Mean-periodic Solutions of Euler Differential Equations

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Abstract

Let $\Phi : C(\mathbb{R}_+) \to \mathbb{C}$ be a given nonzero linear functional. We are looking for mean-periodic solutions for the Euler operator $\delta = t\dfrac{d}{dt}$ and the functional Φ of equations of the form $P(\delta)y(t) = f(t)$ with a polynomial P. A function f is called mean-periodic with respect to Φ iff $\Phi_\tau \{f(t\tau\} = 0$. A necessary condition for existence of such a solution is the requirement the right hand side f to be mean-periodic. Then, the problem is equivalent to the following nonlocal Cauchy problem: $P(\delta)y(t) = f(t)$, $\Phi\left\{\delta^k y\right\} = 0$, $k = 0, 1, \ldots, \deg P - 1$. The solution of the last problem has the following Duhamel-type form $y = \delta(G * f)$, where G is the solution of the nonlocal Cauchy problem for $f(t) \equiv 1$ and $*$ denotes the convolution product in $C(\mathbb{R}_+)$

$$(f * g)(t) = \Phi_\tau \left\{ \int_\tau^t f\left(\frac{t\tau}{\sigma}\right) g(\sigma) \frac{d\sigma}{\sigma} \right\}.$$

1. Mean-periodic functions for the Euler operator with respect to a functional

As it is well-known, the notion of mean-periodic function for the differentiation operator $\dfrac{d}{dt}$ with respect to a linear functional $\Phi : C(\mathbb{R}) \to \mathbb{C}$ is introduced in 1935 by J. Delsarte [1]. An extensive study of it is proposed in 1947 in the L. Schwartz' memoir [4].

Let us remind this basic definition.

Definition 1 *A function $f \in C(\mathbb{R})$ is said to be mean-periodic with respect to a linear functional $\Phi : C(\mathbb{R}) \to \mathbb{C}$ iff $\Phi_\tau\{f(t + \tau)\} = 0$ identically.*

In Dimovski and Skórnik [3] an operational method is considered for solving linear ordinary differential equations with constant coefficients in mean-periodic functions with respect to a given functional Φ.

Here we extend this approach to Euler equations.

Definition 2 *A function $f \in C(\mathbb{R}_+) \to \mathbb{C}$, where $\mathbb{R}_+ = (0, \infty)$, is said to be mean-periodic for the Euler operator $\delta = t\dfrac{d}{dt}$ with respect to a linear functional $\Phi : C(\mathbb{R}_+) \to \mathbb{C}$ iff $\Phi_\tau\{f(t\tau)\} = 0$ identically.*

In the sequel a basic role is played by the following convolution product introduced in Dimovski and Skórnik [3] and Dimovski and Hristov [2]:

Theorem 1 ([3]). *The operation*

$$(f * g)(t) = \Phi_\tau \left\{ \int_\tau^t f\left(\frac{t\tau}{\sigma}\right) g(\sigma) \frac{d\sigma}{\sigma} \right\} \tag{1}$$

converts $C(\mathbb{R}_+)$ into a commutative and associative algebra.

For the sake of completeness we supply a sketch of the proof. The commutativity is almost obvious. Let us verify the associativity. It is possible to do this by a direct check, but an easier way is to verify it at first for polynomials and then to use approximation argument.

Let $f(t) = t^\mu$ and $g(t) = t^\nu$. Then

$$\begin{aligned} \{t^\mu\} * \{t^\nu\} &= \Phi_\tau \left\{ \int_\tau^t \frac{(t\tau)^\mu}{\sigma^\mu} \sigma^\nu \frac{d\sigma}{\sigma} \right\} = t^\mu \Phi_\tau \left\{ \tau^\mu \int_\tau^t \sigma^{\nu-\mu-1} d\sigma \right\} = \\ &= t^\mu \Phi_\tau \left\{ \tau^\mu \frac{t^{\nu-\mu} - \tau^{\nu-\mu}}{\nu - \mu} \right\} = \frac{E(\mu)t^\nu - E(\nu)t^\mu}{\nu - \mu} \end{aligned}$$

with $E(\lambda) = \Phi_\tau\{\tau^\lambda\}$. Using this expression, it follows that

$$(\{t^\mu\} * \{t^\nu\}) * \{t^\varkappa\} = \{t^\mu\} * (\{t^\nu\} * \{t^\varkappa\}) \tag{2}$$

