On Linear Interpolation under Interval Data

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Abstract. Some results related to the problem of interpolation of n vertical segments (x_k, Y_k), $k = 1, \ldots, n$, in the plane with generalized polynomial functions that are linear combinations of m basic functions are presented. It is proved that the set of interpolating functions (if not empty) is bounded in every subinterval (x_k, x_{k+1}) by two unique such functions η_k^- and η_k^+. An algorithm with result verification for the determination of the boundary functions η_k^-, η_k^+ and for their effective tabulation is reported and some examples are discussed.

1 Formulation of the problem

Starting point of this presentation is a problem of system modeling: The structure of relation between quantities characterizing system behavior has to be quantitatively determined, being compatible with measurements of these quantities. The following hypotheses are assumed [7]–[11], [15], [17], [18]:

1. **Assumptions on the modeling function.**

   The modeling function $\eta(\lambda; \cdot)$ defined on some interval $X = [x^-, x^+] \subseteq R$ depends linearly on m parameters:

$$\eta(\lambda; \xi) = \sum_{i=1}^{m} \lambda_i \varphi_i(\xi) = \varphi(\xi)^\top \cdot \lambda, \qquad x^- \le \xi \le x^+, \quad \xi \in R, \tag{1}$$

   where $\lambda = (\lambda_1, \ldots, \lambda_m)^\top \in R^m$ is a vector of (unknown) parameters, and $\varphi(\cdot) = (\varphi_1(\cdot), \ldots, \varphi_m(\cdot))^\top \in C(X, R^m)$ is a vector of m fixed functions continuous on X. (The dot "$\cdot$" for the inner product will be further omitted.) The vector $\varphi(\cdot)$ generates a matrix defined for $(x'_1, \ldots, x'_m)$, $x'_i \in X$, $i = 1, \ldots, n$, by

$$\begin{pmatrix} \varphi_1(x'_1) & \cdots & \varphi_m(x'_1) \\ \vdots & \ddots & \vdots \\ \varphi_1(x'_m)) & \cdots & \varphi_m(x'_m) \end{pmatrix} \tag{2}$$

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We shall assume that the determinant of (2) does not vanish for any $x' = (x'_1, \ldots, x'_m)$ such that $x^- \le x'_1 < \ldots < x'_m \le x^+$. A set φ of functions satisfying the above assumption will be further called a *system of basic functions*. The class of all modeling functions of the form (1) using a system of basic functions is denoted by $\mathcal{L}_m(X)$.

2. **Assumptions on the type of errors in the data.**

   If n is the number of the measurements, then $J = \{1, \ldots, n\}$ denotes the corresponding index set. The input data are error-free and the output data errors are unknown but bounded (UBB) [8], [9]. This means that there are n input data $x_j, j \in J$, in the real interval $[x^-, x^+]$ such that

$$x_0 = x^- \le x_1 < x_2 < \ldots < x_n \le x_n \le x^+ = x_{n+1},$$

   and there are n output interval measurements $Y_j = [y_j^-, y_j^+],\ j \in J$, which contain the correct values of the corresponding measured quantities.

   Denote the input data by $x = (x_1, x_2, \ldots, x_n)^\top \in R^n$ and the output measurements by $Y = (Y_1, \ldots, Y_n)^\top \in IR^n$. The pairs $(x_j, Y_j),\ j \in J$, can be considered as vertical segments in the plane xOy.

Assume that $m \le n$ and consider the problem of finding modeling functions $\eta \in \mathcal{L}_m(X)$ interpolating the vertical segments $(x_j, Y_j),\ j \in J$. More precisely we are interested in the set $\eta(x, Y; \xi)$ of values at ξ of all modeling functions η interpolating the segments $(x_j, Y_j),\ j \in J$, that is in the set:

$$\eta(x, Y; \xi) = \{\eta(\lambda; \xi) \mid \eta \text{ is such that } \eta(\lambda; x_j) \in Y_j,\ j \in J\}, \quad \xi \in X. \tag{3}$$

The requirement that the values of η at x_j range in the corresponding intervals Y_j

$$\eta\,(\lambda; x_j) = \varphi(x_j)^\top \lambda \in Y_j, \quad j \in J, \tag{4}$$

can be written in matrix representation as

$$\Phi(x)\lambda \in Y, \tag{5}$$

where

$$\Phi(x) = \begin{pmatrix} \varphi_1(x_1) & \cdots & \varphi_m(x_1) \\ \vdots & \ddots & \vdots \\ \varphi_1(x_n)) & \cdots & \varphi_m(x_n) \end{pmatrix} = \begin{pmatrix} \varphi(x_1)^\top \\ \vdots \\ \varphi(x_n)^\top \end{pmatrix}.$$

Since φ is a system of basic functions, rank $\Phi(x) = m$.

