Bounding the Response of Mechanical Structures with Uncertainties in All the Parameters *

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Abstract. The application of a general-purpose self-verified parametric iteration for bounding the response of mechanical systems involving rational dependencies between interval parameters is investigated. Based on the availability of self-validated parametric linear solver, a general framework of computer-assisted proof of global and local monotonicity properties is presented. By the discussed methodology and software tools some frame structures with uncertainties in cross-sectional properties, applied loadings, material properties, geometry and connections are analyzed. The results are compared to literature data produced by other methods and a comparison of different measures of overestimation is done.

Keywords: self-verified methods, structural frames, parameter-dependent interval linear systems

1. Introduction

Uncertainty quantification is an emerging discipline which is nowadays well recognized by SIAM and structural engineering community. One of the research directions in this field utilizes intervals for representing the uncertain quantities and interval-based methods for reliable bounding the model response under variations in the uncertain parameters.

Many mechanical problems, e.g. linear static problems, modelled by finite element method, can be described by systems of linear equations involving uncertain model parameters. When the uncertain parameters are introduced by bounded intervals, the problem can be transformed into an interval linear system which should be solved appropriately to bound the mechanical system response. This approach is usually called Interval Finite Element Method. Overview of recent developments in the area of uncertainty treatment using interval finite element methods and their applications in structural engineering mechanics can be found in (Muhanna et al., 2004), (Muhanna et al., 2005). Although known for a decade, a self-validated parametric iteration method (Rump, 1994) is not adopted (even for a comparison purpose) and has single mechanical applications (Dessombz et al., 2001), (Popova et al., 2003). Instead, a construction method, called Element-By-Element approach (Mullen and Muhanna, 1999), is developed which introduces extra variables

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and equations in order to eliminate the dependencies between interval parameters. The penalty and Lagrange multiplier methods are used to impose the necessary constraints for compatibility and equilibrium (Muhanna and Mullen, 2001), (Muhanna et al., 2005). Non-parametric interval fixed-point iteration is modified and used to solve the model parametric interval linear system. During this transformation of the original parametric system, self-verifying properties of the interval iteration are lost or delayed to the final phase of solving non-parametric interval linear system. Recently, accounting for the structure of input data in systems related to truss structures, by splitting the iteration into two parts, Neumaier and Pownuk (2005) achieved an advance in self-verified methods applied to truss structures. Assuming a particular structure of the dependencies their method removes the restriction of most self-validating methods for linear systems to have a strongly regular matrix.

Depending on what model is adopted and which model parameters are considered to be uncertain or how they are involved into the interval linear system to be solved, the latter can be classified into two types: parametric linear systems involving affine-linear dependencies between the parameters and parametric linear systems involving arbitrary nonlinear dependencies between the interval parameters. So far mainly problems involving affine-linear dependencies have been solved. In this work we come back to the *parametric* fixed-point iteration, initially introduced by S. Rump (1994), and first time apply it for bounding the response of structural engineering systems involving nonlinear dependencies between the model parameters. In (Popova, 2005) the inclusion method is combined with a simple interval arithmetic technique providing inner and outer bounds for the range of monotone rational functions. The arithmetic on proper and improper intervals (Gardeñes et al., 2001) is considered as an intermediate computational tool for eliminating the dependency problem in range computation and for obtaining inner estimations by outwardly rounded interval arithmetic. This methodology is implemented into a number of supporting software tools with result verification, developed in the environment of *Mathematica*, (Popova, 2005).

Combinatorial approach and the monotonicity approach have been favored by many authors in solving linear elastic problems involving particular uncertain parameters (Rao and Berke, 1997), (Ganzerli and Pantelides, 1999), (McWilliam, 2000), (Pownuk, 2000). A rigorous application of these approaches requires validation of their assumptions which are not generally valid. In Section 2.2 of this paper we present a general framework of computer-aided proof of global and local monotonicity properties of parametric solutions provided that a self-verified solver of parametric linear systems is available.

A recent work (Corliss et al., 2004), see also (Corliss and Foley, 2005), identified typical parameter uncertainties in finite element models of structural steel frames with partially constrained connections and by applying a sequence of interval-based methods the response of a simple one-bay steel frame to variations in cross-sectional properties, loading, material properties, and connections is bounded. Taking occasion of the appeal at the end of the presentation (Corliss and Foley, 2005) for other reliable methods solving parameter-dependent linear systems, in Section 3.1 this work we expand the structural analysis performed in (Corliss et al., 2004) by application of the self-verified parametric iteration, a rigorous hybrid monotonicity approach, and interval subdivision technique to the same problem and to a larger structural steel frame. The goal is to increase the awareness of the engineering community about the variety of interval-based methods with result verification that can be used in the analysis of mechanical structures involving uncertain parameters.

The paper is organized in two parts. Section 2 briefly describes the methodological and software tools that are used in the second part. Section 3 contains the analysis of of structural frames.

2. Methodology and Software Tools

In this section we give a brief summary of the numerical methods and software tools that will be used in solving linear elastic mechanical problems with uncertainties in all the model parameters. The methods have general purpose and do not assume any particular structure of the input data.

Consider linear algebraic system

$$A(p) \cdot x \ = \ b(p), \tag{1a}$$

where $A(p)$ is an $n \times n$ matrix, $b(p)$ is an n-dimensional vector and $p = (p_1, \ldots, p_k)^\top$ is a k-dimensional parameter vector. The elements of $A(p)$ and $b(p)$ are, in general, nonlinear functions of the parameters

$$a_{ij}(p) \ = \ a_{ij}(p_1, \ldots, p_k), \tag{1b}$$

$$b_i(p) \ = \ b_i(p_1, \ldots, p_k), \qquad i, j = 1, \ldots, n. \tag{1c}$$

The parameters are considered to be unknown or uncertain and varying within prescribed intervals

$$p \ \in \ [p] = ([p_1], \ldots, [p_k])^\top. \tag{1d}$$

When the parameters vary within a box $[p] \in \mathbb{IR}^k$ the set of solutions, called parametric solution set is

$$\Sigma^p = \Sigma\left(A(p), b(p), [p]\right) \ := \ \{x \in \mathbb{R}^n \mid A(p) \cdot x = b(p) \text{ for some } p \in [p]\}\,. \tag{2}$$

In general, a solution set has very complicated structure, and does not need even to be convex. The parametric solution set Σ^p is bounded if $A(p)$ is nonsingular for every $p \in [p]$. For a nonempty bounded set $\Sigma \subseteq \mathbb{R}^n$, define interval hull $\Box : P\mathbb{R}^n \to \mathbb{IR}^n$ by

$$\Box\, \Sigma \ := \ [\inf \Sigma,\ \sup \Sigma] \ = \ \cap\{[x] \in \mathbb{IR}^n \mid \Sigma \subseteq [x]\}.$$

Since it is quite expensive to obtain Σ^p or $\Box\, \Sigma^p$, the solution of interest is seeking an interval vector $[y] \in \mathbb{IR}^n$ such that $[y] \supseteq \Box\, \Sigma^p \supseteq \Sigma^p$, and the goal is $[y]$ to be as narrow as possible.

