Generalization of a Parametric Fixed-Point Iteration

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Based on new sufficient regularity conditions, Rump's method for solving parametric interval linear systems is generalized which expands its scope of applicability over a class of co-called column-dependent parametric matrices.

1 Introduction

Consider a linear system $A(p) \cdot x \; = \; b(p)$ involving affine-linear dependencies on a parameter vector $p \in \mathbb{R}^k$, that is

$$A(p) = A^{(0)} + \sum_{\nu=1}^{k} p_\nu A^{(\nu)}, \quad b(p) = b^{(0)} + \sum_{\nu=1}^{k} p_\nu b^{(\nu)}, \qquad A^{(m)} \in \mathbb{R}^{n\times n},\ b^{(m)} \in \mathbb{R}^n,\ m = 0,\ldots,k. \tag{1}$$

When p varies in $[p] \in \mathbb{IR}^k$ the parametric solution set is $\Sigma\left(A(p), b(p), [p]\right) := \{x \in \mathbb{R}^n \mid A(p)\cdot x = b(p) \text{ for some } p \in [p]\}$.

The following notations are used. $\mathbb{IR}^{n\times n}$ is the set of $n \times n$ interval matrices (cf. [1]). For an interval matrix $[A] = [\underline{A}, \overline{A}] = \{A \mid \underline{A} \le A \le \overline{A}\}$, denote the mid-point matrix by $\check{A} = \frac{1}{2}(\underline{A} + \overline{A})$, the radius matrix by $\text{rad}([A]) = \frac{1}{2}(\overline{A} - \underline{A})$ and the absolute value matrix by $|A| = (|a_{ij}|)$. These notations apply to vectors as well. For a bounded set Σ, define hull $\square(\Sigma) := [\inf \Sigma,\ \sup \Sigma]$. $\varrho(A)$ is the spectral radius of a matrix A. I denotes the unit matrix. Denote by $A([p]) := \square\{A(p) \in \mathbb{R}^{n\times n} \mid p \in [p]\},\ b([p]) := \square\{b(p) \in \mathbb{R}^n \mid p \in [p]\}$ the corresponding non-parametric interval matrix and vector.

An iteration method for verified enclosure $\square(\Sigma(A(p), b(p), [p])) \subseteq [y] \in \mathbb{IR}^n$ is proposed in [4]. Rump's ***parametric*** iteration method and, to our knowledge, most methods for solving parametric interval linear systems require strong regularity of $A([p])$. Strong regularity of a non-parametric interval matrix is introduced in [1] and presents a sufficient condition for its regularity. In [2] we define strong regularity of a parametric interval matrix and give conditions that characterize it.

Definition 1.1 A parametric matrix $A(p) \in \mathbb{R}^{n\times n}$, $p \in [p] \in \mathbb{IR}^k$ is called strongly regular over $[p]$ if some of the matrices

$$[B] := \square\left\{A^{-1}(\check{p})A(p) \mid p \in [p]\right\} \qquad \text{or} \qquad [B'] := \square\left\{A(p)A^{-1}(\check{p}) \mid p \in [p]\right\} \qquad \text{is regular.} \tag{2}$$

The classes of column-, row-dependent parametric matrices, introduced in [2], show that the conditions for strong regularity of a parametric matrix give better estimations for its regularity than the conditions based on the non-parametric matrix.

Definition 1.2 A parametric matrix $A(p) \in \mathbb{R}^{n\times n}$, defined by (1), is called column-dependent parametric matrix if for some $m \in \{1,\ldots,k\}$ and for some $j \in \{1,\ldots,n\}$, $\text{Card}(\mathcal{I}) \ge 2$, where $\mathcal{I} := \{i \mid 1 \le i \le n,\ a_{ij}^{(m)} \ne 0\}$.

A parametric matrix $A(p) \in \mathbb{R}^{n\times n}$ is called row-dependent if $A(p)^\top$ is column-dependent.

2 Verifiable sufficient regularity conditions generalizing Rump's parametric iteration

For the numerical computations, it is advantageous to have regularity conditions in terms of an arbitrary matrix R instead of the exact inverse $A^{-1}(\check{p})$.