In (4) the input data x and the output data Y are known, the parameter λ is unknown. We thus have a system of n algebraic inclusions for the determination of the m-dimensional parameter λ. Any λ satisfying (4) is called a *feasible parameter*. Every feasible parameter λ generates a modeling function $\eta(\lambda; \cdot) \in \mathcal{L}_m(X)$ which is called an adequate model.

Denote by Λ the set of all feasible parameters, by $\eta(\Lambda;\cdot)$ the set of solutions (solution functions) and by $\eta(\Lambda;\xi)$ the set of values of all solutions at $\xi \in X$, respectively

$$\Lambda = \{\lambda \in R^m \mid \Phi(x)\lambda \in Y\}, \tag{6}$$

$$\eta(\Lambda;\cdot) = \left\{\varphi(\cdot)^\top \lambda \mid \lambda \in \Lambda\right\}. \tag{7}$$

$$\eta(\Lambda;\xi) = \left\{\varphi(\xi)^\top \lambda \mid \lambda \in \Lambda\right\}. \tag{8}$$

The set $\eta(\Lambda;\xi)$ is obviously another notation for the set (3) we are looking for, i. e. $\eta(x, Y;\xi) = \eta(\Lambda;\xi)$ so that (3) and (8) are equivalent.

From $\varphi(\cdot) \in C(X, R^m)$ it follows that the set $\eta(\Lambda;\xi)$ defined by (8) is an interval for any fixed $\xi \in X$. Thus (8) defines an interval (interval-valued) function on X. In this paper we shall be particularly concerned with characterizing and computing the boundary lower and upper functions $\eta^-(x, Y;\cdot)$, $\eta^+(x, Y;\cdot)$ of the interval function (3), resp. (8). The latter are also called the enveloping functions of the set of modeling functions $\{\eta(\lambda;\cdot) \mid \lambda \in \Lambda\}$ interpolating (x, Y) [14].

To include the set $\eta(x, Y;\cdot)$ at a point ξ, several approaches are known:

1. The bounds for the sought values are directly determined, e.g. by solving two linear optimization problems [15], [17]
$$\eta(x, Y;\xi) = \left[\min_{\lambda\in\Lambda}\left\{\varphi(\xi)^\top \lambda\right\},\ \max_{\lambda\in\Lambda}\left\{\varphi(\xi)^\top \lambda\right\}\right].$$
2. The parameter set Λ is enclosed by an interval vector (box) Λ^I (cf. [18]). Then the following relation holds:
$$\eta(x, Y;\xi) \subseteq \varphi(\xi)^\top \Lambda^I.$$
3. Suitable parameters $\lambda^- = \lambda^-(\xi)$ and $\lambda^+ = \lambda^+(\xi)$ from the set Λ are determined for given ξ, so that
$$\eta(x, Y;\xi) = \left[\varphi(\xi)^\top \lambda^-(\xi),\ \varphi(\xi)^\top \lambda^+(\xi)\right].$$

All these approaches deal with the set Λ and are not very effective for the determination of $\eta(x, Y;\xi)$ for $\xi \in X$. Numerically, the problem of finding the envelope (8) is quite different from the problem of finding the parameter set Λ defined by (6) (see [8]–[10], [11], [18]). The set Λ is an m-dimensional polytope, which is not easily representable, whereas $\eta(x, Y;\xi)$ is a closed one-dimensional interval that can be efficiently computed for arbitrary ξ. The computation of $\eta(x, Y, \xi)$ can be of practical importance. It seems necessary to seek direct methods for the characterization and computation of $\eta(x, Y;\xi)$ in X. In what follows we propose a direct method for computing the values of the interval function η which does not use the feasible parameter set Λ or any particular values of λ as methods 1–3 above do. Our method uses the fact that the boundary functions are generalized Polynomials in each subinterval $[x_k, x_{k+1}]$ (see Proposition 2). A computer program written in PASCAL-SC [5] is reported, which efficiently computs the interval function $\eta(x, Y;\xi)$ for arbitrary $\xi \in X$.