Below we use the following notations. $\mathbb{R}^n, \mathbb{R}^{n\times m}$ denote the set of real vectors with n components and the set of real $n \times m$ matrices, respectively. By normal (*proper*) interval we mean a real compact interval $[a] = [a^-, a^+] := \{a \in \mathbb{R} \mid a^- \leq a \leq a^+\}$. By $\mathbb{IR}^n, \mathbb{IR}^{n\times m}$ we denote interval n-vectors and interval $n \times m$ matrices. The end-point functionals $(\cdot)^-, (\cdot)^+$, the mid-point function mid$(\cdot)$, where mid$([a^-, a^+]) := (a^- + a^+)/2$, and the diameter (width) function $\omega(\cdot)$, where $\omega([a^-, a^+]) := a^+ - a^-$, are applied to interval vectors and matrices componentwise. The absolute value of a matrix $A = (a_{ij})$ is denoted by $|A| = (|a_{ij}|)$; for $[a] \in \mathbb{IR}$, $|[a]| := \max\{|a| \mid a \in [a]\}$. For two matrices of the same size matrix (vector) inequalities $A \leq B$ and the interval subset relations $[A] \subseteq [B]$ are understood componentwise. $A < B$ if $A \leq B$ and $A \neq B$, analogously $[A] \subset [B]$ if $[A] \subseteq [B]$ and $[A] \neq [B]$. The above matrix notations apply to vectors, considered as one-column matrices, as well. $\varrho(A)$ is the spectral radius of a matrix A, I denotes the identity matrix. For interval quantities $[A], [B]$, operations between them are always interval operations. The result is the smallest interval quantity containing the corresponding result when using power set operations. For example,

$$[A] \in \mathbb{IR}^{n\times n},\ [b] \in \mathbb{IR}^n\ :\ [A]\cdot[b] := \cap\{[c] \in \mathbb{IR}^n \mid \forall\, a \in [A], \forall\, b \in [b]\ :\ a\cdot b \in [c]\}.$$

We assume the reader is familiar with conventional interval arithmetic, cf. (Moore, 1979), (Neumaier, 1990).

2.1. Inclusion Theorems

The inclusion theorems for the solution set of a parametric linear system given here present a direct consequence from the inclusion theory for nonparametric problems developed by S. Rump and discussed in many works, cf. (Rump, 1986; Rump, 1990; Rump, 1994). The basic idea of combining the Krawczyk-operator and the existence test by Moore was further elaborated by S. Rump (1986) who proposed several improvements leading to inclusion theorems for the solution of nonparametric interval linear systems $[A]\cdot x = [b]$. Computing verified inclusions for the solution set of an interval linear system with data dependencies was first considered by C. Jansson (1991). He treated systems with symmetric and skew-symmetric matrices, as well as dependencies in the right hand side, by modifying the general nonparametric inclusion theorem to account for the dependencies in the system. In (Rump, 1994, Theorem 4.8) S. Rump gives a straightforward generalization to affine-linear dependencies in the matrix and the right hand side. The affine-linear dependencies between the parameters in $A(p), b(p)$ allow an explicit representation of the ranges of the residual vector $z(p) := R\cdot(b(p) - A(p)\cdot\tilde{x})$ and the iteration matrix $C(p) := I - R\cdot A(p)$ by interval expressions, as it is stated by the following theorem.

Theorem 2.1. Consider parametric linear system (1a) where $A(p), b(p)$ are defined by

$$a_{ij}(p) := a_{ij}^{(0)} + \sum_{\nu=1}^{k} p_\nu a_{ij}^{(\nu)}, \qquad b_i(p) := b_i^{(0)} + \sum_{\nu=1}^{k} p_\nu b_i^{(\nu)}, \quad i,j = 1,\dots,n.$$

Let $R \in \mathbb{R}^{n\times n}$, $[y] \in \mathbb{IR}^n$, $\tilde{x} \in \mathbb{R}^n$ be given and define $[z] \in \mathbb{IR}^n$, $[C] \in \mathbb{IR}^{n\times n}$ by

$$\begin{aligned}
[z] &:= R\cdot(b^{(0)} - A^{(0)}\tilde{x}) + \sum_{\nu=1}^{k}[p_\nu](R\cdot b^{(\nu)} - R\cdot A^{(\nu)}\cdot\tilde{x}),\\
[C] &:= I - R\cdot A^{(0)} - \sum_{\nu=1}^{k}[p_\nu](R\cdot A^{(\nu)}),
\end{aligned}$$

where $A^{(0)} := \left(a_{ij}^{(0)}\right),\ \dots, A^{(k)} := \left(a_{ij}^{(k)}\right) \in \mathbb{R}^{n\times n},\ b^{(0)} := (b_i^{(0)}),\dots,b^{(k)} := (b_i^{(k)}) \in \mathbb{R}^n$.
Define $[v] \in \mathbb{IR}^n$ by means of the following Einzelschrittverfahren

$$1 \le i \le n\ :\ [v_i] := \{[z] + [C]\cdot[u]\}_i, \qquad u := (v_1, ..., v_{i-1}, y_i, ..., y_n)^\top.$$

If $[v] \subsetneq [y]$, then R and every matrix $A(p), p \in [p]$ are regular, and for every $p \in [p]$ the unique solution $\hat{x} = A^{-1}(p)b(p)$ of (1a) satisfies $\hat{x} \in \tilde{x} + [v]$.

The above theorem generalizes Theorem 4.8 from (Rump, 1994) by requiring computation of the range of $C(p)$ instead of using an interval extension $C([p])$, cf. (Popova, 2004c). Although a sharp enclosure of the iteration matrix is used also by other authors (Dessombz et al., 2001; Muhanna et al., 2005), the necessity of this improvement is not well justified therein. The generalization of Theorem 4.8 from (Rump, 1994) is proven theoretically and demonstrated by several numerical examples in (Popova, 2004b; Popova, 2004c). Indeed, for a class of so-called column-dependent parametric matrices (Popova, 2004b), the following relation holds

$$[Cp] := \square\{C(p) \mid p \in [p]\} \ \subset \ C([p]) =: [C],$$

which implies $|[Cp]| < |[C]|$. If in addition, $|[Cp]| + |[C]|$ is irreducible, from the theory of nonnegative matrices it follows that $\varrho(|[Cp]|) < \varrho(|[C]|)$. Thus the range enclosure of $C(p)$ will provide convergence of the iteration method for $\varrho(|[Cp]|) < 1$, while a worse enclosure (e.g. $C([p])$) may not for some column-dependent parametric matrices and some interval domains for the parameters. Examples demonstrating the expanded scope of application of the generalized Theorem 2.1 can be found in (Popova, 2004b; Popova, 2004c; Popova and Krämer, 2004).

In case of arbitrary nonlinear dependencies between the parameters of a linear system we can give only a general formulation of the inclusion theorem, as bellow.

Theorem 2.2. Consider parametric linear system defined by (1a–1d). Let $R \in \mathbb{R}^{n\times n}$, $[y] \in \mathbb{IR}^n$, $\tilde{x} \in \mathbb{R}^n$ be given and define $[z] \in \mathbb{IR}^n$, $[C] \in \mathbb{IR}^{n\times n}$ by

$$\begin{aligned} [z] \ &:= \ \square\{R\,(b(p) - A(p)\tilde{x}) \mid p \in [p]\}, \\ [C] \ &:= \ \square\{I - R\cdot A(p) \mid p \in [p]\}. \end{aligned}$$

Define $[v] \in \mathbb{IR}^n$ by means of the following Einzelschrittverfahren

$$1 \le i \le n \ : \ [v_i] := \{[z] + [C]\cdot[u]\}_i, \qquad u := (v_1, ..., v_{i-1}, y_i, ..., y_n)^\top.$$

If $[v] \subsetneqq [y]$, then R and every matrix $A(p)$ with $p \in [p]$ are regular, and for every $p \in [p]$ the unique solution $\widehat{x} = A^{-1}(p)b(p)$ of (1a–1d) satisfies $\widehat{x} \in \tilde{x} + [v]$.

