Commutants of $\dfrac{d^2}{dx^2}$ on the Real Half-Line

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Abstract

Let C_h^1 denotes the space of the smooth functions $f(x)$ on the real half-line $\mathbb{R}_{\geq 0} = [0, \infty)$ satisfying the initial value condition $f'(0) - hf(0) = 0$ with fixed real h. We characterize the continuous linear operators $M : C_h^1 \to C_h^1$ which commute with the square $D^2 = \dfrac{d^2}{dx^2}$ of the differentiation operator $D = \dfrac{d}{dx}$ on the subspace C_h^2 of the twice continuously differentiable functions of C_h^1. The explicit representation of such operators is $Mf(x) = \Phi_y\{T^y f(x)\}$, where

$$T^y f(x) = \frac{1}{2}\{f(x+y) + f(|x-y|)\} + \frac{h}{2}\int_{|x-y|}^{x+y} f(t)dt$$

and Φ is a linear functional on C_h^1.

The kernel space of this operator is denoted by MP_Φ and is called the space of the mean-periodic functions for D^2 determined by Φ. It is proved that the space MP_Φ is invariant under the resolvent operator of D^2 with the boundary value conditions $y'(0) - hy(0) = 0$ and $\Phi\{y\} = 0$. A convolution structure $* : C_h^1 \times C_h^1 \to C_h^1$ is introduced in C_h^1, such that the resolvent operator is a continuous operator and MP_Φ is an ideal in the convolution algebra $(C_h^1, *)$. This result is used for effective solution in mean-periodic functions of ordinary differential equations of the form $P(D^2)y = f$ with a polynomial P.

A family of operators commuting with $D^2 = \dfrac{d^2}{dx^2}$

Let C_h^1 be the space of smooth functions f on $\mathbb{R}_{\geq 0} = [0, \infty)$ satisfying the boundary value condition

$$f'(0) - hf(0) = 0 \tag{1}$$

with a fixed $h \in \mathbb{R}$. By C_h^2 we denote the subspace of twice continuously differentiable functions of C_h^1.

Lemma 1 *The operators*

$$T^y f(x) = \frac{1}{2}\{f(x+y) + f(|x-y|)\} + \frac{h}{2}\int_{|x-y|}^{x+y} f(t)dt \tag{2}$$

map C_h^1 onto C_h^1 and have the following properties:

(i) $T^y f(x) = T^x f(y)$;

(ii) $T^0 f(x) = f(x)$;

(iii) $D^2 T^y = T^y D^2$ *on* C_h^2;

(iv) $T^y T^z = T^z T^y$.

Proof: It is seen directly that $(T^y f)(0) - h(T^y f)(0) = 0$ for arbitrary $f \in C^1(\mathbb{R}_{\geq 0})$ and hence $T^y : C_h^1 \to C_h^1$.

The properties (i) and (ii) are obvious.

In order to prove (iii), we verify it first for $y \leq x$ and then for $x < y$. If $y \leq x$, then

$$T^y f(x) = \frac{1}{2}\{f(x+y) + f(x-y)\} + \frac{h}{2}\int_{x-y}^{x+y} f(t)dt$$

and

$$\frac{d^2}{dx^2}T^y f(x) = \frac{1}{2}[f''(x+y) + f''(x-y)] + \frac{h}{2}[f'(x+y) - f'(x-y)] = T^y f''(x).$$

If $x < y$, then the verification of $\frac{d^2}{dx^2}T^y f(x) = T^y f''(x)$ goes in the same way.

For the proof of (iv), one may verify it first for even powers of x, i.e. for $f(x) = x^{2n}$, and then to proceed by approximation of an arbitrary function $f \in C_h^1$ by polynomials of the form $P(x^2)$.

