily fixed m we can use the representation (16) instead of the general expression (2).

In the case $0 \le m \le n-1$ the representation (16) reduces to the first sum and $Mz^m = \sum_{m=0}^{n-1} d_{m,\mu} z^\mu$. Now we calculate $\widetilde{D}Mz^m$ and $M\widetilde{D}z^m$:

$$\widetilde{D}(Mz^m) = \widetilde{D} \sum_{m=0}^{n-1} d_{m,\mu} z^\mu = \sum_{m=0}^{n-1} d_{m,\mu} \widetilde{D} z^\mu = \sum_{m=0}^{n-1} d_{m,\mu}.0 = 0;$$

$$M(\widetilde{D}z^m) = M0 = 0,$$

i.e. $\widetilde{D}Mz^m = M\widetilde{D}z^m = 0$. Here we used the second case in (5).

In the case $m \ge n$ use the first line in (5) and next use (16) with z^{m-n} to represent $M\widetilde{D}z^m$:

$$\begin{aligned} M\widetilde{D}z^m &= M \left(\prod_{j=1}^{n} c_{j,m-n+j} \right) z^{m-n} = \left(\prod_{j=1}^{n} c_{j,m-n+j} \right) M z^{m-n} \qquad (19)\\ &= \left(\prod_{j=1}^{n} c_{j,m-n+j} \right) \left(\sum_{\mu=0}^{n-1} d_{m-n,\mu} z^\mu + \sum_{\mu=n}^{([\frac{m-n}{n}]+1)n-1} \frac{\prod_{\nu=1}^{[\frac{\mu}{n}]} \prod_{j=1}^{n} c_{j,m-n-\nu n+j}}{\prod_{\nu=1}^{[\frac{\mu}{n}]} \prod_{j=1}^{n} c_{j,\mu-\nu n+j}} d_{m-n-[\frac{\mu}{n}]n,\mu-[\frac{\mu}{n}]n} \cdot z^\mu \right). \end{aligned}$$

To represent the inverse commutation $\widetilde{D}Mz^m$, apply $\widetilde{D}$ to (16):

$$\widetilde{D}Mz^m = \sum_{\mu=0}^{n-1} d_{m,\mu} \widetilde{D} z^\mu + \sum_{\mu=n}^{([\frac{m}{n}]+1)n-1} \frac{\prod_{\nu=1}^{[\frac{\mu}{n}]} \prod_{j=1}^{n} c_{j,m-\nu n+j}}{\prod_{\nu=1}^{[\frac{\mu}{n}]} \prod_{j=1}^{n} c_{j,\mu-\nu n+j}} d_{m-[\frac{\mu}{n}]n,\mu-[\frac{\mu}{n}]n} \cdot \widetilde{D} z^\mu. \qquad (20)$$

The first sum will vanish because the second case in (5) gives $\widetilde{D}z^\mu = 0$ for $0 \le \mu \le n-1$.
Now use (5) for $\mu \ge n$:

$$\widetilde{D}Mz^m = \sum_{\mu=n}^{([\frac{m}{n}]+1)n-1} \frac{\prod_{\nu=1}^{[\frac{\mu}{n}]}\prod_{j=1}^{n} c_{j,m-\nu n+j}}{\prod_{\nu=1}^{[\frac{\mu}{n}]}\prod_{j=1}^{n} c_{j,\mu-\nu n+j}} d_{m-[\frac{\mu}{n}]n,\mu-[\frac{\mu}{n}]n} \left(\prod_{j=1}^{n} c_{j,\mu-n+j}\right) z^{\mu-n}. \tag{21}$$

It is suitable to separate the sum as $\sum_{\mu=n}^{2n-1} + \sum_{\mu=2n}^{([\frac{m}{n}]+1)n-1}$. In the first sum the whole denominator will be canceled with the product in brackets since $\left[\frac{\mu}{n}\right] = 1$, but in the second sum, after canceling $\prod_{j=1}^{n} c_{j,\mu-n+j}$, the denominator will have n factors less than the numerator (without $\nu = 1$):

$$\begin{aligned} \widetilde{D}Mz^m &= \sum_{\mu=n}^{2n-1} \left(\prod_{j=1}^{n} c_{j,m-n+j}\right) d_{m-n,\mu-n}\ z^{\mu-n} \\ &+ \sum_{\mu=2n}^{([\frac{m}{n}]+1)n-1} \frac{\left(\prod_{j=1}^{n} c_{j,m-n+j}\right)\left(\prod_{\nu=2}^{[\frac{\mu}{n}]}\prod_{j=1}^{n} c_{j,m-\nu n+j}\right)}{\prod_{\nu=2}^{[\frac{\mu}{n}]}\prod_{j=1}^{n} c_{j,\mu-\nu n+j}} d_{m-[\frac{\mu}{n}]n,\mu-[\frac{\mu}{n}]n}\ z^{\mu-n}. \end{aligned} \tag{22}$$