In case of arbitrary nonlinear dependencies between the uncertain parameters in a system, computing $[z]$ and $[C]$ in Theorem 2.2 requires sharp range enclosure for nonlinear functions. This is a key problem in interval analysis and there exists a variety of methods and techniques devoted to this problem. The quality of the range enclosure for $z(p) := R\cdot(b(p) - A(p)\cdot\tilde{x})$ will determine the sharpness of the parametric solution set enclosure. The verification iteration based on Theorem 2.2 will be convergent if the interval matrix $\square\{R\cdot A(p) \mid p \in [p]\}$ is regular which we call strong regularity of the parametric matrix $A(p)$ in the domain $[p]$, following the term initially introduced in (Neumaier, 1990). Since the left preconditioning introduces an affine transformation on the columns of $A(p)$, only systems with column-dependent parametric matrices may benefit from a sharper enclosure of $C(p) = I - R\cdot A(p)$.

In (Popova, 2005) the above inclusion theorem is combined with a simple interval arithmetic technique providing inner and outer bounds for the range of monotone rational functions. The arithmetic of generalised (proper and improper) intervals is considered as an intermediate computational tool for eliminating the dependency problem in range computation and for obtaining inner estimations by outwardly rounded interval arithmetic (Gardeñes et al., 2001). A detailed presentation of this technique and the corresponding algorithm with result verification, which solves linear systems whose input data are rational functions of interval parameters, can be found in (Popova, 2005). This methodology, rigorously implemented in software tools presented in Section 2.4, will be used in Section 3 for solving linear systems obtained by FE modelling of mechanical structures with uncertainties in all the parameters determining the structure behavior.

The above theorems define how to compute an outer enclosure of the solution set of an interval linear system, i.e. an interval vector which is verified to contain the exact solution set hull, respectively the true solution set of the system. However, it is important to know the quality of the computed enclosure, in other words: how much such an enclosure overestimates the exact hull of the solution set. The amount of overestimation can be approximated by an inner inclusion of the solution set hull which is a componentwise inner estimation of the solution set (Neumaier, 1987; Rump, 1990).

Definition 2.1. An interval vector $[x] \in \mathbb{IR}^n$ is called componentwise inner approximation for some set $\Sigma \in \mathbb{R}^n$ if

$$\inf_{\sigma\in\Sigma} \sigma_i \leq x_i^- \quad \text{and} \quad x_i^+ \leq \sup_{\sigma\in\Sigma} \sigma_i, \qquad \text{for every } 1 \leq i \leq n.$$

The interval vector $[x]$ from the above definition is an inner inclusion of the solution set hull and should be distinguished from an inner inclusion of the solution set, that is $[x] \subseteq [\inf(\Sigma), \sup(\Sigma)]$ but $[x] \not\subseteq \Sigma$.

Basing on ideas developed in (Neumaier, 1987), a cheap method for computing rigorous inner inclusion of the solution set hull is proposed in (Rump, 1990). The next theorem establishes how to compute the componentwise inner estimation of the parametric solution set.

Theorem 2.3. Let $A(p) \cdot x = b(p)$, where $A(p) \in \mathbb{R}^{n\times n}$, $b(p) \in \mathbb{R}^n$, $p \in [p] \in \mathbb{IR}^k$, and $R \in \mathbb{R}^{n\times n}$, $\tilde{x} \in \mathbb{R}^n$, $[y] \in \mathbb{IR}^n$ be given. Define

$$\begin{aligned} [z] &:= \square\{R\cdot(b(p) - A(p)\cdot\tilde{x}) \mid p \in [p]\}, \\ [\Delta] &:= [C]\cdot[y], \quad \text{where } [C] := \square\{I - R\cdot A(p) \mid p \in [p]\}. \end{aligned}$$

Let the solution set $\Sigma^p = \Sigma(A(p), b(p), [p])$ be defined as in (2) and assume

$$[z] + [\Delta] \subsetneqq [y].$$

Then

$$[\tilde{x} + [z]^- + [\Delta]^+,\ \tilde{x} + [z]^+ + [\Delta]^-] \subseteqq \square\Sigma^p \subseteqq \tilde{x} + [z] + [\Delta]$$

or, in coordinate notations, for all $i = 1, \ldots, n$ there exists $x^-, x^+ \in \Sigma^p$ with

$$\begin{aligned} \tilde{x}_i + [z_i]^- + [\Delta_i]^- \leqq x_i^- \leqq \tilde{x}_i + [z_i]^- + [\Delta_i]^+ \quad &\text{and} \\ \tilde{x}_i + [z_i]^+ + [\Delta_i]^- \leqq x_i^+ \leqq \tilde{x}_i + [z_i]^+ + [\Delta_i]^+. & \end{aligned}$$

In order to have a guaranteed inner inclusion all the computations should be done in computer arithmetic with directed roundings, cf. (Popova, 2005).

The method from Theorem 2.3 has its limits. When widening the intervals for the parameters, respectively the interval components of the linear system, the inner inclusion becomes smaller and smaller, and finally vanishes. The latter means that no quantitative measure for the quality of the outer enclosure can be given. For wide parameter intervals empty inner inclusion usually means bad outer enclosure and, when further widening the input intervals, the outer solution enclosure will fail at a certain point. Numerical examples demonstrating this effect can be found in (Popova and Krämer, 2004). The same result of empty inner inclusion intervals can be obtained also for very tight parameter intervals due to the rounding errors in computing inner approximations. A necessary and sufficient condition for non-empty inner inclusions is provided by the relation $\omega([\Delta_i]) \leqq \omega([z_i])$, where the notations are as in Theorem 2.3, $[\Delta_i]$ is computed with outward rounding and $[z_i]$ is computed with inward rounding.

When somehow we have sharpen the outer solution enclosure $\square\,\Sigma^p \subseteq [\hat{v}] \subseteq [v] = \tilde{x} + [z] + [\Delta]$, then the improved outer estimation $[\hat{v}]$ can replace $[v]$ in Theorem 2.3 to get an improved inner estimation of $\square\,\Sigma^p$. Numerical example demonstrating this property can be found in (Popova, 2001).

2.2. Rigorous Monotonicity Approach

For many mechanical systems the exact bounds of the system response can be obtained by the so-called combinatorial approach. The combinatorial solution is computed as a convex hull of the solutions to all point linear systems corresponding to an exhaustive combination of the bounds of the interval parameters. Combinatorial hull is a quality of particular parametric solution sets which is not valid in general. The combinatorial approach gives the exact solution set hull in exact arithmetic in the special case when the parametric solution is monotone with respect to all the parameters. If the combinatorial hull property is not proven theoretically (as by Neumaier and Pownuk (2005)) or numerically (as below), any other non-rigorous application of combinatorial or monotonicity approach would result in an interval box underestimating the true parametric solution set. This is the reason by which combinatorial and monotonicity approaches are usually referred as methods giving inner inclusion of the solution set hull (Muhanna et al., 2005; Neumaier and Pownuk, 2005).

In this section we briefly sketch a rigorous application of the combinatorial/monotonicity approach within a general framework for solving parametric linear systems. The rigorousness is provided by computer-assisted numerical proofs of global and local monotonicity properties of the parametric solution. Since an essential ingredient of this approach is a self-verified solver for parametric linear systems, we call this approach a rigorous hybrid monotonicity approach (Popova, 2004a).

The general framework of the rigorous hybrid monotonicity approach consists of three basic components:

1. self-verified solver for parametric linear systems;
2. computer-assisted proof of global and local monotonicity properties of a parametric solution;
3. guaranteed solution enclosure for point linear systems.