Since the operators (2) are a very special case of the generalized translation operators of B. M. Levitan (see [3]), one may rely also on a general proof in this book. $\square$

Theorem 1 *Let $M : C_h^1 \to C_h^1$ be a continuous linear operator, such that $M : C_h^2 \to C_h^2$. Then the following assertions are equivalent:*

(i) $MD^2 = D^2M$ *in* C_h^2*;*

(ii) $MT^y = T^yM$ *for each* $y \geq 0$*;*

(iii) M *has the explicit representation*

$$Mf(x) = \Phi_y\{T^y f(x)\} = \Phi_y\left\{\frac{f(x+y)+f(|x-y|)}{2} + \frac{h}{2}\int_{|x-y|}^{x+y} f(t)dt\right\} \tag{3}$$

with a linear functional Φ *in* C_h^1*.*

Proof:
(i)$\Rightarrow$(ii)
Let $f(x)$ be an even polynomial. Then the Maclaurin expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} D^{2n} f(0)$$

gives the following representation of the translated function:

$$T^y f(x) = T^x f(y) = \sum_{n=0}^{\infty} \frac{y^{2n}}{(2n)!} T^x D^{2n} f(0)$$
$$= \sum_{n=0}^{\infty} \frac{y^{2n}}{(2n)!} T^0 D^{2n} f(x) = \sum_{n=0}^{\infty} \frac{y^{2n}}{(2n)!} D^{2n} f(x).$$

Now (ii) will follow if we apply M to both sides and use $MD^{2n}f(x) = D^{2n}Mf(x)$ which follows immediately from (i) for each $n \in \mathbb{N}$:

$$MT^y f(x) = \sum_{n=0}^{\infty} \frac{y^{2n}}{(2n)!} MD^{2n} f(x) = \sum_{n=0}^{\infty} \frac{y^{2n}}{(2n)!} D^{2n} Mf(x) = T^y Mf(x).$$

(ii)$\Rightarrow$(iii)
Let us define a continuous linear functional Φ in C_h^1 by $\Phi\{f\} = (Mf)(0)$. Substituting $y = 0$ in

$$T^y Mf(x) = MT^y f(x) = MT^x f(y),$$

we obtain

$$T^0 Mf(x) = MT^x f(0).$$

The left hand side is $Mf(x)$ and the right hand side is the value of the functional Φ for the function $T^x f$. Hence

$$Mf(x) = \Phi_y\{T^x f(y)\} = \Phi_y\{T^y f(x)\}.$$

Thus the implication is proved using y as the "dumb" variable of the functional.

(iii)$\Rightarrow$(i)

Let $Mf(x) = \Phi_y\{T^y f(x)\}$. Then $D^2 Mf(x) = \Phi_y\{D^2 T^y f(x)\}$. Using $D^2 T^y = T^y D^2$ from Lemma 1, we have

$$D^2 Mf(x) = \Phi_y\{T^y D^2 f(x)\} = MD^2 f(x).$$

Hence (iii)$\Rightarrow$(i). $\square$

Theorem 2 *The commutant of* $D^2 = \dfrac{d^2}{dx^2}$ *in* C_h^1 *is a commutative ring.*

Proof: Let $M : C_h^1 \to C_h^1$ and $N : C_h^1 \to C_h^1$ commute with $D^2 = \dfrac{d^2}{dx^2}$ in C_h^2.

According to (iii) from Theorem 1, there are linear functionals Φ and Ψ in C_h^1, such that

$$Mf(x) = \Phi_y\{T^y f(x)\} \quad \text{and} \quad Nf(x) = \Psi_z\{T^z f(x)\}.$$

Then

$$MNf(x) = \Phi_y \Psi_z\{T^y T^z f(x)\} \quad \text{and} \quad NMf(x) = \Psi_z \Phi_y\{T^z T^y f(x)\}.$$

By (iv) from Lemma 1, $T^z T^y = T^y T^z$, and hence

$$NMf(x) = \Psi_z \Phi_y\{T^z T^y f(x)\} = \Psi_z \Phi_y\{T^y T^z f(x)\}.$$

It remains to use the Fubini property $\Psi_z \Phi_y g(y,z) = \Phi_y \Psi_z g(y,z)$ for functions $g(y,z) \in C^1(\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0})$ in order to assert that $MN = NM$. $\square$

Mean-periodic functions for $D^2 = \dfrac{d^2}{dx^2}$ in C_h^1

Definition 1 *The kernel space* $\ker M$ *of an operator of the form* $Mf(x) = \Phi_y\{T^y f(x)\}$ *is called the space of the mean-periodic functions for* $D^2 = \dfrac{d^2}{dx^2}$, *associated with the linear functional* Φ.