It remains to replace μ by $\mu+n$ and ν by $\nu+1$:

$$\begin{aligned} \widetilde{D}Mz^m &= \sum_{\mu=0}^{n-1} \left(\prod_{j=1}^{n} c_{j,m-n+j}\right) d_{m-n,\mu+n-n}\ z^{\mu} \\ &+ \sum_{\mu=n}^{([\frac{m}{n}]+1)n-1-n} \frac{\left(\prod_{j=1}^{n} c_{j,m-n+j}\right)\left(\prod_{\nu=1}^{[\frac{\mu+n}{n}]-1}\prod_{j=1}^{n} c_{j,m-(\nu+1)n+j}\right)}{\prod_{\nu=1}^{[\frac{\mu+n}{n}]-1}\prod_{j=1}^{n} c_{j,\mu+n-(\nu+1)n+j}} d_{m-[\frac{\mu+n}{n}]n,\mu+n-[\frac{\mu+n}{n}]n}\ z^{\mu}. \end{aligned} \tag{23}$$

After the obvious simplifications this representation of $\widetilde{D}Mz^m$ coincides with the representation (19) of $M\widetilde{D}z^m$ which proves the sufficiency of (2) and thus the theorem. ∎

3 Particular cases

Example 1. Let us note that as a simplest particular case of the Dunkl operator, when all parameters k_j, $j = 1, 2, \dots, n$, of the Dunkl operators $D_j = D_{k_j}$ are taken to be 0, one can have the n-th power D^n of the classical differentiation operator $D_0 f(z) = Df(z) = \frac{df(z)}{dz}$. Then $c_{j,m} = m$, $j = 1, 2, \dots, n$, and Theorem 3 describes the commutant of D^n as:

$$Mf(z) = \sum_{\mu=0}^{n-1}\sum_{m=0}^{\infty} a_m d_{m,\mu} z^\mu + \sum_{\mu=n}^{\infty}\sum_{m=[\frac{\mu}{n}]n}^{\infty} a_m \frac{m \dots (m-[\frac{\mu}{n}]n+1)}{\mu \dots (\mu-[\frac{\mu}{n}]n+1)} d_{m-[\frac{\mu}{n}]n,\mu-[\frac{\mu}{n}]n} \cdot z^\mu. \tag{24}$$

Similar results for D, its powers, and generalizations of D are given by some Russian mathematicians. In particular, in [4] (§5.1) one can find such theorem and also additional bibliography.

Example 2. If we take not a composition, but a single Dunkl operator with parameter $k > 0$, i.e. $n = 1$, then the representation of $\mathrm{Comm}(D_k)$ given by M.S. Hristova from [5] is obtained:

$$Mf(z) = \sum_{m=0}^{\infty} a_m d_m z^\mu + \sum_{\mu=1}^{\infty}\sum_{m=\mu}^{\infty} a_m \frac{c_m \cdots c_{m-\mu+1}}{c_\mu \dots c_1} d_{m-\mu}\ z^\mu. \tag{25}$$

Example 3. If $n \geq 1$ is arbitrary, but all parameters of the Dunkl operators $D_j = D_{k_j}$ in the composition $\widetilde{D} = D_1 D_2 \dots D_n$ are equal, i.e. $k_1 = k_2 = \dots = k_n = k > 0$, then our result reduces to the representation due to M.S. Hristova in [6]:

$$Mf(z) = \sum_{\mu=0}^{n-1} \sum_{m=0}^{\infty} a_m d_{m,\mu} z^\mu + \sum_{\mu=n}^{\infty} \sum_{m=\left[\frac{\mu}{n}\right]n}^{\infty} a_m \frac{c_m \dots c_{m-\left[\frac{\mu}{n}\right]n+1}}{c_\mu \dots c_{\mu-\left[\frac{\mu}{n}\right]n+1}} d_{m-\left[\frac{\mu}{n}\right]n, \mu-\left[\frac{\mu}{n}\right]n} \cdot z^\mu. \tag{26}$$

Final notes. A different description of the commutant $\mathrm{Comm}(D_k)$ of the first power of the Dunkl operator in the space of the continuous functions on the real line $\mathbb{R}$ is given in [2], based on the convolutional approach (see Dimovski [1]). It depends on an arbitrary continuous linear functional $\Phi : C(\mathbb{R}) \to \mathbb{C}$. Note, that Theorem 3 also allows in the case of composition of n Dunkl operators to choose arbitrarily n systems of constants $d_{m,\mu}$, $0 \leq \mu \leq n-1$, $m = 0, 1, 2, \dots$.

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