Provided that we have a self-verified solver for parametric linear systems, we can verify the global and local monotonicity properties of the parametric solution $x(p) = A(p)^{-1} \cdot b(p)$. Below we use the following notations. For $[a] = [a^-, a^+] \in \mathbb{IR}$, define $\mathrm{sign}([a]) = \{1 \text{ if } a^- \geq 0,\ -1 \text{ if } a^+ \leq 0, 0 \text{ if } a^-a^+ < 0\}$. For a set of indices $\mathcal{I} = \{i_1, \ldots, i_n\}$, the vector $(x_{i_1}, \ldots, x_{i_n})^\top$ will be denoted by $x_\mathcal{I}$ and $[x_\mathcal{I}] = [x_\mathcal{I}^-, x_\mathcal{I}^+]$ where $x_\mathcal{I}^- = (x_{i_1}^-, \ldots, x_{i_n}^-)^\top$, $x_\mathcal{I}^+ = (x_{i_1}^+, \ldots, x_{i_n}^+)^\top$.

The global monotonicity properties are verifiable by solving k parametric linear systems in the global domain $[p] \in \mathbb{IR}^k$

$$A(p)\frac{\partial x}{\partial p_\nu} \ = \ \frac{\partial b(p)}{\partial p_\nu} \ - \ \frac{\partial A(p)}{\partial p_\nu} \cdot [x^*], \qquad \nu = 1, \ldots, k, \tag{3}$$

where $[x^*] \supseteq \Sigma^p$ is an initial enclosure of the parametric solution set. Let us suppose that for fixed i, $1 \leq i \leq n$ there exist index sets

$$L_+ = \{\nu \mid \mathrm{sign}\left[\frac{\partial x_i}{\partial p_\nu}\right] = 1\}, \qquad L_- = \{\nu \mid \mathrm{sign}\left[\frac{\partial x_i}{\partial p_\nu}\right] = -1\}.$$

If $L_- \cup L_+ = \{1, \ldots, k\}$, then

$$[\inf \Sigma_i^p,\ \sup \Sigma_i^p] \ = \ [\{A^{-1}(p_{L_+}^-, p_{L_-}^+) \cdot b(p_{L_+}^-, p_{L_-}^+)\}_i,\ \{A^{-1}(p_{L_+}^+, p_{L_-}^-) \cdot b(p_{L_+}^+, p_{L_-}^-)\}_i].$$

Monotonicity can also be used even when some solution components are not globally monotonic with respect to some parameters. Suppose that for some i, $1 \leq i \leq n$, there exist index sets

$$L_+ = \{\nu \mid \mathrm{sign}\left[\frac{\partial x_i}{\partial p_\nu}\right] = 1\}, \qquad L_- = \{\nu \mid \mathrm{sign}\left[\frac{\partial x_i}{\partial p_\nu}\right] = -1\}, \qquad L_0 = \{\nu \mid \mathrm{sign}\left[\frac{\partial x_i}{\partial p_\nu}\right] = 0\},$$

such that $L_0 \neq \{1, \ldots, k\}$ and $L_0 \neq \emptyset$. Consider two new parametric linear systems

$$A^-(p_{L_0}) \cdot y \ = \ b^-(p_{L_0}) \tag{4}$$

$$A^+(p_{L_0}) \cdot z \ = \ b^+(p_{L_0}), \tag{5}$$

wherein

$$\begin{aligned} a_{ij}^-(p_{L_0}) \ &:= \ a_{ij}(p_{L_+}^-, p_{L_-}^+, p_{L_0}), \qquad & b_i^-(p_{L_0}) \ &:= \ b_i(p_{L_+}^-, p_{L_-}^+, p_{L_0}) \\ a_{ij}^+(p_{L_0}) \ &:= \ a_{ij}(p_{L_+}^+, p_{L_-}^-, p_{L_0}) \qquad & b_i^+(p_{L_0}) \ &:= \ b_i(p_{L_+}^+, p_{L_-}^-, p_{L_0}) \end{aligned}$$

for $i, j = 1, \ldots, n$ and $p_{L_0} \in [p_{L_0}]$.

Let $[y^*] \supseteq \Sigma\,(A^-(p_{L_0}), b^-(p_{L_0}), [p_{L_0}])$ and $[z^*] \supseteq \Sigma\,(A^+(p_{L_0}), b^+(p_{L_0}), [p_{L_0}])$. In general,

$$[\inf \Sigma_i^p,\ \sup \Sigma_i^p] \ \subseteq \ [y_i^*] \cup [z_i^*].$$

However, we may prove some monotonicity properties of the parametric solutions to (4), (5) by solving the corresponding parametric derivative systems in a considerably reduced interval domain $[p_{L_0}]$.

$$\begin{aligned} A^-(p_{L_0})\frac{\partial y}{\partial p_\nu} \ &= \ \frac{\partial b^-(p_{L_0})}{\partial p_\nu} \ - \ \frac{\partial A^-(p_{L_0})}{\partial p_\nu} \cdot [y^*] \\ A^+(p_{L_0})\frac{\partial z}{\partial p_\nu} \ &= \ \frac{\partial b^+(p_{L_0})}{\partial p_\nu} \ - \ \frac{\partial A^+(p_{L_0})}{\partial p_\nu} \cdot [z^*], \end{aligned}$$

for all $\nu \in L_0$, where $[y^*], [z^*]$ are initial enclosures of the solution sets of (4), resp. (5), or an initial enclosure of $\Sigma(A(p), b(p), [p])$.

This way, a computer-aided proof of global and local monotonicity properties of the parametric solution can be performed by self-validated solving of parametric linear systems. The success of the numerical proof depends very much on the quality of the parametric solution enclosure and on the quality of the initial enclosure (Popova, 2004a). Some specific issues related to this approach will be discussed in a separate work.

2.3. Measures of Overestimation

The quality of a solution enclosure is measured by estimating how much an outer solution enclosure overestimates the true parametric solution set or an inner inclusion of the solution set hull (since the true hull is usually not known). A discussion about different methods used for obtaining inner hull estimations can be found in (Neumaier and Pownuk, 2005). The inclusion method, presented in Section 2.1, is equipped with an easy computable guaranteed inner estimation of the solution set hull. In this work we shall measure the overestimation of the outer solution enclosure with respect to a combinatorial solution and to the guaranteed inner estimation of the solution hull provided by the method.

Provided that we have computed the exact solution set hull or some inner estimation(s) of the hull, the amount of overestimation should be quantified. The endeavor of providing sharper solution enclosures has resulted in utilization of different measures of overestimation. In the next section we shall use and compare the quality quantifications provided by the following measures of overestimation.

For two intervals $[a], [b] \in \mathbb{IR}$ such that $[a] \subseteq [b]$, the standard measure of overestimation that is usually applied is the percentage by which $[b]$ overestimates the interval $[a]$, defined as $\mathcal{O}_\omega$: $\mathbb{IR} \times \mathbb{IR} \longrightarrow \mathbb{R}_+$

$$\mathcal{O}_\omega([a], [b]) \ := \ 100(1 - \omega([a])/\omega([b])).$$

Distance-based measures of overestimation are sometimes used in the engineering literature, e.g. (Muhanna et al., 2005). $\mathcal{O}_d : \mathbb{IR} \times \mathbb{IR} \longrightarrow \mathbb{R} \times \mathbb{R}$ is defined by

$$\mathcal{O}_d([a], [b]) \ := \ 100\left(1 - a^-/b^-,\ 1 - a^+/b^+\right).$$

Since we will compare part of our results to those obtained in (Corliss et al., 2004), we will need the measure of overestimation used therein. For $[a], [b] \in \mathbb{IR}$, $[a] \subseteq [b]$ and $c \in \mathbb{R}$, $c \in [a]$, define $\mathcal{O}_c : \mathbb{IR} \times \mathbb{IR} \times \mathbb{R} \longrightarrow \mathbb{R}_+$ by

$$\mathcal{O}_c([a], [b], c) \ := \ 100\left(b^- - a^- + a^+ - b^+\right)/c.$$

Overestimation measures are applied to interval vectors componentwise.