We use the notation $MP_\Phi = \ker M$, i.e. $MP_\Phi = \{f \in C_h^1 : \Phi_y\{T^y f(x)\} = 0\}$.

In order to reveal some specific properties of MP_Φ, let us introduce the resolvent operator $R_{-\lambda^2}$ of the operator $D^2 = \dfrac{d^2}{dx^2}$, defined by the the boundary value conditions $y'(0) - hy(0) = 0$ and $\Phi\{y\} = 0$. In other words, $y(x) = R_{-\lambda^2} f(x)$ is the solution of the differential equation

$$y'' + \lambda^2 y = f(x)$$

satisfying the boundary value conditions $y'(0) - hy(0) = 0$ and $\Phi\{y\} = 0$.

Lemma 2 *The resolvent operator $R_{-\lambda^2}$ of $D^2 = \dfrac{d^2}{dx^2}$ has the following explicit form*

$$
\begin{aligned}
R_{-\lambda^2} f(x) &= \frac{1}{\lambda}\int_0^x \sin\lambda(x-t) f(t)dt \\
&\quad - \frac{\lambda\cos\lambda x + h\sin\lambda x}{\lambda E(\lambda)} \Phi_y\left\{\int_0^y \sin\lambda(y-t) f(t)dt\right\},
\end{aligned}
$$

where $E(\lambda) = \Phi_t\left\{\dfrac{\lambda\cos\lambda t + h\sin\lambda t}{\lambda}\right\}$, is an entire function of exponential type.

The proof is a matter of a direct check.

Lemma 3 *$R_{-\lambda^2}$ maps MP_Φ into itself, i.e. $R_{-\lambda^2}(MP_\Phi) \subset MP_\Phi$.*

Proof: Let $f \in MP_\Phi$, i.e. $\Phi_y\{T^y f(x)\} = 0$. We are to prove that $\varphi(x) = \Phi_y\{T^y R_{-\lambda^2} f(x)\} \equiv 0$. Indeed, we have

$$
\begin{aligned}
(D^2+\lambda^2)\varphi(x) &= \Phi_y\{(D^2+\lambda^2)T^y R_{-\lambda^2} f(x)\} \\
= \Phi_y\{T^y (D^2+\lambda^2) R_{-\lambda^2} f(x)\} &= \Phi_y\{T^y f(x)\} \equiv 0,
\end{aligned}
$$

since $(D^2+\lambda^2)R_{-\lambda^2} f(x) = f(x)$. Hence $\varphi(x)$ belongs to the kernel space of $D^2+\lambda^2$, i.e. $\varphi(x) = A\cos\lambda x + B\sin\lambda x$ with constants A and B. φ satisfies the condition $\varphi'(0) - h\varphi(0) = 0$ and hence $B\lambda - hA = 0$. In other words, $\varphi(x)$ is a function of the form $\varphi(x) = A\left(\cos x + \dfrac{h\sin\lambda}{\lambda}\right)$. Using the boundary value condition $\Phi\{f\} = 0$, we obtain

$$
0 = A\Phi_t\left\{\cos x + \frac{h\sin\lambda}{\lambda}\right\} = AE(\lambda).
$$

But $E(\lambda) \neq 0$ and hence $A = 0$. Thus we proved that $\varphi(x) \equiv 0$. □

For the sake of simplicity, from now on we restrict our considerations to the case $h = 0$, i.e. to the space

$$
C_0^1 = \{f \in C^1(\mathbb{R}_{\geq 0}), f'(0) = 0\}.
$$

This is possible due to an explicit isomorphism between C_h^1 and C_0^1.