2 Characterization and computation of the set of modeling functions

Every feasible parameter λ generates a vector $y = (y_1, \ldots, y_n)^\top \in R^n$ by

$$y_j = \eta(\lambda; x_j), \quad j \in J, \tag{9}$$

in matrix form $y = \Phi(x)\lambda$. The set of all vectors y defined by (9) can be written as

$$Y' = \left\{ y = (\eta(\lambda; x_1), \ldots, \eta(\lambda; x_n))^\top \mid \lambda \in \Lambda \right\}. \tag{10}$$

Let us first consider the case $n = m$ when the number of data equals the number of parameters. A modeling function from $\mathcal{L}_m(X)$, that is of the form $\eta(\lambda; \cdot) = \varphi(\cdot)^\top \lambda$, which interpolates a set of m data (x, Y), satisfy a system $\Phi(x)\lambda \in Y$ of m algebraic inclusions for the m unknown parameters, or

$$\Phi(x)\lambda = y, \quad y \in Y.$$

Since $\det \Phi(x) \neq 0$, for any $y \in Y$ we have $\lambda = \Phi^{-1}(x)y$. For the set of values of all modeling functions interpolating (x, Y) at ξ we obtain respectively

$$\begin{aligned} \eta(x, Y; \xi) = \left\{ \varphi(\xi)^\top \lambda \mid \lambda \in \Lambda \right\} &= \left\{ \varphi(\xi)^\top \left(\Phi^{-1}(x) y \right) \mid y \in Y \right\} \\ &= \left\{ \left(\varphi(\xi)^\top \Phi^{-1}(x) \right) y \mid y \in Y \right\} \\ &= \left(\varphi(\xi)^\top \Phi^{-1}(x) \right) Y, \end{aligned}$$

where the last expression uses interval arithmetic [1].

We thus obtain the following

Proposition 1 *For $m = n$ we have for the solution set (3), resp. (8)*

$$\eta(x, Y; \xi) = \left(\varphi(\xi)^\top \Phi^{-1}(x) \right) Y, \quad \xi \in X. \tag{11}$$

The interval function (11) will be further called *simple interval interpolation function.* Proposition 1 shows that it can be computed for every ξ in interval arithmetic using (11). Such an exact presentation can not be given for Λ because of the so-called wrapping effect. Indeed for the set Λ of feasible parameters we have

$$\Lambda = \left\{ \Phi^{-1}(x) y \mid y \in Y' \right\} \subseteq \left\{ \Phi^{-1}(x) y \mid y \in Y \right\},$$

where Y' is defined by (10). From this using interval arithmetic we obtain the inclusion $\Lambda \subseteq \Phi^{-1}(x) Y = \Lambda^I$, which may be very rough. This is due to the fact that Λ is a convex polytope, whereas Y, resp. Λ^I, is a box (orthotope), n-dimensional interval.

We now consider the general case $m \leq n$. We first give a definition which will be used in the next proposition.

Definition 1 *(see [16]) A l-face of Λ is a subset of Λ defined by*

$$y_j^- \le \varphi(\xi_j)^\top \lambda \le y_j^+, \quad j \in J,$$

where $m - l$ of the above linear independent inequalities transform into equalities.

The following proposition holds [12], [13]:

Proposition 2 *Let $\eta(x, Y; \cdot)$ be the envelope for the set of all modeling functions from $\mathcal{L}_m(X)$ which interpolate (x, Y). If $\Lambda \neq \emptyset$, then for all $k \in \{0, 1, \ldots, n\} = \{0\} \cup J$ the upper and lower boundary functions of $\eta\,(x, Y; \xi)$ are unique elements of $\mathcal{L}_m(X)$ for every $\xi \in (x_k, x_{k+1})$. In other words, there exist parameters $\lambda_k^-, \lambda_k^+ \in \Lambda$ such that*

$$\eta^-\,(x, Y; \xi) = \eta(\lambda_k^-; \xi) = \varphi(\xi)^\top \lambda_k^-, \quad \xi \in (x_k, x_{k+1})\,,$$

$$\eta^+\,(x, Y; \xi) = \eta(\lambda_k^+; \xi) = \varphi(\xi)^\top \lambda_k^+, \quad \xi \in (x_k, x_{k+1})\,.$$

Proof. We show the correctness only for the upper function — for the lower function the proof is analogous.