The presented parametric fixed-point method provides a guaranteed inner estimation $[v]$ of the solution hull $[h]$ at no additional cost. Since the computation of $[v]$ uses the computed outer enclosure $[u]$ in a "symmetric " way, it can be expected that $[v]$ is almost symmetric to $[u]$ with respect to the exact solution set hull. That is why, $\frac{1}{2}\mathcal{O}_w([v], [u]) \approx \mathcal{O}_w([h], [u])$ will be used for measuring the quality of a solution enclosure.

2.4. Software Tools

Interval methods discussed in this paper and elsewhere are implemented in the environment of *Mathematica* (Wolfram, 1999). The *Mathematica* package `IntervalComputations 'LinearSystems'` contains a collection of functions which compute guaranteed inclusions for the solution set of an interval linear system (Popova, 2004a). The particular solvers differ upon the type of the linear system to be solved and the implemented solution method. Except for a C-XSC module solving parametric linear systems with affine-linear dependencies (Popova and Krämer, 2004), the above *Mathematica* package is the only by now public software for solving parameter-dependent interval linear systems.

`ParametricNSolve[Ap, bp, tr]` is the function which solves linear systems involving affine-linear dependencies between interval parameters. The function is based on entirely numerical computations and therefore it is fast. The function is updated to handle sparse arrays as input data.

`ParametricSSolve[Ap, bp, tr]` computes a guaranteed enclosure of the solution set to a parametric linear system `Ap.x = bp` involving rational dependencies by the algorithm presented in (Popova, 2005). The parameters and their interval values are specified by a list `tr` of transformation rules[1]. All iterative solvers can take two optional arguments affecting the computational process, respectively the output of the function. `InnerEstimation` is an option which when set to `True` specifies the computing of component-wise inner approximation of the solution set in addition to the outer enclosure. The option is set to `False` by default. Even set to `True`, the option is active only if the *Mathematica* package `IntervalComputations 'GeneralisedIntervals'` is available. `Refinement` is an option which set to `True` implies an iterative refinement procedure applied to the computed outer solution enclosure. The default setting is `False`.

Due to a previous improvement of the inclusion theory, new functions generating guaranteed inclusions of the solutions to nonsquare over-/underdetermined (parametric) linear systems are developed. Several functions supporting the hybrid monotonicity approach and a subdivision strategy ar also part of the package.

Approaching to parametric linear systems with rational dependencies, the integration of symbolic-algebraic and self-validating numerical computations based on interval arithmetic is found to be a fruitful synergism. The power of *Mathematica* to support rigorous exact and/or variable precision interval computations, the functionality of a generalized interval arithmetic package and the tools provided by the other interval packages, make a suitable environment for exploration and solving parametric problems with interval uncertainties.

In order to provide a broad access to solvers for parametric interval linear systems a web interface for the available *Mathematica* software is developed which can be found at

http://cose.math.bas.bg/webComputing/

Accessing the webComputing pages users enter or upload data, choose between different options, and submit data to build up a sequence of results in a numeric, symbolic, graphics or combined form. The end-users do not need to buy, install, and maintain software; they do not need to develop user software or to learn different software applications training time being considerably reduced. They can be certain that use the most recent version. The technical professionals and

[1] *Mathematica* transformation rules have the form `name -> value`.

interval researchers can easily explore newly developed methods; compare the efficiency of different methods and software tools; teach interval methods involving students in an active exploration by doing. Since algebraic computations are time consuming and web*Mathematica* applications have a fixed time limit for using the *Mathematica* kernel, the nonlinear parametric web solver is suitable only for small size problems, while large problems involving affine-linear dependencies can be solved remotely. The parametric web solvers allow uploading data files from the client machine onto the server. For a parametric system, 3 data files (containing the matrix, the right-hand side vector and the rules for the parameters) are required. Present restriction to the maximum size of a data file is 4MB. Matrix/vector data in a file presently should be specified by *Mathematica* lists, or as sparse arrays (Wolfram, 1999). Future enhancement of the solvers include different data formats, downloading the generated results on the client machine and combining/reusing the results from different pages.

3. Numerical Examples

3.1. One-Bay Steel Frame

In this section we consider a simple one-bay structural steel frame, shown in Figure 1, that was initially considered and analyzed by Corliss et al. (2004). In their work the authors survey typical uncertainties for the parameters characterizing the structural behavior and apply the Muhanna-Mullen Element-by-Element approach (Muhanna and Mullen, 2001), interval subdistributivity properties, scaling, and constraint propagation in order to demonstrate the feasibility of interval techniques for bounding structural responses in the presence of interval parameters. Here the analysis of Corliss et al. is expanded by the methods presented in Section 2.

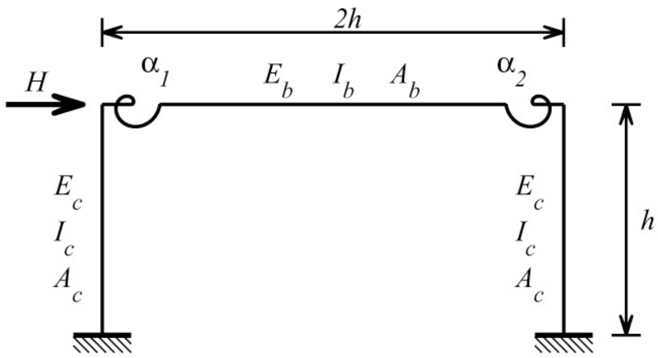

Figure 1. One-bay structural steel frame (after Corliss et al. (2004)).

In order to compare the results generated by the different methods, we strictly follow the structure system and the uncertainties for the parameters considered in(Corliss et al., 2004). Following the usual practice, the authors have assembled the following linear system corresponding to the portal structure in Figure 1.

$$\begin{pmatrix}
\frac{A_bE_b}{L_b}+\frac{12E_cI_c}{L_c^3} & 0 & \frac{6E_cI_c}{L_c^2} & 0 & 0 & -\frac{A_bE_b}{L_b} & 0 & 0 \\
0 & \frac{A_cE_c}{L_c}+\frac{12E_bI_b}{L_b^3} & 0 & \frac{6E_bI_b}{L_b^2} & \frac{6E_bI_b}{L_b^2} & 0 & -\frac{12E_bI_b}{L_b^3} & 0 \\
\frac{6E_cI_c}{L_c^2} & 0 & \alpha+\frac{4E_cI_c}{L_c} & -\alpha & 0 & 0 & 0 & 0 \\
0 & \frac{6E_bI_b}{L_b^2} & -\alpha & \alpha+\frac{4E_bI_b}{L_b} & \frac{2E_bI_b}{L_b} & 0 & -\frac{6E_bI_b}{L_b^2} & 0 \\
0 & \frac{6E_bI_b}{L_b^2} & 0 & \frac{2E_bI_b}{L_b} & \alpha+\frac{4E_cI_c}{L_c} & 0 & -\frac{6E_bI_b}{L_b^2} & -\alpha \\
-\frac{A_bE_b}{L_b} & 0 & 0 & 0 & 0 & \frac{A_bE_b}{L_b}+\frac{12E_cI_c}{L_c^3} & 0 & \frac{6E_cI_c}{L_c^2} \\
0 & -\frac{12E_bI_b}{L_b^3} & 0 & -\frac{6E_bI_b}{L_b^2} & -\frac{6E_bI_b}{L_b^2} & 0 & \frac{A_cE_c}{L_c}+\frac{12E_bI_b}{L_b^3} & -\frac{6E_bI_b}{L_b^2} \\
0 & 0 & 0 & 0 & -\alpha & \frac{6E_cI_c}{L_c^2} & -\frac{6E_bI_b}{L_b^2} & \alpha+\frac{4E_cI_c}{L_c}
\end{pmatrix}
\begin{pmatrix} d2_x \\ d2_y \\ r2_z \\ r5_z \\ r6_z \\ d3_x \\ d3_y \\ r3_z \end{pmatrix}
=
\begin{pmatrix} H \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{6}$$

It is readily seen that this is a linear system involving rational dependencies between the frame parameters. Typical nominal parameter values and the corresponding worst case uncertainties, as proposed in (Corliss et al., 2004), are shown in Table I.