Lemma 4 *The linear operator*

$$\tau f(x) = f(x) + h\int_0^x e^{-h(x-t)} f(t)dt \tag{4}$$

maps C_h^1 onto C_0^1 and its inverse is

$$\tau^{-1} f(x) = f(x) + h\int_0^x f(t)dt. \tag{5}$$

If $f \in C_h^2$, then $\tau f \in C_0^2$ and $(\tau f)'' = \tau f''$.

The proof is a matter of simple check (see Dimovski [1], p.153).

Due to Lemma 4, instead of the resolvent operator $R_{-\lambda^2}$ of D^2 with boundary value conditions $y'(0) - hy(0) = 0$ and $\Phi\{y\} = 0$, we may consider the resolvent operator $\widetilde{R_0}$ of D^2, defined by the boundary value conditions $y'(0) = 0$ and $\widetilde{\Phi}\{y\} = 0$, where $\widetilde{\Phi} = \Phi \circ \tau^{-1}$.

From now on we will use the notation Φ instead of $\widetilde{\Phi}$, assuming that we are all the time in the case $h = 0$.

For a further simplification we assume that $\lambda = 0$ is not an eigenvalue of the eigenvalue problem $y'' + \lambda^2 y = 0$, $y'(0) = 0$, $\Phi\{y\} = 0$. This means that there exists a right inverse operator R of D^2, such that $(Rf)'(0) = 0$, $\Phi\{Rf\} = 0$ which is possible when $\Phi\{1\} \neq 0$. If so, we may assume additionally that $\Phi\{1\} = 1$ without any loss of generality. Then the right inverse of D^2 has the form

$$Rf(x) = \int_0^x (x-t)f(t)dt - \Phi_y\left\{\int_0^y (y-t)f(t)dt\right\}.$$

In Dimovski [1], pp. 148-151, the following theorem is proved:

Theorem 3 *The operation*

$$(f * g)(x) = \int_0^x dt\int_0^t f(t-\tau)g(\tau)d\tau + \frac{1}{2}\Phi_t\left\{\int_0^t \psi(x,\tau)d\tau\right\}, \tag{6}$$

where

$$\psi(x,t) = \int_x^t f(t+x-\tau)g(\tau)dz + \int_{-x}^t f(|t-x-\tau|)g(|\tau|)d\tau,$$

is an inner operation in C_0^1, which is bilinear, commutative, and associative, and the operator R is the convolution operator $R = \{1\}*$, i.e. $Rf = \{1\} * f$.

Theorem 4 *The subspace MP_Φ of mean-periodic functions for D^2 associated with the linear functional Φ form an ideal in the convolution algebra $(C_0^1, *)$.*

Proof: By Lemma 3, if $f \in MP_\Phi$, then $Rf \in MP_\Phi$. But from Theorem 2 $Rf = \{1\} * f$ and $R^k f = \{Q_k(x^2)\} * f$, where Q_k is a polynomial of degree k.

Choose a polynomial sequence $\{P_n(x)\}_{n=1}^{\infty}$ converging to $g(\sqrt{x})$ uniformly on each segment $[a, b] \subset [0, \infty)$. Then $\{P_n(x^2)\}_{n=1}^{\infty}$ converges to $g(x)$ in C_0^1. But $P_n(x^2) = \sum_{k=0}^{n} \alpha_k Q_k(x^2)$ with some constants $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n$. Then $\{P_n(x^2)\} * f \in MP_\Phi$ since $\{Q_k(x^2)\} * f \in MP_\Phi,\ k = 0, 1, 2, \ldots, n$. Obviously the limit of a sequence of mean-periodic functions is also mean-periodic.

Hence $g * f \in MP_\Phi$ for arbitrary $g \in C_0^1$ and therefore MP_Φ is an ideal in $(C_0^1, *)$. □

Theorem 4 may be used to study the problem of solution of ordinary differential equations with constant coefficients of the form

$$P\left(\frac{d^2}{dx^2}\right) y = f(x)$$

in mean-periodic functions of the space MP_Φ and to extend the Heaviside algorithm for obtaining such solutions in explicit form. This will be left for a subsequent publication, but analogous considerations for the Dunkl operator D_k instead of D^2 can be seen in Dimovski, Hristov, and Sifi [2].

References

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