Let the assertion of the proposition be wrong. Then a point $\xi_s \in (x_k, x_{k+1})$ and two parameters $\lambda^1, \lambda^2 \in \Lambda$ exist such that

$$\eta^+(x, Y; \xi_s) = \varphi(\xi_s)^\top \lambda^1 = \varphi(\xi_s)^\top \lambda^2$$

and $\lambda^1 \neq \lambda^2$. On the other hand we have

$$\eta^+(x, Y; \xi_s) = \max_{\lambda \in \Lambda} \varphi(\xi_s)^\top \lambda. \tag{12}$$

Because the set of optimal points of the linear programming problem (12) is convex all points of the segment $[\lambda^1, \lambda^2]$ are optimal points. The set of all optimal points of (12) is a l-face of Λ, where $l \geq 1$ and the vector $\varphi(\xi_s)$ is perpendicular to this l-face of Λ.

The l-face is an intersection of $(m - l)$ hyperplanes with linear independent normal vectors $a^1, \ldots, a^{m-l} \in \{\varphi(\xi_j), j \in J\}$ because they are linear independent (the modeling function is from $\mathcal{L}_m\,(X)$). Thus the vector $\varphi(\xi_s)$ is a linear combination of $a^1, \ldots, a^{m-l}$. This is a contradiction to the assertion that φ is a system of basic functions.

We thus proved that for every point $\xi_s \in (x_k, x_{k+1})$ the optimum value of $\varphi^\top(\xi_s)\lambda$ is reached for an unique λ. This λ is a 0-face (that is a vertex) of the polytope Λ (if λ is an l-face with $l \geq 1$, it will not be unique). Now, take $\xi' \neq \xi$", such that $\xi', \xi" \in (x_k, x_{k+1})$. We shall show that ξ' and ξ" generate same solutions of the optimization problem (12).

Indeed, assume that the corresponding solutions are two different vertices λ', λ". Let $\xi^* = (1 - \alpha)\xi' + \alpha\xi$" and let λ^* be the optimum corresponding to ξ^*. Since φ is continuous there is at least one $\alpha \in (0, 1)$ such that ξ^* is an l-face with $l \geq 1$. This is a contradiction. $\square$

Proposition 2 shows, that under the given assumptions the upper and lower boundary functions $\eta\,(x, Y; \xi)$ for all $\xi \in (x_k, x_{k+1})$ are themselves elements of $\mathcal{L}_m\,(X)$, and, therefore, to find $\eta\,(x, Y; \xi)$ for $\xi \in [x_k, x_{k+1}]$ we have to determine expressions for these two functions.

Let the index set Q be a subset of the index set J with m elements: $Q \subseteq J$, $card(Q) = m$. Let Q be ordered in increasing order and let $q(i)$ be the i-th element of Q. Denote by $x^Q = \left(x_{q(1)}, \ldots, x_{q(m)}\right)^\top$ the vector x reduced to the index set Q. Analogously $Y^Q = \left(Y_{q(1)}, \ldots, Y_{q(m)}\right)^\top$ is the vector Y reduced to Q.

To find the set of functions from $\mathcal{L}_m(X)$ interpolating a reduced set of m data (x^Q, Y^Q) we consider the corresponding system of m algebraic inclusions for m unknown parameters:

$$\Phi(x^Q)\lambda \in Y^Q,$$

and applying (11), obtain for the corresponding simple interpolation polynomials

$$\eta(x^Q, Y^Q; \xi) = \left(\varphi(\xi)^\top \Phi^{-1}(x^Q)\right) Y^Q.$$

Proposition 3 *The value of $\eta(x, Y; \cdot)$ at a point ξ is given by [6]*

$$\eta(x, Y; \xi) = \bigcap_{Q \subseteq J} \eta(x^Q, Y^Q; \xi) = \bigcap_{Q \subseteq J} \left(\varphi(\xi)^\top \Phi^{-1}(x^Q)\right) Y^Q.$$

Proposition 3 shows that the value of $\eta(x, Y; \cdot)$ at ξ can be determined by an intersection of $\left(\begin{array}{c} n \\ m \end{array}\right)$ simple interval interpolating functions.

Definition 2 *The set of data (x, Y) is said to be compatible (with the basic functions $\varphi_1(\xi), \ldots, \varphi_m(\xi)$), if*

$$Y_j = \eta\left(x, Y; x_j\right), \quad j \in J.$$

In other words, for a compatible set of data (x, Y) the interval Y_j is the projection of the set Y' defined by (10) on the j-th coordinate axis.

If the set of data (x, Y) is not compatible, it can be reduced to a compatible set $(x, \tilde{Y})$, generating the same solution set $\eta(x, Y; \cdot) = \eta(x, \tilde{Y}; \cdot)$, using the following

Proposition 4 *For the compatible intervals we have*

$$\tilde{Y}_j = Y_j \cap \bigcap_{Q \subseteq J \quad j \notin Q} \eta(x^Q, Y^Q; x_j).$$

The next proposition shows that, if the set of data (x, Y) is compatible, then $\eta(x, Y; \xi)$ may be determined by an intersection of a reduced number of simple interval interpolating functions.