Initially, the system (6), where L_b, L_c are replaced by their nominal values, is solved with parameter uncertainties which are 1% of the values presented in the last column of Table I,

$$\begin{aligned}
&E_b, E_c \in [28965200,\ 29034800],\ I_b \in [509.49,\ 510.51],\ I_c \in [271.728,\ 272.272], \\
&A_b \in [10.287,\ 10.313],\ A_c \in [14.3856,\ 14.4144],\quad \alpha \in [276195960,\ 278726040], \\
&H \in [5283.465,\ 5327.535].
\end{aligned} \tag{7}$$

Applying the rigorous monotonicity approach we have found the monotonicity profile of the system response presented in Table II which proves that the combinatorial approach gives the exact hull in exact arithmetic. Note, that all solution components are only locally monotone with respect to Ab. The exact hull $[h]$ of the solution set for this problem, computed in rational arithmetic and then rounded outwardly to 10 digits accuracy, is presented in (Popova, 2005).

The parametric linear system (6) is solved by the presented general parametric fixed-point iteration. The system involves eight uncertain parameters which are considered to vary independently within tolerance intervals (7). The guaranteed outer enclosure $[u]$ of the system response and an inner estimation $[v]$ of the outer enclosure, obtained in just one single execution of the parametric solver function, are presented with 10 digits accuracy in (Popova, 2005). The quality of the obtained

Table I. Parameters involved in the steel frame example, their nominal values, and worst case uncertainties.

parameter		nominal value	uncertainty
Young modulus	E_b	$29 * 10^6$ lbs/in^2	$\pm 348 * 10^4$
	E_c	$29 * 10^6$ lbs/in^2	$\pm 348 * 10^4$
Second moment	I_b	510 in^4	± 51
	I_c	272 in^4	± 27.2
Area	A_b	10.3 in^2	± 1.3
	A_c	14.4 in^2	± 1.44
External force	H	5305.5 lbs	± 2203.5
Joint stiffness	α	$2.77461 * 10^9$ lb-in/rad	$\pm 1.26504 * 10^9$
Length	L_c	144 in, L_b 288 in	

Table II. One-bay steel frame example with uncertain parameters (7): monotonicity properties of the system response.

solution component	parameter E_b	E_c	I_b	I_c	A_b	A_c	α	H
1. $d2_x$	-1	-1	-1	-1	-1, -1	-1	-1	1
2. $d2_y$	1	-1	1	-1	-1, -1	-1	1	1
3. $r2_z$	1	1	1	1	1 1	1	1	-1
4. $r5_z$	1	1	1	1 1	1 1	1	-1	-1
5. $r6_z$	1	1	1	1	-1 -1	1	-1	-1
6. $d3_x$	-1	-1	-1	-1	1 1	-1	-1	1
7. $d3_y$	-1	1	-1	1	-1 -1	1	1	-1
8. $r3_z$	1	1	1	1	-1 -1	1	1	-1

enclosure is measured by the three measures of overestimation, defined in Section 2.3, and also compared to the quality of the solution enclosures for the same problem obtained by alternative methods used in (Corliss et al., 2004), see Table III.

The second and third columns in Table III demonstrate the relation $\frac{1}{2}\mathcal{O}_w([v],[u]) \approx \mathcal{O}_w([h],[u])$. The distance-based measure $\mathcal{O}_d$ gives two numbers with different signs corresponding to the endpoints of the intervals. As demonstrated by the results in Table III, this measure yields values which are two orders of magnitude less than the overestimation measure $\mathcal{O}_\omega([h],[u])$. The other overestimation measure $\mathcal{O}_c([h],[u],\mu)$ is also not comparable to $\mathcal{O}_\omega([h],[u])$ giving values with one order of magnitude less than the latter.

Table III. One-bay steel frame example with uncertain parameters (7): comparison of overestimation measures in %. $\mathcal{O}_c([h],[u_i])$ are after (Corliss et al., 2004), $i=3$ – Table V, $i=2$ – Table IV, $i=1$ – Table III, respectively, dash means no available data.

solution comp.	$\frac{1}{2}\mathcal{O}_\omega$ $([v],[u])$	$\mathcal{O}_\omega$ $([h],[u])$	$10^2\mathcal{O}_d$ $([h],[u])$	$\mathcal{O}_c$ $([h],[u],\mu)$	$\mathcal{O}_c$ $([\tilde{h}],[u_3],\mu)$	$\mathcal{O}_c$ $([\tilde{h}],[u_2],\mu)$	$\mathcal{O}_c$ $([\tilde{h}],[u_1],\mu)$
1. $d2_x$	0.83	0.83	-0.75, 0.38	0.011	0.29	0.40	78.02
2. $d2_y$	0.57	0.57	-0.86, 0.20	0.011	0.004	0.13	85.38
3. $r2_z$	4.58	4.31	3.01, -3.53	0.065	0.75	0.84	81.18
4. $r5_z$	8.65	7.73	5.89, -6.31	0.122	1.62	1.63	85.32
5. $r6_z$	13.54	11.99	9.81, -10.32	0.201	–	–	–
6. $d3_x$	0.84	0.84	-0.76, 0.39	0.011	–	–	–
7. $d3_y$	0.79	0.79	0.33, -1.21	0.015	–	–	–
8. $r3_z$	3.40	3.23	2.19, -2.70	0.049	–	–	–

The last three columns in Table III present the quality of the solution enclosures obtained in (Corliss et al., 2004) by the application of EBE approach (Muhanna and Mullen, 2001) to the system (6)–(7). The application of the EBE approach was successively improved in (Corliss et al., 2004) by applying subdistributivity property and scaling which has resulted in improved solution enclosures measured by $\mathcal{O}_c([\tilde{h}],[u_i],\tilde{\mu})$, where $[\tilde{h}]$ is the solution set hull reported in (Corliss et al., 2004), and $[u_i]$ is the corresponding solution enclosure. Comparing the best solution enclosure, obtained by the EBE approach — $\mathcal{O}_c([\tilde{h}],[u_3],\tilde{\mu})$, to the quality $\mathcal{O}_c([h],[u],\mu)$ of the solution enclosure obtained by the present parametric method, we see the superiority of the present method by one order of magnitude. The results in Table III show also that the different components of the system response have different sensitivity to variations in the system parameters.

It is well-known that the parametric fixed-point iteration gives sharper solution enclosures for smaller interval tolerances. To illustrate this effect we have subdivided the ranges (7) of some interval-valued parameters and obtain enclosure of the system response as a hull of the solution enclosures in all sub-domains. The results obtained after the application of the subdivision approach, reported in (Popova, 2005), show an improvement between 0.37% and 3.05% in the solution enclosure obtained by subdivision of the intervals. The overestimation for the different components of the system response is different ranging from 0.2% to 9.22%.

