Inner Estimation of the Parametric Tolerable Solution Set

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Abstract

We consider a linear algebraic system $A(p)x = b(q)$, where the elements of the matrix and the right-hand side vector are linear functions of uncertain parameters varying within given intervals. The linear tolerance problem for the so-called parametric tolerable solution set $\Sigma_{tol}(A(p), b(q), [p], [q]) = \{x \in \mathbb{R}^n \mid (\forall p \in [p])(\exists q \in [q])(A(p)x = b(q))\}$ requires an inner estimation of this solution set, that is an interval vector $[y]$, such that $[y] \subseteq \Sigma_{tol}(A(p), b(q), [p], [q])$. In this paper we consider the first methods for finding inner estimation of the parametric tolerable solution set, namely, we propose parametric generalization of the so-called centered approach and of the vertex approach. The results obtained by the two approaches are compared on some numerical examples. The advantages of the parametric approach are demonstrated on problems with independent nonparametric entries and in controllability analysis of linear dynamical systems involving interval uncertainties.

Keywords: interval linear equations, dependent data, tolerable solution set

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1. Introduction

Consider a linear algebraic system

\[ A(p)x = b(p) \]  

having linear uncertainty structure

\[ A(p) := A_0 + \sum_{k=1}^{K} p_k A_k, \quad b(p) := b_0 + \sum_{k=1}^{K} p_k b_k, \]  

where \( A_k \in \mathbb{R}^{m \times n}, b_k \in \mathbb{R}^m, k = 0, \ldots, K \) and the parameters \( p = (p_1, \ldots, p_K)^\top \) are considered to be uncertain, varying within given intervals

\[ p \in [p] = ([p_1], \ldots, [p_K])^\top. \]  

Above, \( \mathbb{R}^n \) and \( \mathbb{R}^{m \times n} \) denote the set of real vectors with \( n \) components and the set of real \( m \times n \) matrices, respectively. A real compact interval is \( [a] = [\underline{a}, \overline{a}] := \{a \in \mathbb{R} \mid \underline{a} \leq a \leq \overline{a}\} \). By \( \mathbb{R}^n, \mathbb{R}^{m \times n} \) we denote the sets of interval \( n \)-vectors and interval \( m \times n \) matrices, respectively. For \( [a] = [\underline{a}, \overline{a}] \), define mid-point \( \text{mid}([a]) = \overline{a} := (\underline{a} + \overline{a})/2 \), radius \( \text{rad}([a]) = \hat{a} := (\overline{a} - \underline{a})/2 \) and absolute value (magnitude) \( |[a]| := \max\{|\underline{a}|, |\overline{a}|\} \). These functions are applied to interval vectors and matrices componentwise. Without loss of generality and in order to have a unique representation (1b), we assume that \( \hat{p}_k > 0 \) for all \( 1 \leq k \leq K \).

The parameter dependent linear system (1a)–(1c) presents a generalization of the nonparametric interval linear system \([A]x = [b], [A] \in \mathbb{IR}^{m \times n}, [b] \in \mathbb{IR}^m\), which was mainly studied until recently. Thus a nonparametric interval system \([A]x = [b], [a] \in \mathbb{IR}^n\), can be considered as a special case of a parametric one involving \( mn + m \) interval parameters \( a_{ij} \in [a_{ij}], b_i \in [b_i], 1 \leq i \leq m, 1 \leq j \leq n. \) However the practical problems, which require solving linear algebraic systems, usually are described by complicated dependencies between the uncertain model parameters. Therefore, the parameter dependent linear systems (1a)–(1c) usually provide a more precise setting and more precise results for the real-life problems involving uncertainties, cf. [11].