since both sides of (2) have one and the same symmetric form

$$t^\mu \frac{E(\nu)E(\varkappa)}{(\mu-\nu)(\mu-\varkappa)} + t^\nu \frac{E(\varkappa)E(\mu)}{(\nu-\varkappa)(\nu-\mu)} + t^\varkappa \frac{E(\mu)E(\nu)}{(\varkappa-\mu)(\varkappa-\nu)}.$$

with respect to μ, ν, and $\varkappa$. Then, (2), differentiated m, n, and k times with respect to μ, ν, and $\varkappa$ correspondingly, gives

$$(\{t^\mu(\ln t)^m\} * \{t^\nu(\ln t)^n\}) * \{t^\varkappa(\ln t)^k\} = \{t^\mu(\ln t)^m\} * (\{t^\nu(\ln t)^n\} * \{t^\varkappa(\ln t)^k\}).$$

Next, passing to the limits $\mu \to +0, \nu \to +0$ and $\varkappa \to +0$, one gets

$$(\{(\ln t)^m\} * \{(\ln t)^n\}) * \{(\ln t)^k\} = \{(\ln t)^m\} * (\{(\ln t)^n\} * \{(\ln t)^k\}).$$

But the bilinearity of (1) implies for arbitrary polynomials P, Q and R

$$(\{P(\ln t)\} * \{Q(\ln t)\}) * \{R(\ln t)\} = \{P(\ln t)\} * (\{Q(\ln t)\} * \{R(\ln t)\}).$$

To finish the proof, note that if $t \in \mathbb{R}_+$, then $\ln t$ covers the whole real line $\mathbb{R}$. Then Weierstrass' theorem allows any function in $C(\mathbb{R}_+)$ to be approximated almost uniformly by polynomials of $\ln t, t > 0$, i.e. by a sequence uniformly convergent to the function on each segment $[a, b] \subset \mathbb{R}_+$. Due to the continuity of the functional Φ the desired identity $(f * g) * h = f * (g * h)$ holds for every $f, g, h \in C(\mathbb{R}_+)$.

Further, we restrict the functional Φ by $\Phi\{1\} \neq 0$ and, without essential loss of generality, we may assume $\Phi\{1\} = 1$.

Let $L : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ be the right inverse operator of $\delta = t\dfrac{d}{dt}$, defined by the boundary value condition $\Phi\{Lf\} = 0$. It is easy to find $Lf(t)$ explicitly:

$$Lf(t) = \int_1^t \frac{f(\tau)}{\tau} d\tau - \Phi_\sigma \left\{ \int_1^\sigma \frac{f(\tau)}{\tau} d\tau \right\}. \qquad (3)$$

Moreover, Lf has the convolution representation

$$Lf = \{1\} * f$$

and $L^n f = \{Q_n(\ln t)\} * f$, where Q_n is a polynomial of degree exactly n.

Let MP_Φ^δ denote the space of the mean-periodic functions for δ with respect to Φ.

Lemma 1 *If $f \in MP_\Phi^\delta$, then $Lf \in MP_\Phi^\delta$.*

Proof: Let $f \in MP_\Phi^\delta$, i.e. $\Phi_\tau\{f(t\tau)\} = 0$. Consider the function $\varphi(t) = \Phi_\eta\{(Lf)(t\eta)\}$. Then

$$\delta\varphi(t) = \delta\Phi_\eta\{(Lf)(t\eta)\} = \Phi_\eta\{\delta(Lf)(t\eta)\} = \Phi_\eta\{(\delta L)f(t\eta)\} = \Phi_\eta\{f(t\eta)\} = 0,$$

since f is mean-periodic. Then $t\dfrac{d\varphi(t)}{dt} = 0$ and $t > 0$ imply $\varphi(t) \equiv C$, where C is a constant. In order to determine C, let us take $t = 1$. Then

$$\varphi(1) = \Phi_\eta\{Lf(\eta)\} = \Phi_\eta \left\{ \int_1^\eta \frac{f(\tau)}{\tau} d\tau - \Phi_\sigma \left\{ \int_1^\sigma \frac{f(\tau)}{\tau} d\tau \right\} \right\} =$$

$$= \Phi_\eta \left\{ \int_1^\eta \frac{f(\tau)}{\tau} d\tau \right\} - \Phi_\sigma \left\{ \int_1^\sigma \frac{f(\tau)}{\tau} d\tau \right\} \Phi\{1\} = 0,$$

which means $Lf \in MP_\Phi^\delta$.