The presented parametric fixed-point iteration fails in solving the parametric linear system (6) for the worst case (over 40%) parameter uncertainties given in Table I. For very large uncertainties the parametric matrix is not strongly regular as required by the method. But we can solve the problem by subdividing the parameter intervals. As small are the sub-domains as better will be the solution enclosure. Inclusions (inner and outer) of the solution set hull are obtained by subdivision of the worst-case parameter intervals $(E_b, E_c, I_b, I_c, A_b, A_c, \alpha, H)^\top$ correspondingly into

Table IV. One-bay steel frame example with worst-case parameter uncertainties (Table I) solved by subdivision of the parameter intervals $(E_b, E_c, I_b, I_c, A_b, A_c, \alpha, H)^\top$ correspondingly into $(2,2,2,2,1,1,1,1)^\top$ equal subintervals. Inner $[v_s]$ and outer $[u_s]$ inclusions of the solution set hull are compared to the combinatorial solution $[\tilde{h}]$.

	$d2_x$	$d2_y$	$r2_z$	$r5_z$	$r6_z$	$d3_x$	$d3_y$	$r3_z$
$\frac{1}{2}\mathcal{O}_\omega([v_s],[u_s])$	19.97	15.87	–	–	–	20.12	23.50	–
$\mathcal{O}_\omega([\tilde{h}],[u_s])$	18.41	12.23	26.23	41.43	41.84	18.56	18.77	26.70

$(2,2,2,2,1,1,1,1)^\top$ equal subintervals. The quality of the obtained outer enclosure is presented in Table IV. Although the inner estimations for the most sensitive solution components are empty set intervals, a minimal number of subdivisions provided an outer enclosure overestimating the combinatorial solution with 12% to 42%. These results show that even for comparatively large parameter intervals, the presented parametric fixed-point iteration is able to enclose the solution. Although the parametric matrix is strongly regular (which provides convergence of the method) even for the large parameter uncertainties that are chosen, a poor accuracy of the residual vector enclosure may be the reason for overestimating the system response.

3.2. Two-Bay Two-Story Frame

As large frame examples we consider rectangular multi-story multi-bay frames. We model the two-bay two-story steel frame with IPE 400 beams and HE 280 B columns as shown in Figure 2.

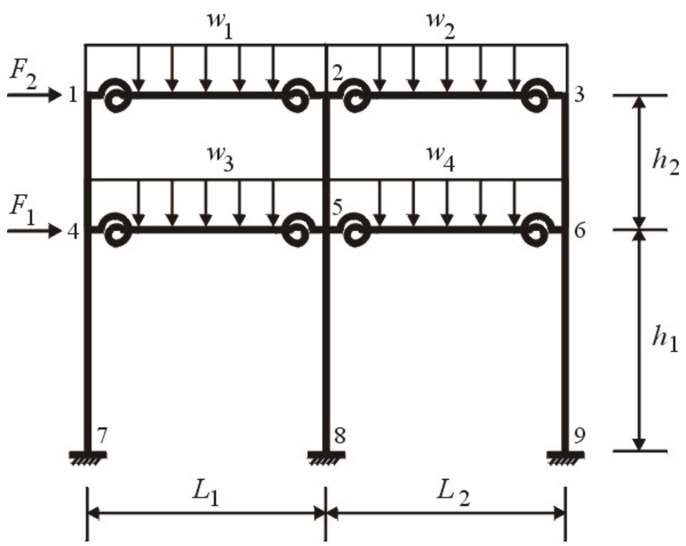

Figure 2. Two-bay two-story steel frame.

The frame is subjected to lateral static forces and vertical uniform loads. Beam-to-column connections are considered to be semi-rigid and are modelled by single rotational spring elements. The use of spring models is better fitted to steel frames with bolted connections, for example beam-to column connections of extended-end-plate system. Semi-rigid steel and reinforced concrete frames have been widely used to reduce the seismic loading. However many structures of this type have been strongly damaged or collapsed during the Northridge earthquake, which struck Southern California in 1994. The main reason has been found to be the increased flexibility of entire frame being strongly influenced by P-Δ effect. Semi-rigid frames are in large extent sensitive to physical properties of the beam-to-column connections and this was the main reason to direct our research in this direction.

Structure elements are specified to be beam or column. Columns are chosen to be traditional 2D frame elements for the elastic analysis having three degrees of freedom per node – two translations and one rotation. Beam elements also have three degrees of freedom at each node – two translations and one rotation. The rotational springs are added to both ends and internal rotations are eliminated. Beam elements allow for application of traditional finite element procedure which requires matrices of order 6×6. Basement nodes are fixed and are not able to displace. Applying conventional methods for analysis of frame structures, cf. (Zienkiewicz, 1971), a system of 18 linear equations is composed where the coefficients are rational functions of the model parameters. The distributed beam loading is transformed to the equivalent nodal forces. In this manner the parameters related to the geometric properties are included in the global loading vector.

In contrast to the system considered in Section 3.1 the linear system describing present two-bay two-story frame in Figure 2 has the following right-hand side vector whose components depend also on parameters of the beams, not only on the applied loadings

$$\Bigg(f_2,\ -\frac{1}{2}w_1Lb_1,\ -\frac{w_1Lb_1^2}{12(1+\frac{2Eb_1Ib_1}{cLb_1})},\ 0,\ -\frac{w_1Lb_1}{2}-\frac{w_2Lb_2}{2},\ \frac{w_1Lb_1^2}{12(1+\frac{2Eb_1Ib_1}{cLb_1})}-\frac{w_2Lb_2^2}{12(1+\frac{2Eb_2Ib_2}{cLb_2})},$$
$$0,\ -\frac{w_2Lb_2}{2},\ \frac{w_2Lb_2^2}{12(1+\frac{2Eb_2Ib_2}{cLb_2})},\ f_1,\ -\frac{1}{2w_3Lb_3},\ -\frac{w_3Lb_3^2}{12(1+\frac{2Eb_3Ib_3}{cLb_3})},$$
$$0,\ -\frac{w_3Lb_3}{2}-\frac{w_4Lb_4}{2},\ \frac{w_3Lb_3^2}{12(1+\frac{2Eb_3Ib_3}{cLb_3})}-\frac{w_4Lb_4^2}{12(1+\frac{2Eb_4Ib_4}{cLb_4})},\ 0,\ -\frac{w_4Lb_4}{2},\ \frac{w_4Lb_4^2}{12(1+\frac{2Eb_4Ib_4}{cLb_4})}\Bigg)^\top.$$

The following data, taken according to the European Standard (Eurocode 3, 2003), are used in the model.

	Columns (HE 280 B)	Beams (IPE 400)	
Cross-sectional area	$A_c = 0.01314\ \mathrm{m}^2$,	$A_b = 0.008446\ \mathrm{m}^2$	
Moment of inertia	$I_c = 19270 * 10^{-8}\ \mathrm{m}^4$,	$I_b = 23130 * 10^{-8}\ \mathrm{m}^4$	
Modulus of elasticity	$E_c = 2.1 * 10^8\ \mathrm{kN/m}^2$,	$E_b = 2.1 * 10^8\ \mathrm{kN/m}^2$	
Length	$L_c = 3$ m,	$L_b = 2L_c$ m	
Rotational spring stiffness	$c = 10^8$ kN		(8)

Uniform vertical load	$w_1 = \ldots = w_4 = 30$ kN/m
Concentrated lateral forces	$f_1 = f_2 = 100$ kN

As a first problem a system structure having 13 uncertain parameters: A_c, I_c, E_c, A_b, I_b, E_b, c, $w_1, \ldots, w_4$, f_1, f_2 was considered. The system parameters were initially taken to vary within 1% tolerance intervals $[p - p/200, p + p/200]$ where p is the corresponding parameter nominal value from (8).

