Optimization of Algorithms by Continued Fractions

Frithjof Blomquist
University of Wuppertal
Scientific Computing / Software Technology
Gaußstr. 20
D-42097 Wuppertal
Germany
blomquist@math.uni-wuppertal.de

Abstract

In C-XSC the elementary functions are typically implemented using an argument reduction to rather small intervals containing the origin. For the Special Functions of Mathematical Physics such argument reductions are not available. So in wider intervals best approximations (Remes algorithm) using rational functions are recommended. Guaranteed upper bounds of relative approximation errors can be calculated, see [1],[2]. However, the relative evaluation error of such a rational function $R(x-x_0) = P_M(x-x_0)/Q_N(x-x_0)$ is often too large. To avoid this disadvantage $R(x-x_0)$ can be converted to a finite continued fraction, where principally two different results are possible. For example with $x_0 = 0$ and the simple rational function $R(x) := (3+4x-2x^2+x^3)/(1+x-2x^2-2x^3)$ it holds

$$R(x) = CF_5(x) := 3 + \cfrac{x}{1 + \cfrac{x}{-\frac{1}{3} + \cfrac{x}{-3 + \cfrac{x}{\frac{1}{10} + \cfrac{x}{\frac{75}{49} - \frac{15}{49}\cdot x}}}}}$$

and the second possible finite continued fraction for $R(x)$ is given by

$$R(x) = CF_3(x) := -\frac{1}{2} + \cfrac{3/2}{x + \frac{5}{2} + \cfrac{53/12}{x - \frac{217}{106} - \frac{392}{8427}\cdot\cfrac{1}{x + \frac{29}{53}}}}$$

It holds $CF_5(x) \to 3$ for $x \to 0$ and $CF_3(x) \to -1/2$ for $|x| \to +\infty$, so $CF_5(x)$ and $CF_3(x)$ deliver only small evaluation errors for $x \to 0$ and $|x| \to +\infty$, respectively. However, the advantage of the last term $CF_3(x)$ is that for its evaluation only nine elementary operations are needed whereas $CF_5(x)$ and $R(x)$ require twelve and thirteen operations, respectively.

To use this advantage of $CF_3(x)$, a rational approximation $R(u)$ with $u := (x - x_0)^k, k \in N$ is first calculated with a computer algebra system. Then with the transformation

$$(x - x_0)^k = u = \frac{1}{\frac{1}{u}} = \frac{1}{v}$$

a new rational approximation function $R(1/v) =: T(v)$ is available, and for $x \to x_0$ it holds $|v| \to \infty$. Then calculating the associated continued fraction, for example with the length 5, delivers

$$T(v) = K_5(v) := b_0 + \cfrac{a_1}{v + b_1 + \cfrac{a_2}{v + b_2 + \cfrac{a_3}{v + b_3 + \cfrac{a_4}{v + b_4 + \cfrac{a_5}{v + b_5}}}}}$$

The advantage of $K_5(v)$ is now the relatively small evaluation error for $x \to x_0$ and the smaller number of necessary elementary operations in comparison to the original rational approximation function $R(u)$.

For calculating a guaranteed upper bound of the relative approximation error first an inclusion of $K_5(v)$ by a rational function $\hat{R}(u)$ is needed and can be realized using the recursion formulas in [3, pp. 176], and with $\hat{R}(u)$ again the approximation error is computable, [1].

References:

[1] Krämer, W.; *Sichere und genaue Abschätzung des Approximationsfehlers bei rationalen Approximationen*, Report of the Institut für Angewandte Mathematik, Universität Karlsruhe, Germany, 1996.

[2] Blomquist, F.; *Automatische a priori Fehlerabschätzungen zur Entwicklung optimaler Algorithmen und Intervallfunktionen in C-XSC*, Fachbereich 7, Mathematik, Bergische Universität Wuppertal, 2006.

[3] Press W.H. et al. *Numerical Recipes in C, The Art of Scientific Computing*, second edition, Cambridge University Press, 1992.

Keywords: C-XSC, continued fractions, computer algebra systems, Special Functions.