Theorem 2.1 *Let $A(p) \in \mathbb{R}^{n\times n}$ with $p \in [p] \in \mathbb{IR}^k$ be a parametric matrix and $R \in \mathbb{R}^{n\times n}$. If some of the matrices $[U] := \square\{R \cdot A(p) \mid p \in [p]\}$, $[U'] := \square\{A(p) \cdot R \mid p \in [p]\}$ is an H-matrix, then $A(p)$ is strongly regular over $[p]$.*

Proof. If $[U]$ is an H-matrix, then by [1], Th. 3.7.5 (ii), $\check{U}$ is an H-matrix and $\varrho\left(\langle\check{U}\rangle^{-1}\text{rad}([U])\right) < 1$, wherein $\langle\check{U}\rangle$ denotes the Ostrowski comparison matrix for $\check{U}$ (see [1]). By [1], Th. 3.7.5 (iii), we have $|\check{U}^{-1}| \leqq \langle\check{U}\rangle^{-1}$. Let $[V] := \square\left\{A^{-1}(\check{p})A(p) \mid p \in [p]\right\}$, then with $[p_\nu] = [\check{p} - \delta_\nu, \check{p} + \delta_\nu]$, $0 < \delta_\nu \in \mathbb{R}$, $\nu = 1,\ldots,k$,

$$\begin{aligned}
\text{rad}([V]) \;&=\; \sum_{\nu=1}^{k} \delta_\nu |A^{-1}(\check{p})A^{(\nu)}| \;=\; \sum_{\nu=1}^{k} \delta_\nu |A^{-1}(\check{p})R^{-1}RA^{(\nu)}| \;=\; \sum_{\nu=1}^{k} \delta_\nu |(RA(\check{p}))^{-1}RA^{(\nu)}| \\
&\leqq\; \sum_{\nu=1}^{k} \delta_\nu |(RA(\check{p}))^{-1}| \cdot |RA^{(\nu)}| \;=\; |(RA(\check{p}))^{-1}| \sum_{\nu=1}^{k} \delta_\nu |RA^{(\nu)}| \;=\; |\check{U}^{-1}|\text{rad}([U]) \;\leqq\; \langle\check{U}\rangle^{-1}\text{rad}([U]).
\end{aligned}$$

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Hence $\varrho(\mathrm{rad}([V])) \leqq \varrho(\langle\check{U}\rangle^{-1}\mathrm{rad}([U])) < 1$ and, by Theorem 1 from [2], $A(p)$ is strongly regular over $[p]$. If U' is an H-matrix the proof goes analogously with the consideration $\varrho(AB) = \varrho(BA)$ for $A, B \in \mathbb{R}^{n\times n}$ ([1], Prop. 3.2.4 (9)). □

The next theorem, which in fact verifies strong regularity, can be used for the computational verification of the regularity of a parametric interval matrix.

Theorem 2.2 *Let $A(p) \in \mathbb{R}^{n\times n}$ with $p \in [p] \in \mathbb{IR}^k$. Let $R \in \mathbb{R}^{n\times n}$ be given and let*

$$[C(p)] := \{I - \square\{A(p)\cdot R \mid p \in [p]\} \text{ if } A(p) \text{ is row-dependent}, \quad I - \square\{R\cdot A(p) \mid p\in[p]\} \text{ otherwise}\}. \tag{3}$$

For $[x^0] \in \mathbb{IR}^n$ define the iteration $[x^{l+1}] := [C(p)] \diamondsuit [x^l] \oplus [e^l]$ for $l \in \mathbb{N}$, where $[e^l] \in \mathbb{IR}^n$, $[e^l] \to [e] \in \mathbb{IR}^n$ with $0 \in \mathrm{int}([e])$, int$([x])$ denoting the topological interior, and all operations $\oplus$, $\diamondsuit$ are outwardly-rounded computer interval operations. If $[C(p)]\diamondsuit[x^l] \subseteq \mathrm{int}([x^l])$ for some $l \in \mathbb{N}$, then R and every $A(p)$ with $p \in [p]$ are regular.