Table V. Solutions for displacements and rotations of two-bay two-story frame system with 13 parameters having 1% uncertainties.

	dx_1(m)	dy_1 (m)	θ_1 (rad)	dx_3(m)	dy_3 (m)	θ_3 (rad)
$10^3[v]$	[12.80, 13.20]	[-.2143, -.2062]	[-2.168, -2.099]	[12.21, 12.60]	[-.3439, -.3333]	[-.2079, -.1554]
$10^3[u]$	[12.78, 13.21]	[-.2145, -.2060]	[-2.175, -2.092]	[12.20, 12.62]	[-.3441, -.3331]	[-.2146, -.1487]
$\mathcal{O}_\omega([h],[u])$	4.90	3.20	9.24	4.98	2.93	11.04
$\mathcal{O}_c([h],[u])$	0.16	0.13	0.35	0.17	0.09	3.99

The parametric solver, presented in this paper, found a guaranteed outer enclosure $[u]$ of the system response and a corresponding inner estimation $[v]$ of the solution set hull. The results for displacements and rotations of selected nodes are given in Table V. The system response at the first three nodes is most sensitive to the variations in model parameters. The bounds for the solution are captured by sharp intervals. Applying rigorously the monotonicity approach based on verified parametric solver, it was numerically proven that the combinatorial approach gives the exact solution set hull. That is why, the last two rows of Table V list the percentage by which the outer enclosures produced by the parametric solver overestimate the true bounds of the system response. The results in Table V show that the rotations are about three times more sensitive to the variations in model parameters than the displacements. The same behavior was observed during the analysis of the portal structure in Section 3.1.

Further, we solve the same parametric system where the element material properties are taken to vary within 1% tolerances while the spring stiffness and all applied loadings are taken to vary within large 10% tolerance intervals. Table VI presents the results obtained for the nodes one and three. The first row in Table VI gives the combinatorial solution which is used for measuring the overestimation produced by the parametric solver. Except for θ_3, interval bounds for the system response are reasonable although not quite sharp. The percentage of overestimation increases with increasing the width of the parameter intervals. The lower quality of the solution enclosures for large parameter intervals is probably due to a poor range estimation of the residual vector in the algorithm. Proving monotonicity properties of the system response with respect to the loadings parameters $w_1, \ldots, w_4$, f_1, f_2 and solving corresponding parametric systems involving reduced number of parameters results in a quite sharp solution enclosure presented in the second part of Table VI.

It should be noted that for 10% tolerance intervals of the model parameters even the combinatorial solution is such that the interval for θ_3 contains zero.

Table VI. Interval solutions for displacements and rotations of two-bay two-story frame system with 13 parameters. The material properties have 1% uncertainties while the spring stiffness and the applied loadings have 10% uncertainties.

	dx_1(m)	dy_1 (m)	θ_1 (rad)	dx_3(m)	dy_3 (m)	θ_3 (rad)
$10^3[h]$	[12.16, 13.85]	[-.2308, -.1902]	[-2.316, -1.956]	[11.60, 13.24]	[-.3604, -.3174]	[-.3545, -.0100]
$10^3[u]$	[11.92, 13.89]	[-.2311, -.1850]	[-2.333, -1.896]	[11.36, 13.28]	[-.3607, -.3119]	[-.3724, .04663]
$\mathcal{O}_\omega([h],[u])$	14.51	12.10	17.60	14.61	12.04	17.82
$\mathcal{O}_c([h],[u])$	2.19	2.65	3.59	2.25	1.73	41.05
solutions after applying the monotonicity properties w.r.t. the applied loadings						
$10^3[u]$	[12.13, 13.88]	[-.2314, -.1896]	[-2.323, -1.949]	[11.56, 13.28]	[-.3613, -.3165]	[-.3599, -4.e^{-6}]
$\mathcal{O}_\omega([h],[u])$	3.99	2.75	3.89	3.99	3.83	3.05
$\mathcal{O}_c([h],[u])$	0.54	0.54	0.68	0.55	0.50	5.95

As a second larger problem of this type we consider the same system structure as above but assuming that each structure element has properties varying independently within 1% tolerance intervals. This leads to an 18×18 parametric system involving 37 interval parameters. The results for displacements and rotations of the selected nodes, listed in Table VII, are similar to those obtained for the system involving 13 parameters having the same uncertainties.

Table VII. Solutions for displacements and rotations of two-bay two-story frame system with 37 parameters having 1% uncertainties.

	dx_1(m)	dy_1 (m)	θ_1 (rad)	dx_3(m)	dy_3 (m)	θ_3 (rad)
$[v] * 10^3$	[12.67, 13.32]	[-.224, -.1964]	[-2.222, -2.045]	[12.09, 12.73]	[-.3571, -.3199]	[-.2569, -.1062]
$[u] * 10^3$	[12.62, 13.37]	[-.2249, -.1954]	[-2.237, -2.030]	[12.04, 12.77]	[-.3584, -.3186]	[-.2716, -.0915]
$\frac{1}{2}\mathcal{O}_\omega([v],[u])$	6.05	3.44	7.28	6.16	3.19	8.18
$\frac{1}{2}\mathcal{O}_c([v],[u])$	0.34	0.48	0.70	0.36	0.37	8.09

While the combinatorial solution for the problem involving 37 uncertain parameters requires solving $2^{37} \approx 1.37 * 10^{11}$ point linear systems (in rational arithmetic), or applying Monte Carlo simulation usually takes 10^6 trials in order to assess the quality of a solution enclosure, just one single execution of our parametric solver yields both guaranteed outer solution enclosure $[u]$ and its inner estimation $[v]$, based on which $1/2\mathcal{O}_\omega([v],[u])$ measures the quality of the obtained solution bounds.

4. Conclusion

The application of a self-verified parametric iteration method for bounding the response of uncertain mechanical structures modelled by finite element method is presented. The method can solve linear systems involving arbitrary non-linear dependencies between the uncertain input data, provided that it is combined with good tools for range enclosure. It is demonstrated that very sharp solution enclosures are generated for small parameter tolerances. Powerful range enclosing techniques are necessary to provide good accuracy of the solution enclosure when the system parameters are subjected to large uncertainties which retain the strong regularity property of the parametric matrix.

We have demonstrated the feasibility of the general-purpose parametric iteration method for bounding structure responses in the presence of uncertainties in all model parameters. It was illustrated by the numerical examples that for small intervals the method is superior to other, although not self-verified, methods like the EBE approach. Even for quite large parameter uncertainties, the interval subdivision guarantee the feasibility of the method and the accuracy of the inclusions.

The most attractive feature of the discussed methodology and software tools consists in the fact that they yield validated inclusions computed by a finite precision arithmetic. To provide this feature a rigorous computer implementation by interval arithmetic with directed roundings is necessary. Any self-verified parametric solver can be incorporated in a general framework for computer-assisted proof of global and local monotonicity properties of a parametric solution. Basing on these properties, a guaranteed and highly accurate enclosure of the solution set hull can be computed.

Contrary to other approaches for modelling uncertain mechanical systems that apply special techniques at the level of constructing the linear system to be solved in order to reduce the dependencies, the present method requires no preliminary specialized construction methods. For example, there is no need to overcome the coupling as in the EBE approach. Present method is highly automated since engineers need to apply only conventional methods for obtaining the linear system in a parametric form by software tools widely available in modern computing environments (Matlab, *Mathematica*, etc.). Uncertainties in all the system parameters (e.g., material, load and geometry properties) can be considered and handled simultaneously. A combination of interval methods can ensure very sharp bounds for the system response. Furthermore, the present method and all the methods combined to obtain sharp bounds for the system response, are implemented in software tools which are freely available and ready for application. When the construction methods, used for assembling the global stiffness matrix and the global loading vector, cannot eliminate all the dependencies between the input parameters, a parametric iteration, respectively the implemented parametric solver, should be used instead of a non-parametric one.

Being the only general-purpose parametric linear solver, the presented methodology and software tools are applicable in the context of any problem which requires solving of linear systems whose input data depend on uncertain (interval) parameters.

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