Proposition 5 *If the set of data (x, Y) is compatible with the basic functions, then for every $k \in \{0\} \cup J$ the following formula holds*

$$\eta\left(x, Y; \xi\right) = \bigcap_{Q(k)} \eta(x^{Q(k)}, Y^{Q(k)}; \xi) \quad \textit{for} \quad \xi \in [x_k, x_{k+1}],$$

where $Q(k) \subseteq J$, $card(Q(k)) = m$, is such that

$$\begin{array}{rl} k, k+1 \in Q(k), & \textit{if} \;\; 0 < k < n, \\ 1, n \in Q(k), & \textit{if} \;\; k = 0 \;\textit{or}\; k = n. \end{array}$$

Proof. We shall prove that for each index set Q there exists an index set $Q(k)$ such that $\eta\left(x^{Q(k)}, Y^{Q(k)}; \xi\right) \subseteq \eta\left(x^{Q}, Y^{Q}; \xi\right)$, if $\xi \in [x_k, x_{k+1}]$.

We construct such a suitable index set $Q(k)$ for a given index set Q as follows. There are four cases:

1. $x_{k+1} \leq x_{q(1)}$ and $k \neq 0$. We take $Q(k) = \{k, k+1, q(2) ..., q(m-1)\}$.
2. $x_{q(m)} \leq x_k$ and $k \neq n$. We take $Q(k) = \{q(2), \ldots, q(m-1), k, k+1\}$.
3. $k = 0$ or $k = n$. We take $Q(k) = \{1, q(2), \ldots, q(m-1), n\}$.
4. $x_{q(1)} \leq x_k$ and $x_{k+1} \leq x_{q(m)}$. Let $l = \max\left\{i : x_{q(i)} \leq x_k\right\}$, then $x_{q(l)} \leq x_k$ and $x_{k+1} \leq x_{q(l+1)}$. We take $Q(k) = \{q(1), \ldots, q(l-1), k, k+1, q(l+2), \ldots, q(m)\}$.

We see that the sets Q and $Q(k)$ have at least $m-2$ common points. We show that $\eta\left(x^{Q(k)}, Y^{Q(k)}; \xi\right) \subseteq \eta\left(x^{Q}, Y^{Q}; \xi\right)$ only for the upper function. The case of lower function can be proved analogously.

Because the data (x, Y) are compatible with the basic functions we have $\eta^+\left(x^{Q}, Y^{Q}; x_k\right) \geq \eta^+\left(x^{Q(k)}, Y^{Q(k)}; x_k\right) = y_k^+$ and $y_{k+1}^+ = \eta^+\left(x^{Q(k)}, Y^{Q(k)}; x_{k+1}\right) \leq \eta^+\left(x^{Q}, Y^{Q}; x_{k+1}\right)$.

Let for some $\xi_s \in (x_k, x_{k+1})$

$$\eta^+\left(x^{Q(k)}, Y^{Q(k)}; \xi_s\right) > \eta^+\left(x^{Q}, Y^{Q}; \xi_s\right). \tag{13}$$

Then in the interval $[x_k, x_{k+1}]$ there are at least two intersection points ξ_1 and ξ_2 of $\eta^+\left(x^{Q(k)}, Y^{Q(k)}; \cdot\right)$ and $\eta^+\left(x^{Q(k)}, Y^{Q(k)}; \cdot\right)$. The graphs of the latter functions have m common points. Thus they are identical. This is a contradiction to (13).□

For $m = 2$ there is only one possible set $Q(k)$. Namely for $\xi \in [x_k, x_{k+1}]$, $0 \leq k \leq n$,

$$Q(k) = \begin{cases} \{k, k+1\}, & \text{if } 0 < k < n, \\ \{1, n\}, & \text{if } k = 0 \text{ or } k = n. \end{cases}$$

$$\eta(x, Y; \xi) = \eta(x^{Q(k)}, Y^{Q(k)}; \xi) \quad \text{for} \quad \xi \in [x_k, x_{k+1}].$$

Propositions 1–5 provide possibilities for piecewise computation of $\eta(\xi; x, Y)$ in the segments (x_k, x_{k+1}). The interval function $\eta(\xi; x, Y)$ may be determined at $\xi \in (x_k, x_{k+1})$ by intersecting the values of certain simple interval interpolating functions at ξ.