P r o o f. Theorem 1.5 from [4], with $[z] = 0$, implies that $\varrho(|[C(p)]|) < 1$. Therefore for every $p \in [p]$, $\varrho(I - R\cdot A(p)) \leq \varrho(|I - R\cdot A(p)|) < 1$. Hence R and every $A(p)$, $p \in [p]$ are regular. □

We use the above results to generalize Rump's parametric fixed-point iteration method [4].

Theorem 2.3 *Let $A(p)\cdot x = b(p)$ with $p \in [p] \in \mathbb{IR}^k$ be a parametric linear system, where $A(p), b(p)$ are defined by (1). Let $R \in \mathbb{R}^{n\times n}$, $[Y] \in \mathbb{IR}^n$, $\tilde{x} \in \mathbb{R}^n$ and define $[Z] \in \mathbb{IR}^n$, $[C(p)] \in \mathbb{IR}^{n\times n}$ by*

$$[Z] := R\cdot(b^{(0)} - A^{(0)}\tilde{x}) + \sum_{\nu=1}^{k}[p_\nu](R\cdot b^{(\nu)} - R\cdot A^{(\nu)}\cdot\tilde{x}), \qquad [C(p)] := I - R\cdot A^{(0)} - \sum_{\nu=1}^{k}[p_\nu](R\cdot A^{(\nu)}).$$

Define $[V] \in \mathbb{IR}^n$ by means of $1 \leq i \leq n \;:\; V_i := \{[Z] + [C(p)]\cdot[U]\}_i, \quad [U] := (V_1, ..., V_{i-1}, Y_i, ..., Y_n)^\top$. If $[V] \subsetneqq [Y]$, then R and every matrix $A(p), p \in [p]$ are regular, and for every $p \in [p]$ the unique solution $\widehat{x} = A^{-1}(p)b(p)$ of the system satisfies $\widehat{x} \in \tilde{x} + [V]$.

With $[D] := [C(p)]\cdot[V] \in \mathbb{IR}^n$, the following inner estimation holds

$$[\tilde{x} + \inf([Z]) + \sup([D]),\ \tilde{x} + \sup([Z]) + \inf([D])] \subseteq \square(\Sigma(A(p), b(p), [p])).$$

P r o o f. It can be easily verified that $[C(p)] := I - \square\{A(p)\cdot R \mid p \in [p]\} = I - R\cdot A^{(0)} - \sum_{\nu=1}^{k}[p_\nu](R\cdot A^{(\nu)})$. Then Theorem 2.2 above and Theorem 4.8 from [4] finish the proof. □

Example 2.4 Consider $n\times n$ column-dependent parametric linear systems $Q(2,p)\cdot x = b(p)$, where for $i, j = 1, \ldots, n$,

$$q_{ij}(2,p) := \begin{cases} p_j & \text{if } i \leqq j, \\ 0 & \text{if } i = j+2, \\ 1 & \text{otherwise}, \end{cases} \qquad b(p) = (p_1, \ldots, p_n)^\top, \qquad p_i \in [(i+1) \pm 10\%].$$

Table 1 presents the growing spectral radius of $|[C]|$, where $[C] = I - R\cdot A([p])$ is the non-parametric iteration matrix. For $n = 2$ and $n = 3$, Rump's method converges in 3, resp. 6 iterations. For $n \leq 50$, $\varrho(|[C(p)]|) = 0.2$ and the generalized method converges in 2 iterations.

Table 1 The spectral radius of the absolute value of the non-parametric iteration matrix for the Example 2.4.

n	2	3	4	5	10	15	20	25	30	35	40	45	50
$\varrho(\|[C]\|)$	0.46	0.81	1.69	1.95	3.07	3.24	3.40	3.58	3.76	3.96	4.17	4.38	4.59

The new narrow class of parametric iteration matrices $[C(p)]$, defined by (3), extends the class of matrices $A(p)$ that can be proven to be regular. This way, the generalized parametric fixed-point iteration gets an expanded scope of applications. An implementation of the generalized method is described in [3].

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