Corollary 1 *Let $P(\lambda)$ be a polynomial. If $f \in MP^{\delta}_{\Phi}$, then*

$$\{P(\ln t)\} * f \in MP^{\delta}_{\Phi}.$$

Indeed, if $P(\lambda) = \sum_{k=0}^{\deg P} \beta_k \lambda^k$, then λ_k can be expressed as linear combination $\lambda^k = \sum_{j=0}^{k} \gamma_j Q_j(\lambda)$, where $Q_j(\lambda)$ are the polynomials from the proof of Theorem 1. Hence

$$\{P(\ln t)\}* = \left\{ \sum_{k=0}^{\deg P} \nu_k Q_k(\ln t) \right\} * = \sum_{k=0}^{\deg P} \nu_k L^k$$

with some constants ν_k. Then the lemma implies $\{P(\ln t)\} * f \in MP^{\delta}_{\Phi}$ provided $f \in MP^{\delta}_{\Phi}$.

Theorem 2 *MP^{δ}_{Φ} is an ideal in $(C(\mathbb{R}_+), *)$.*

Proof: Let $f \in MP^{\delta}_{\Phi}$ and $g \in C(\mathbb{R}_+)$. If $P(\lambda)$ is an arbitrary polynomial, it follows from Lemma 1 that $P(\ln t) * f \in MP^{\delta}_{\Phi}$. According to Weierstrass' approximation theorem, we can find a polynomial sequence $\{P_n\}_{n=1}^{\infty}$, for which $P_n(x) \rightrightarrows g(e^x)$ on each segment $[a, b] \subset \mathbb{R} = (-\infty, \infty)$. Then $P_n(\ln t) \rightrightarrows g(t)$ on each segment $[\alpha, \beta] \subset \mathbb{R}_+ = (0, \infty)$. But from Corollary 1

$$P_n(\ln t) * f \in MP^{\delta}_{\Phi}, \quad \forall n \in \mathbb{N}.$$

Since the space MP^{δ}_{Φ} is closed with respect to the uniform convergence, the limit $g * f = f * g$ is mean-periodic, too.

2. Nonlocal Cauchy problems for Euler equations

Let $\Phi : C(\mathbb{R}_+) \to \mathbb{C}$ be a linear functional. According to Riesz-Markov theorem Φ has a representation of the form $\Phi\{f\} = \int_{\alpha}^{\beta} f(t) d\gamma(t)$, where $0 < \alpha < \beta < +\infty$ and γ is a function with bounded variation.

Definition 3 *$P(\lambda)$ be a polynomial with $\deg P \geq 1$. The boundary value problem*

$$P(\delta)y = f, \tag{4}$$

$$\Phi\{\delta^k y\} = \alpha_k, \quad k = 0, 1, 2, \ldots, \deg P - 1 \tag{5}$$

with given $\alpha_k \in \mathbb{C}$ is said to be a nonlocal Cauchy problem for the Euler equation (4).

In [3] an operational method for solution of such nonlocal boundary value problems is developed. Here we reproduce the basic elements of this approach.

First, a Mikusiński-type operational calculus for the right inverse operator L, defined by (3), is developed. Without any loss of generality we may assume that $\Phi\{1\} = 1$. Then (3) becomes

$$Lf(t) = \int_1^t \frac{f(\tau)}{\tau} d\tau - \Phi_\sigma \left\{ \int_1^\sigma \frac{f(\tau)}{\tau} d\tau \right\}. \tag{6}$$

In fact, L is the convolution operator $L = \{1\}*$, i.e. $Lf = \{1\} * f$, in the convolution algebra $(C(\mathbb{R}_+), *)$ with the multiplication (1).

Let $\mathfrak{M}$ be the ring of convolution fractions of the form $\dfrac{f}{g}$ where $f \in C(\mathbb{R}_+)$ and $g \in C(\mathbb{R}_+)$ but g being a non-divisor of zero in $(C(\mathbb{R}_+), *)$.

Then the operator L can be identified with the constant function $\{1\}$, i.e. $L = \{1\}$. By 1 we will denote the unit element of $\mathfrak{M}$ and hence $1 \neq \{1\}$. The basic element of the operational calculus we are to develop, is played by the element $S = \dfrac{1}{L}$ which may be called the *algebraic Euler operator*.