Numerical algorithm. Propositions 1–5 above suggest two possible algorithms for the effective computation of $\eta(x, Y; \cdot)$ at $\xi \in X$, which can be used for the tabulation of η (that is computation of η in a given number of points ξ_i, $i = 1, \ldots, l$).

A. Compute $\eta(x, Y; \cdot)$ at ξ by means of Proposition 3. If a tabulation in a series of knots $\xi_1, \ldots, \xi_l$ is needed, then the computation can be reduced using the following approach based on Proposition 2. When evaluating η at a point ξ which is the first point to be computed in a new interval $[x_k, x_{k+1}]$, store the information about the expressions for the boundary functions η^-, η^+. For instance, store the end-points of Y_i, lying on

η^-, resp. η^+, such as $Y_1^{\alpha_1}, Y_2^{\alpha_2}, \ldots, Y_n^{\alpha_n}$, $\alpha_i \in \{+,-\}$). According to Proposition 2 the functions η^-, η^+ produce the values of $\eta(x, Y; \cdot)$ for all ξ in the subinterval $[x_k, x_{k+1}]$, so that once we find functions η^-, η^+ for a fixed $\xi \in [x_k, x_{k+1}]$ we can use these functions when tabulating $\eta(x, Y; \cdot)$ in the whole subinterval $[x_k, x_{k+1}]$.

B. Compute first the compatible intervals $\tilde{Y}_i$ generating the same solution set $\eta(x, \tilde{Y}; \cdot) = \eta(x, Y; \cdot)$ by means of Proposition 4, that is by intersection of (interval) values of simple interval interpolating polynomials. The latter are computed by means of Proposition 1. Then use Proposition 5 to compute $\eta(x, Y; \cdot)$ at arbitrary ξ. If a tabulation of $\eta(x, Y; \cdot)$ in several points from a subinterval $[x_k, x_{k+1}]$ is needed, then the same approach as in **A.** can be exploited.

3 The polynomial case

If the basic functions are polynomials of $(i-1)$-th degree of the form $\varphi_i(x) = x^{i-1}$, $i = 1, ..., m$, then the determinant of (2) is the Vandermond's determinant:

$$\det \Phi(x') = \prod_{i>j} (x_i - x_j),$$

which does not vanish and therefore $\{\varphi_i\}_{i=1}^m$ is a system of basic functions. In this section we take for $\mathcal{L}_m(X)$ the class of polynomial functions of $(m-1)$-st degree of the form $\eta_{m-1}(\lambda; \xi) = \lambda_1 + \lambda_2 \xi + ... + \lambda_m \xi^{m-1}$, which are defined on $X = R$.

In the case $n = m$ formula (11) for the simple interval interpolation function obtains the form

$$\eta_{m-1}(x, Y; \xi) = l(x; \xi)^\top Y, \quad l_i(x; \xi) = \prod_{k=1,\ldots,m,\ k \neq i} \frac{\xi - x_k}{x_i - x_k},$$

(see [3], [4]). An explicit evaluation without using interval arithmetic is given in [2].

If $n > m$ using Proposition 3 we can present the value of $\eta_{m-1}(x, Y; \xi)$ at a point ξ by intersecting $\binom{n}{m}$ simple interval interpolating polynomials [6]

$$\eta_{m-1}(x, Y; \xi) = \bigcap_{Q \subseteq J} l\left(x^Q; \xi\right)^\top Y^Q, \quad l_i(x^Q; \xi) = \prod_{k=1,\ldots,m,\ k \neq i} \frac{\xi - x_{q(k)}}{x_{q(i)} - x_{q(k)}}.$$

The intervals Y_j can be reduced to compatible intervals $\tilde{Y}_j$ generating the same solution set $\eta_{m-1}(x, Y; \cdot)$ as follows [6]

$$\tilde{Y}_j = Y_j \cap \bigcap_{Q \subseteq J,\ \ j \notin Q} l\left(x^Q; x_j\right)^\top Y^Q.$$

For $m = 2$ applying Proposition 5 we obtain that there is only one possible set $Q(k)$, therefore for every $\xi \in [x_k, x_{k+1}]$, $0 \leq k \leq n$, the following expression holds [6]

$$\eta_1(x, Y; \xi) = \begin{cases} \frac{\xi - x_{k+1}}{x_k - x_{k+1}} Y_k + \frac{\xi - x_k}{x_{k+1} - x_k} Y_{k+1}, & \text{if } \ 0 < k < n, \\ \frac{\xi - x_n}{x_1 - x_n} Y_1 + \frac{\xi - x_1}{x_n - x_1} Y_n, & \text{if } \ k = 0 \ \text{ or } \ k = n, \end{cases}$$

where again (x, Y) is assumed compatible.

