Generalization of a Parametric Fixed-Point Iteration

Evgenija D. Popova*

Institute of Mathematics & Informatics, Bulgarian Academy of Sciences, Acad. G. Bonchev Str., bl. 8, 1113 Sofia, Bulgaria

Based on new sufficient regularity conditions, Rump's method for solving parametric interval linear systems is generalized which expands its scope of applicability over a class of co-called column-dependent parametric matrices.

1 Introduction

Consider a linear system $A(p) \cdot x = b(p)$ involving affine-linear dependencies on a parameter vector $p \in \mathbb{R}^k$, that is

$$A(p) = A^{(0)} + \sum_{\nu=1}^{k} p_\nu A^{(\nu)}, \quad b(p) = b^{(0)} + \sum_{\nu=1}^{k} p_\nu b^{(\nu)}, \qquad A^{(m)} \in \mathbb{R}^{n\times n},\ b^{(m)} \in \mathbb{R}^n,\ m = 0,\ldots,k. \tag{1}$$

When p varies in $[p] \in \mathbb{IR}^k$ the parametric solution set is $\Sigma\left(A(p), b(p), [p]\right) := \{x \in \mathbb{R}^n \mid A(p)\cdot x = b(p) \text{ for some } p \in [p]\}$.

The following notations are used. $\mathbb{IR}^{n\times n}$ is the set of $n \times n$ interval matrices (cf. [1]). For an interval matrix $[A] = [\underline{A}, \overline{A}] = \{A \mid \underline{A} \le A \le \overline{A}\}$, denote the mid-point matrix by $\check{A} = \frac{1}{2}(\underline{A} + \overline{A})$, the radius matrix by $\mathrm{rad}([A]) = \frac{1}{2}(\overline{A} - \underline{A})$ and the absolute value matrix by $|A| = (|a_{ij}|)$. These notations apply to vectors as well. For a bounded set Σ, define hull $\square(\Sigma) := [\inf\Sigma,\ \sup\Sigma]$. $\varrho(A)$ is the spectral radius of a matrix A. I denotes the unit matrix. Denote by $A([p]) := \square\{A(p) \in \mathbb{R}^{n\times n} \mid p \in [p]\}$, $b([p]) := \square\{b(p) \in \mathbb{R}^n \mid p \in [p]\}$ the corresponding non-parametric interval matrix and vector.

An iteration method for verified enclosure $\square(\Sigma(A(p), b(p), [p])) \subseteq [y] \in \mathbb{IR}^n$ is proposed in [4]. Rump's ***parametric*** iteration method and, to our knowledge, most methods for solving parametric interval linear systems require strong regularity of $A([p])$. Strong regularity of a non-parametric interval matrix is introduced in [1] and presents a sufficient condition for its regularity. In [2] we define strong regularity of a parametric interval matrix and give conditions that characterize it.

Definition 1.1 A parametric matrix $A(p) \in \mathbb{R}^{n\times n}$, $p \in [p] \in \mathbb{IR}^k$ is called strongly regular over $[p]$ if some of the matrices

$$[B] := \square\left\{A^{-1}(\check{p})A(p) \mid p \in [p]\right\} \qquad \text{or} \qquad [B'] := \square\left\{A(p)A^{-1}(\check{p}) \mid p \in [p]\right\} \qquad \text{is regular.} \tag{2}$$

The classes of column-, row-dependent parametric matrices, introduced in [2], show that the conditions for strong regularity of a parametric matrix give better estimations for its regularity than the conditions based on the non-parametric matrix.

Definition 1.2 A parametric matrix $A(p) \in \mathbb{R}^{n\times n}$, defined by (1), is called column-dependent parametric matrix if for some $m \in \{1,\ldots,k\}$ and for some $j \in \{1,\ldots,n\}$, $\mathrm{Card}(\mathcal{I}) \ge 2$, where $\mathcal{I} := \{i \mid 1 \le i \le n,\ a_{ij}^{(m)} \ne 0\}$.

A parametric matrix $A(p) \in \mathbb{R}^{n\times n}$ is called row-dependent if $A(p)^\top$ is column-dependent.