The “solution” of a problem involving uncertain (interval) data can be defined in a vast variety of ways. More attention is paid to the parametric \( AE \)-solution sets of the system (1a)–(1c), which are defined by

\[ \Sigma_{AE}^p := \{x \in \mathbb{R}^n \mid (\forall p_A \in [p_A])(\exists p_E \in [p_E])(A(p)x = b(p))\}, \]  

where \( \mathcal{A} \) and \( \mathcal{E} \) are sets of indexes such that \( \mathcal{A} \cup \mathcal{E} = \{1, \ldots, K\}, \mathcal{A} \cap \mathcal{E} = \emptyset. \) For a given index set \( \Pi = \{\pi_1, \ldots, \pi_k\}, p_\Pi \) denotes \( (p_{\pi_1}, \ldots, p_{\pi_k}) \). There
are exactly $2^K$ parametric $AE$-solution sets. Among the $AE$-solution sets most studied and of particular practical interest are: the (parametric) united solution set

$$\Sigma_{uni}(A(p), b(p), [p]) := \{x \in \mathbb{R}^n | (\exists p \in [p])(A(p)x = b(p))\},$$

the (parametric) tolerable solution set

$$\Sigma(A(p_A), b(p_E), [p]) := \{x \in \mathbb{R}^n | (\forall p_A \in [p_A])(\exists p_E \in [p_E])(A(p_A)x = b(p_E))\}$$

and the (parametric) controllable solution set

$$\Sigma(A(p_E), b(p_A), [p]) := \{x \in \mathbb{R}^n | (\forall p_E \in [p_E])(\exists p_A \in [p_A])(A(p_E)x = b(p_A))\}.$$

Although the non-parametric $AE$-solution sets are studied to some extent (see, e.g., [17, 19, 15, 3, 4] and the references given therein), there are only a few results on the more general case of linear parameter dependency. A special case of parametric tolerable solution sets is dealt with in [16]. A full characterization of the parametric $AE$-solution sets is given in [11] and some numerical methods for outer estimation of such solution sets are presented in [12].

In this paper we focus on the parametric tolerable solution set and consider the problem for finding inner estimation of this parametric solution set, that is finding an interval vector $[y]$ such that $[y] \subseteq \Sigma(A(p_A), b(p_E), [p])$.

To our knowledge, this is the first paper which considers methods for inner estimation of the parametric tolerable solution set. In the special case of nonparametric tolerable solution set $\Sigma_{tol}([A], [b])$ this problem was studied in many details in [7, 17]. These articles provide also practical interpretation of the considered problem in terms of tolerance analysis and contain appropriate references to the historical development of the problem and some practical applications. Among the practical applications of the considered problem we will mention the engineering design [8], Leontiev’s input-output models in mathematical economy [14], for parameter identification and controllability analysis in control engineering [20]. It is worth mentioning that $AE$-solution sets, in particular tolerable solution sets, are of particular interest also in the field of fuzzy sets, see [1].

The aim of our study is to give a parametric generalization of the so-called “centered” approach, developed in [7] for the nonparametric case, and to compare the results of this approach to the results obtained by the so-called “vertex” approach, studied in [17] for the nonparametric case and considered in [12] for the parametric outer problem. Section 2 below contains some background results on the explicit description of parametric $AE$-solution sets. A parametric generalization of the centered approach for inner
estimation of the parametric tolerable solution set is presented in Section 3 together with a discussion of its computer implementation. Two forms of the vertex approach for the considered problem are discussed in Section 4 and the obtained approximations are compared to the approximation obtained by the centered approach on some illustrative numerical examples. In Section 5 we present the advantages of the parametric approach for problems with independent nonparametric entries and an application to controllability analysis of linear dynamical systems involving interval uncertainties. The paper ends with some concluding remarks.

2. Theoretical Background

**Definition 1 ([10]).** A parameter \( p_k \), \( 1 \leq k \leq K \), is of 1st class if it occurs in only one equation of the system (1a).

It does not matter how many times a 1st class parameter appears within an equation. A parameter \( p_k \) is of 1st class iff the vector \( b_k - A_k x \), where \( x \) is a vector of formal variables, has only one nonzero component.

**Definition 2 ([10]).** A parameter \( p_k \), \( 1 \leq k \leq K \), is of 2nd class if it is involved in more than one equation of the system (1a).

A parameter \( p_k \) is of 2nd class iff the vector \( b_k - A_k x \), where \( x \) is a vector of formal variables, has more than one nonzero components.

How to obtain explicit description of parametric \( AE \)-solution sets by means of Fourier-Motzkin like parameter elimination of existentially quantified parameters is discussed in [11]. The description is explicit if the system involves only 1st class existentially quantified parameters, called \( E \)-parameters for short\(^1\), and in some other special cases. In the general case the explicit description of a parametric \( AE \)-solution set can be obtained by an algorithmic procedure. The next theorem presents a special case of parametric \( AE \)-solution sets and its explicit description in two equivalent forms: by interval inclusions and by absolute-value inequalities. An early result of S. Shary [18, Proposition 1] gives another equivalent description of the \( AE \)-solution set for 1st class \( E \)-parameters in the matrix and independent right-hand side \([b]\).