We next give two examples for polynomial functions. The computations were performed by a program written in PASCAL-SC [5], based on approach **A.** described at the end of the previous section.

Example 1. Let the following set of data be given

$$(x, Y) \quad = \quad \begin{pmatrix} 1, & [1, 3] \\ 2, & [1, 2] \\ 4, & [1.5, 2.5] \\ 6, & [2, 3] \end{pmatrix}$$

Let the modeling functions be second order polynomials of the form

$$\eta_2(\lambda; \xi) = \lambda_1 + \lambda_2 \xi + \lambda_3 \xi^2.$$

The graph of the interval function $\eta_2(x, Y; \cdot)$ is presented on Fig. 1. For comparison the simple interval polynomial $\eta_3(x, Y; \cdot)$ is also presented. In order to recognize both interval interpolating functions we should keep in mind that $\eta_2 \subseteq \eta_3$.

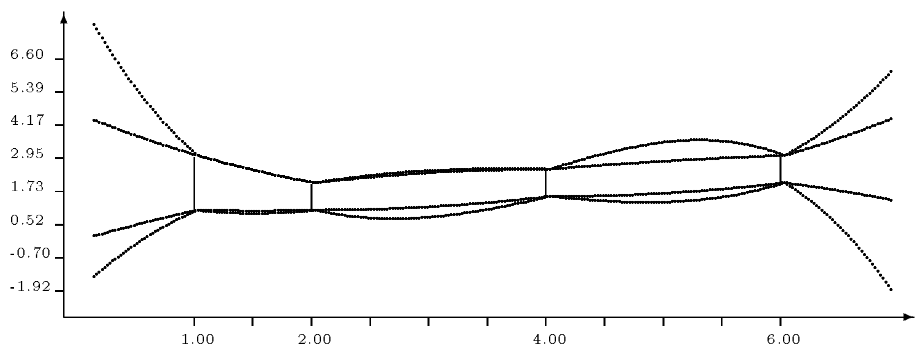

Fig. 1

The program gives the following results for $\eta_2(x, Y; \cdot)$.

Subinterval	Bounding Functions: Lower	Bounding Functions: Upper	Compatible Intervals
$[x_{-\infty}, x_1]$	$Y_1^- Y_3^+ Y_4^-$	$Y_2^+ Y_3^- Y_4^+$	$Y_1 = [1, 3]$
$[x_1, x_2]$	$Y_1^- Y_2^- Y_4^+$	$Y_2^+ Y_3^- Y_4^+$	$Y_2 = [1, 2]$
$[x_2, x_3]$	$Y_2^- Y_3^- Y_4^+$	$Y_2^+ Y_3^+ Y_4^-$	$Y_3 = [1.5, 2.5]$
$[x_3, x_4]$	$Y_2^+ Y_3^- Y_4^-$	$Y_1^- Y_3^+ Y_4^+$	$Y_4 = [2, 3]$
$[x_4, x_\infty]$	$Y_1^- Y_3^+ Y_4^-$	$Y_2^+ Y_3^- Y_4^+$	

We see that the computed compatible intervals coincide with the input intervals, that is the input data are compatible.

Example 2. For the same set of data and for the set of linear modeling functions $\eta_1(\lambda;\xi) = \lambda_1 + \lambda_2\xi$ we obtain

	Bounding Functions		Compatible
Subinterval	Lower	Upper	Intervals
$[x_{-\infty}, x_1]$	$Y_1^- Y_4^+$	$Y_2^+ Y_4^-$	$Y_1 = [1, 2]$
$[x_1, x_2]$	$Y_1^- Y_4^-$	$Y_2^+ Y_4^-$	$Y_2 = [1.2, 2]$
$[x_2, x_3]$	$Y_1^- Y_4^-$	$Y_2^+ Y_4^+$	$Y_3 = [1.6, 2.5]$
$[x_3, x_4]$	$Y_1^- Y_4^-$	$Y_3^+ Y_4^+$	$Y_4 = [2, 3]$
$[x_4, x_\infty]$	$Y_2^+ Y_4^-$	$Y_1^- Y_4^+$	

The interval function $\eta_1(x, Y; \cdot)$, comprising the set of linear modeling functions is presented on Fig. 2. For comparison the function $\eta_2(x, Y; \cdot)$ is given (the latter also appears in Fig. 1). To recognize both functions on Fig. 2 recall that $\eta_1 \subseteq \eta_2$.