2 Verifiable sufficient regularity conditions generalizing Rump's parametric iteration

For the numerical computations, it is advantageous to have regularity conditions in terms of an arbitrary matrix R instead of the exact inverse $A^{-1}(\check{p})$.

Theorem 2.1 *Let $A(p) \in \mathbb{R}^{n\times n}$ with $p \in [p] \in \mathbb{IR}^k$ be a parametric matrix and $R \in \mathbb{R}^{n\times n}$. If some of the matrices $[U] := \square\{R\cdot A(p) \mid p \in [p]\}$, $[U'] := \square\{A(p)\cdot R \mid p \in [p]\}$ is an H-matrix, then $A(p)$ is strongly regular over $[p]$.*

Proof. If $[U]$ is an H-matrix, then by [1], Th. 3.7.5 (ii), $\check{U}$ is an H-matrix and $\varrho\left(\langle\check{U}\rangle^{-1}\mathrm{rad}([U])\right) < 1$, wherein $\langle\check{U}\rangle$ denotes the Ostrowski comparison matrix for $\check{U}$ (see [1]). By [1], Th. 3.7.5 (iii), we have $|\check{U}^{-1}| \leqq \langle\check{U}\rangle^{-1}$. Let $[V] := \square\left\{A^{-1}(\check{p})A(p) \mid p \in [p]\right\}$, then with $[p_\nu] = [\check{p} - \delta_\nu, \check{p} + \delta_\nu]$, $0 < \delta_\nu \in \mathbb{R}$, $\nu = 1,\ldots,k$,

$$\begin{aligned}\mathrm{rad}([V]) &= \sum_{\nu=1}^{k}\delta_\nu|A^{-1}(\check{p})A^{(\nu)}| = \sum_{\nu=1}^{k}\delta_\nu|A^{-1}(\check{p})R^{-1}RA^{(\nu)}| = \sum_{\nu=1}^{k}\delta_\nu|(RA(\check{p}))^{-1}RA^{(\nu)}| \\ &\leqq \sum_{\nu=1}^{k}\delta_\nu|(RA(\check{p}))^{-1}|\cdot|RA^{(\nu)}| = |(RA(\check{p}))^{-1}|\sum_{\nu=1}^{k}\delta_\nu|RA^{(\nu)}| = |\check{U}^{-1}|\mathrm{rad}([U]) \leqq \langle\check{U}\rangle^{-1}\mathrm{rad}([U]).\end{aligned}$$

* e-mail: epopova@bio.bas.bg, Phone: +359 2 979 3704, Fax: +359 2 971 3649

Hence $\varrho(\mathrm{rad}([V])) \leqq \varrho(\langle \check{U} \rangle^{-1}\mathrm{rad}([U])) < 1$ and, by Theorem 1 from [2], $A(p)$ is strongly regular over $[p]$. If U' is an H-matrix the proof goes analogously with the consideration $\varrho(AB) = \varrho(BA)$ for $A, B \in \mathbb{R}^{n\times n}$ ([1], Prop. 3.2.4 (9)). □

The next theorem, which in fact verifies strong regularity, can be used for the computational verification of the regularity of a parametric interval matrix.

Theorem 2.2 *Let $A(p) \in \mathbb{R}^{n\times n}$ with $p \in [p] \in \mathbb{IR}^k$. Let $R \in \mathbb{R}^{n\times n}$ be given and let*

$$[C(p)] := \{I - \square\{A(p)\cdot R \mid p \in [p]\} \text{ if } A(p) \text{ is row-dependent}, \quad I - \square\{R\cdot A(p) \mid p \in [p]\} \text{ otherwise}\}. \tag{3}$$

For $[x^0] \in \mathbb{IR}^n$ define the iteration $[x^{l+1}] := [C(p)] \diamondsuit [x^l] \oplus [e^l]$ for $l \in \mathbb{N}$, where $[e^l] \in \mathbb{IR}^n$, $[e^l] \to [e] \in \mathbb{IR}^n$ with $0 \in \mathrm{int}([e])$, int$([x])$ denoting the topological interior, and all operations $\oplus$, $\diamondsuit$ are outwardly-rounded computer interval operations. If $[C(p)]\diamondsuit [x^l] \subseteq \mathrm{int}([x^l])$ for some $l \in \mathbb{N}$, then R and every $A(p)$ with $p \in [p]$ are regular.