Lemma 2 *If $f \in C^1(\mathbb{R}_+)$, then*

$$\delta f = Sf - \Phi\{f\}, \tag{7}$$

where Sf is the product $S.f$ in $\mathfrak{M}$ and $\Phi\{f\}$ is to be understood as a "numerical" operator, i.e. as the convolution fraction $\dfrac{\{\Phi\{f\}\}}{\{1\}}$.

Proof: By an immediate check it is seen that

$$L(\delta f)(t) = f(t) - f(1) - \Phi_\tau\{f(\tau) - f(1)\} = f(t) - \Phi\{f\}.$$

This identity can be written as

$$L(\delta f) = f - \Phi\{f\}.L.$$

Applying δ to both sides, we obtain (7).

Corollary 2 *For arbitrary $k \in \mathbb{N}$ and $f \in C^k(\mathbb{R}_+)$ we have*

$$\delta^k f = S^k f - \Phi\{f\}S^{k-1} - \Phi\{\delta f\}S^{k-2} - \ldots - \Phi\{\delta^{k-1} f\}. \tag{8}$$

The proof proceeds by induction using (7).

Theorem 3 *Let $\lambda \in \mathbb{C}$ be such that $E\{\lambda\} = \Phi_\tau\{\tau^\lambda\} \neq 0$. Then*

$$\frac{1}{S-\lambda} = \left\{\frac{t^\lambda}{E(\lambda)}\right\} \tag{9}$$

and

$$\frac{1}{(S-\lambda)^k} = \left\{\frac{1}{(k-1)!}\frac{d^{k-1}}{d\lambda^{k-1}}\left(\frac{t^\lambda}{E(\lambda)}\right)\right\}. \tag{10}$$

The proof is given in [3]. In fact

$$\frac{1}{S-\lambda} = L_\lambda,$$

where L_λ is the *resolvent operator*, for which $L_\lambda f(x) = y$ is the solution of the boundary value problem

$$\delta y - \lambda y = f, \quad \Phi\{y\} = 0.$$

It has the form

$$L_\lambda f(t) = \int_1^t \left(\frac{t}{\tau}\right)^\lambda f(\tau)\frac{d\tau}{\tau} - \frac{t^\lambda}{E(\lambda)}\Phi_\sigma\left\{\int_1^\sigma \left(\frac{\sigma}{\tau}\right)^\lambda f(\tau)\frac{d\tau}{\tau}\right\}.$$

L_λ can be represented as the convolution operator

$$L_\lambda f = \left\{\frac{t^\lambda}{E(\lambda)}\right\} * f.$$

Theorem 4 *Let $P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n$ be a polynomial of degree n. Then the boundary value problem* (4)-(5) *is equivalent in M to the linear algebraic equation*

$$P(S)y = f + Q(S), \tag{11}$$

where

$$Q(S) = \sum_{k=0}^{n-1}\sum_{m=0}^{n-k-1} a_k\alpha_m S^{n-k-m-1} = \sum_{\mu=0}^{n-1}\left(\sum_{\nu=0}^{n-\mu-1} a_\nu\alpha_{n-\mu-\nu-1}\right)S^\mu.$$

Proof: First, let $y \in C^n(\mathbb{R}_+)$ be a solution of (4)-(5). Then using (8) we obtain (11).

Conversely, let $y \in C^n(\mathbb{R}_+)$ satisfies (11). Let us involve the right inverse L of δ substituting $S = \dfrac{1}{L}$. Then, L^n, applied to both sides of (11), gives

$$L^n P\left(\frac{1}{L}\right) y = L^n f + L^n Q\left(\frac{1}{L}\right).$$

This can be written as

$$\widetilde{P}(L)y - \widetilde{Q}(L) = L^n f. \tag{12}$$

where $\widetilde{P}(\lambda) = \lambda^n P\left(\dfrac{1}{\lambda}\right)$ and $\widetilde{Q}(\lambda) = \lambda^n Q\left(\dfrac{1}{\lambda}\right)$ are the reciprocal polynomials of P and Q respectively. It remains to apply δ^n to both sides of (12):

$$\delta^n \widetilde{P}(L)y - \delta^n \widetilde{Q}(L) = \delta^n L^n f = f.$$

Since L is a right inverse of δ, then $\delta^n L^k = \delta^{n-k}\delta^k L^k = \delta^{n-k}$ for $k = 0, 1, 2, \ldots, n$. The first term $\delta^n \widetilde{P}(L)y$ becomes $P(\delta)y$, while the second term $\delta^n \widetilde{Q}(L)$ is zero due to the fact that $\deg Q \leq n-1$ and there will always be at least first power of δ acting on the constant function $\{1\}$. Thus $P(\delta)y = f$ is proved.