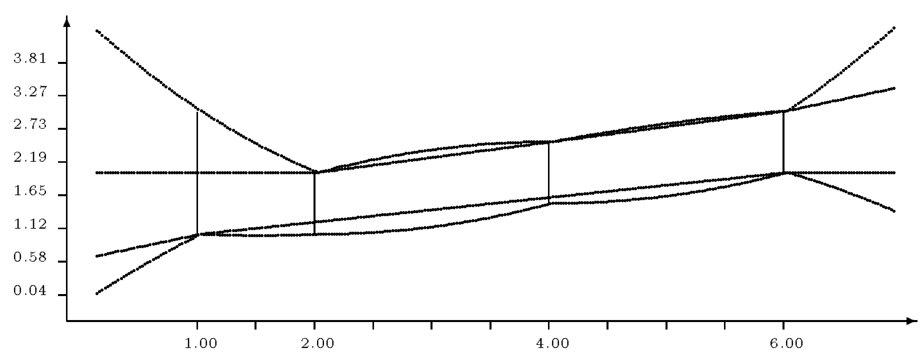

Fig. 2

Remark. To demonstrate the advantages of our method of direct computation of the interval function $\eta_2(x, Y; \cdot)$ let us compute the solution set through the parameter set Λ. Assume that we have computed Λ exactly. We then optimally enclose Λ to obtain an interval vector Λ^I. Then the best result for the upper function is

$$\eta_2^+(\Lambda^I; \xi) = 4.5 + \xi + 0.25\xi^2,$$

and for the lower function

$$\eta_2^-(\Lambda^I; \xi) = -0.1 - \xi - 0.15\xi^2.$$

The width of $\eta_2(\Lambda^I; \xi)$ at $\xi = 6$ is $\omega(\eta_2(\Lambda^I; 6)) = \eta_2^+(\Lambda^I; 6) - \eta_2^-(\Lambda^I; 6) = 31$, whereas the width of $\eta_2(\Lambda; 6) = \eta_2(x, Y; 6)$ as computed by our method is

$$\eta_2^+(x, Y; 6) - \eta_2^-(x, Y; 6) = 1.$$

Example 3. We next consider an example using 6 knots

$$(x, Y) \quad = \quad \begin{pmatrix} 0, & [1, 1.02] \\ 1, & [0.99, 1.25] \\ 2, & [1.04, 1.06] \\ 3, & [1.07, 1.09] \\ 4, & [1.16, 1.18] \\ 5, & [1.23, 1.25] \end{pmatrix}$$

Fig. 3 presents the corresponding polynomials $\eta_5(x, Y; \cdot)$ and $\eta_4(x, Y; \cdot)$. Of course, we have $\eta_4 \subseteq \eta_5$.

4 Conclusion

Our approach and programing tools can be used from experimental scientists, when checking various hypotheses with respect to the type of the modeling functions. Suppose that an interval interpolating function is found which interpolates a given data set and that in the course of experiment some new data are obtained. The new data can be easily checked whether they intersect the available interval solution set. If some of these intersections are empty then it follows that the chosen family of modeling functions is wrong. Then another family of modeling function (possibly involving more parameters or other set of basic functions) should be taken in consideration.

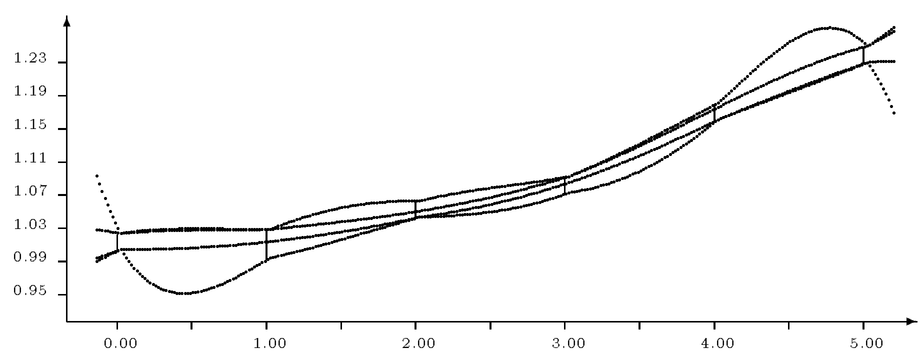

Fig. 3

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