P r o o f. Theorem 1.5 from [4], with $[z] = 0$, implies that $\varrho(|[C(p)]|) < 1$. Therefore for every $p \in [p]$, $\varrho(I - R\cdot A(p)) \leq \varrho(|I - R\cdot A(p)|) < 1$. Hence R and every $A(p)$, $p \in [p]$ are regular. □

We use the above results to generalize Rump's parametric fixed-point iteration method [4].

Theorem 2.3 *Let $A(p)\cdot x = b(p)$ with $p \in [p] \in \mathbb{IR}^k$ be a parametric linear system, where $A(p), b(p)$ are defined by (1). Let $R \in \mathbb{R}^{n\times n}$, $[Y] \in \mathbb{IR}^n$, $\tilde{x} \in \mathbb{R}^n$ and define $[Z] \in \mathbb{IR}^n$, $[C(p)] \in \mathbb{IR}^{n\times n}$ by*

$$[Z] := R\cdot(b^{(0)} - A^{(0)}\tilde{x}) + \sum_{\nu=1}^{k}[p_\nu](R\cdot b^{(\nu)} - R\cdot A^{(\nu)}\cdot\tilde{x}), \qquad [C(p)] := I - R\cdot A^{(0)} - \sum_{\nu=1}^{k}[p_\nu](R\cdot A^{(\nu)}).$$

Define $[V] \in \mathbb{IR}^n$ by means of $1 \leq i \leq n$: $V_i := \{[Z] + [C(p)]\cdot[U]\}_i$, $[U] := (V_1, ..., V_{i-1}, Y_i, ..., Y_n)^\top$. If $[V] \subsetneqq [Y]$, then R and every matrix $A(p), p \in [p]$ are regular, and for every $p \in [p]$ the unique solution $\widehat{x} = A^{-1}(p)b(p)$ of the system satisfies $\widehat{x} \in \tilde{x} + [V]$.

With $[D] := [C(p)]\cdot[V] \in \mathbb{IR}^n$, the following inner estimation holds

$$[\tilde{x} + \inf([Z]) + \sup([D]),\ \tilde{x} + \sup([Z]) + \inf([D])] \subseteq \square(\Sigma(A(p), b(p), [p])).$$

P r o o f. It can be easily verified that $[C(p)] := I - \square\{A(p)\cdot R \mid p \in [p]\} = I - R\cdot A^{(0)} - \sum_{\nu=1}^{k}[p_\nu](R\cdot A^{(\nu)})$. Then Theorem 2.2 above and Theorem 4.8 from [4] finish the proof. □

Example 2.4 Consider $n\times n$ column-dependent parametric linear systems $Q(2,p)\cdot x = b(p)$, where for $i, j = 1, \ldots, n$,

$$q_{ij}(2,p) := \begin{cases} p_j & \text{if } i \leqq j, \\ 0 & \text{if } i = j+2, \\ 1 & \text{otherwise,} \end{cases} \qquad b(p) = (p_1, \ldots, p_n)^\top, \qquad p_i \in [(i+1) \pm 10\%].$$

Table 1 presents the growing spectral radius of $|[C]|$, where $[C] = I - R\cdot A([p])$ is the non-parametric iteration matrix. For $n = 2$ and $n = 3$, Rump's method converges in 3, resp. 6 iterations. For $n \leq 50$, $\varrho(|[C(p)]|) = 0.2$ and the generalized method converges in 2 iterations.

Table 1 The spectral radius of the absolute value of the non-parametric iteration matrix for the Example 2.4.

n	2	3	4	5	10	15	20	25	30	35	40	45	50
$\varrho(\lvert[C]\rvert)$	0.46	0.81	1.69	1.95	3.07	3.24	3.40	3.58	3.76	3.96	4.17	4.38	4.59

The new narrow class of parametric iteration matrices $[C(p)]$, defined by (3), extends the class of matrices $A(p)$ that can be proven to be regular. This way, the generalized parametric fixed-point iteration gets an expanded scope of applications. An implementation of the generalized method is described in [3].

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