The verification of $\Phi\{\delta^k y\} = \alpha_k$, $k = 0, 1, 2, \ldots, n-1$, is more complicated but again straightforward.

Theorem 5 *Let $y \in MP^\delta_\Phi$ be a mean-periodic solution of the Euler differential equation $P(\delta)y = f$. Then a necessary condition for existence of such a solution is $f \in MP^\delta_\Phi$. The problem of solving this equation in MP^δ_Φ is equivalent to the nonlocal Cauchy boundary value problem* (4)-(5) *with the homogeneous initial conditions $\Phi\{\delta^k y\} = 0, \quad k = 0, 1, 2, \ldots, \deg P - 1$.*

Proof: Let y be a mean-periodic solution of (4), i.e. $\Phi_\tau\{y(t\tau)\} = 0$. According to Dimovski and Hristov [2] the operator $M : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ given by $Mf(t) = \Phi_\tau\{y(t\tau)\},\ f \in C(\mathbb{R}_+)$, belongs to the commutant of δ, i.e. $M\delta = \delta M$. Applying M to both sides of $P(\delta)y = f$ and using $My(t) = 0$, we get

$$Mf = MP(\delta)y = P(\delta)My = 0.$$

Hence $f \in MP^\delta_\Phi$.

Now we continue with the proof of the equivalence.

First, let $P(\delta)y = f,\ f \in MP^\delta_\Phi,\ \Phi\{\delta^k y\} = 0,\ k = 0, 1, 2, \ldots, n-1$. Let M be the operator from the commutant of δ, which corresponds to the functional Φ as above. Consider the function $u = My$. We need to prove that $u = 0$. One has

$$\delta^k u(t) = \delta^k My(t) = M\delta^k y(t) = \Phi_\tau\{(\delta^k y)(t\tau)\}.$$

Substituting $t = 1$ in this equality, we obtain

$$\delta^k u(1) = \Phi_\tau\{(\delta^k y)(\tau)\} = 0.$$

Thus u is a solution of the ordinary Cauchy problem

$$P(\delta)u = 0, \quad \delta^k u(1) = 0, k = 0, 1, 2, \ldots, n-1,$$

which has the unique solution $u = 0$, i.e. $My = 0$, which means that y is mean-periodic.

Conversely, let $P(\delta)y = f$ with a mean-periodic solution y, i.e. $My = \Phi_\tau\{y(t\tau)\} = 0$. Applying δ^k, one has

$$0 = \delta^k My = M\delta^k y,$$

which means that $\delta^k y$ is mean-periodic for every $k \geq 0$. This can be written also as

$$\Phi_\tau\{(\delta^k y)(t\tau)\} = 0.$$

Finally, we substitute $t = 1$ and get

$$\Phi_\tau\{(\delta^k y)(\tau)\} = \Phi\{\delta^k y\} = 0, \; k = 0, 1, 2, \ldots, n-1.$$

Now we may assert that the problem of solving the Euler equations in mean-periodic functions is equivalent to solving the algebraic problem $P(S)y = f$ in $\mathfrak{M}$.

Its formal solution in $\mathfrak{M}$ is

$$y = \frac{1}{P(S)} f \tag{13}$$

provided $P(S)$ is non-divisor of zero.

Theorem 6 *$P(S)$ is a non-divisor of* 0 *iff $\Phi_\tau\{\tau^\lambda\} \neq 0$ for each zero of the polynomial $P(\lambda)$, i.e. if $P(\lambda) = 0$ implies $\Phi_\tau\{\tau^\lambda\} \neq 0$.*

Proof: Let $P(\lambda) = a_0(\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n)$, $a_0 \neq 0$. $P(S)$ is a divisor of 0 if and only if at least one of the multipliers $S - \lambda_k$ is a divisor of zero. Let $S - \lambda$ be a divisor of zero, i.e. there exists an element (convolution fraction) $\frac{u}{v} \in \mathfrak{M}$, such that $(S - \lambda)\frac{u}{v} = 0$. The last equality is equivalent to

$$(1 - \lambda L)u = 0,$$

i.e. $\delta u - \lambda u = 0$, with the solution $u = Ct^\lambda$, $C \neq 0$ and $\Phi\{u\} = 0$, i.e. $\Phi_\tau\{\tau^\lambda\} = 0$.

An extension of the Duhamel principle to nonlocal Cauchy problems for Euler equations

Assuming that $\dfrac{1}{SP(S)} = G(t) \in C^{(n)}(\mathbb{R}_+)$, we see that G is the solution of the nonlocal Cauchy boundary value problem:

$$P(\delta)G = 1, \qquad \Phi\{\delta^k G\} = 0, \qquad k = 0, 1, 2, \ldots, n-1.$$

From the equation

$$a_0\delta^n G + a_1\delta^{n-1}G + \ldots + a_n G = 1$$

we obtain $a_0\Phi\{\delta^n G\} = 1$, or

$$\Phi\{\delta^n G\} = \frac{1}{a_0} \tag{14}$$

Then the formal representation (13) becomes

$$y = \delta(G * f) = \delta(f * G) = (\delta f) * G.$$

This is an extension of the classical Duhamel principle to nonlocal Cauchy problems for Euler equations.

Heaviside algorithm for interpreting $\dfrac{1}{P(S)}$ as a function

Let $P(S)$ be a non-divisor of zero in $\mathfrak{M}$. First, decompose $\dfrac{1}{P(S)}$ in simple fractions. Factorizing $P(S)$ as

$$P(S) = a_0(S-\mu_1)^{k_1}(S-\mu_2)^{k_2}\ldots(S-\mu_l)^{k_l}, \quad k_1 + k_2 + \ldots + k_l = n,$$

we have

$$\frac{1}{P(S)} = \sum_{j=1}^{l}\sum_{m=1}^{k_j}\frac{A_{jm}}{(S-\mu_j)^m},$$

where A_{jm} are constants. It remains to replace each fraction $\dfrac{A_{jm}}{(S-\mu_j)^m}$ by the explicit expressions given in Theorem 3.

Example: Let all the zeros of the polynomial P be simple, i.e. $P(\mu) = a_0(\mu-\mu_1)(\mu-\mu_2)\ldots(\mu-\mu_n)$ with $\mu_\nu \neq \mu_\kappa$ for $\nu \neq \kappa$. Then

$$\frac{1}{P(S)} = \sum_{k=1}^{n}\frac{1}{P'(\mu_k)}\,\frac{1}{S-\mu_k} = \sum_{k=1}^{n}\frac{t^{\mu_k}}{P'(\mu_k)E(\mu_k)}.$$

As a particular case, let us take the functional

$$\Phi\{f\} = \frac{f(1) + f(e)}{2}.$$

Then $E(\lambda) = \dfrac{1 + e^{\lambda}}{2}$ and $\dfrac{1}{S - \lambda} = \dfrac{2t^{\lambda}}{1 + e^{\lambda}}$. We can write

$$G(t) = \frac{1}{P(S)} = \sum_{k=1}^{n} \frac{2t^{\mu_k}}{P'(\mu_k)(1 + e^{\mu_k})}$$

and then the solution of the equation $P(\delta)y(t) = f(t)$ with boundary value conditions $\delta^k y(1) + \delta^k y(e) = 0$, $k = 0, 1, \ldots, , n-1$, is

$$y = G * f = \sum_{k=1}^{n} \frac{2}{P'(\mu_k)(1 + e^{\mu_k})} (t^{\mu_k} * f),$$

where

$$t^{\mu_k} * f = \frac{1}{2} \left\{ \int_1^t \left(\frac{t}{\tau}\right)^{\mu_k} f(\tau) \frac{d\tau}{\tau} + \int_e^t \left(\frac{te}{\tau}\right)^{\mu_k} f(\tau) \frac{d\tau}{\tau} \right\}$$

One can proceed in a similar way in the case of multiple zeros of $P(\mu)$. The only difference is that, according to (10), any fraction of the form $\dfrac{1}{(S - \mu)^l}$ should be replaced by the function

$$\frac{1}{(l-1)!} \frac{d^{l-1}}{d\mu^{l-1}} \left(\frac{t^{\mu}}{E(\mu)} \right).$$

As for the resonance case, then additional considerations are needed. They are left for a next publication.

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