\(^1\)similarly, universally quantified parameters are called \( A \)-parameters
Theorem 1 ([11]). Let $A(p)x = b(p)$ involves only 1st class $E$-parameters. A point $x \in \mathbb{R}^n$ belongs to $\Sigma_{AE}$, if and only if
\[
\sum_{\nu \in \mathcal{A}} (A_\nu x - b_\nu)[p_\nu] \subseteq b_0 - A_0 x + \sum_{\mu \in \mathcal{E}} (b_\mu - A_\mu x)[p_\mu],
\]
equivalently
\[
|A(\hat{p})x - b(\hat{p})| \leq \sum_{k=1}^{K} \delta_k |A_k x - b_k| \hat{p}_k,
\]
where $\delta_k := \{1 \text{ if } k \in \mathcal{E}, -1 \text{ if } k \in \mathcal{A}\}$.

Corollary 1. Let all $p_\mu$, $\mu \in \mathcal{E}$, be 1st class parameters. Then for the parametric tolerable solution set we have
\[
\Sigma(A(p_A), b(p_E), [p]) = \left\{ x \in \mathbb{R}^n \mid |A(\hat{p})x - b(\hat{p})| \leq \sum_{\mu \in \mathcal{E}} \hat{p}_\mu |b_\mu| - \sum_{\nu \in \mathcal{A}} \hat{p}_\nu |A_\nu x| \right\},
\]
where $\sum_{\mu \in \mathcal{E}} \hat{p}_\mu |b_\mu| = \text{rad}(b([p_\mathcal{E}]))$. Equivalently,
\[
\Sigma(A(p_A), b(p_E), [p]) = \left\{ x \in \mathbb{R}^n \mid A_0 x + \sum_{\nu \in \mathcal{A}} (A_\nu x)[p_\nu] \subseteq b([p_\mathcal{E}]) \right\}. \tag{3}
\]

To obtain a complete explicit description of a parametric $AE$-solution set in general, one has to eliminate first all 2nd class $E$-parameters by the Fourier-Motzkin like parameter elimination procedure, see [10]. The elimination of every 2nd class $E$-parameter introduces, in general, new characterizing inequalities/inclusions, besides those given in Theorem 1. The number of characterizing inequalities/inclusions grows exponentially with the number of 2nd class $E$-parameters. The degree of the polynomials involved in the characterizing inequalities/inclusions also depends on the 2nd class $E$-parameters involved in the linear system. However, any parametric tolerable solution set is a convex polyhedron, see [11].

In what follows we assume that the parametric tolerable solution set is not empty non-degenerate (having nonempty interior) and bounded. The first two conditions follow from the methodology for inner estimation we consider; it requires knowledge of a point which belongs to the interior of the parametric solution set. Denote by $\text{int } \mathcal{X}$ the topological interior of the set $\mathcal{X}$ in $\mathbb{R}^n$ with the standard topology. The special case of an unbounded tolerable solution set will be considered in much details in a separate paper.
We need also the so-called inner interval subtraction, defined for $[a], [b] \in \mathbb{IR}$ by

$$[a] \ominus [b] := [a - b, \overline{a} - \overline{b}],$$

where $[a] \ominus [b] = \emptyset$ if $\underbar{a} - \underbar{b} > \overline{a} - \overline{b}$. Inner interval subtraction can be represented by the arithmetic operations and functions in the extended interval space of proper and improper intervals [6, 2].

3. Inner Estimation of $\Sigma_{tol}^P$

An inner estimation of the parametric tolerable solution set can be computed by modifying the so-called centered approach, developed by Neumaier in [7] with much details for the nonparametric united solution set and for the nonparametric tolerable solution set. For simplicity, we first consider in Section 3.1 linear systems having in the right-hand side an interval vector $[b]$ with independent components. Then in Section 3.2 we generalize the presented approach for linear systems, where the right-hand side vector $b(p_E)$ involves arbitrary linear dependencies between some existentially quantified parameters $p_E$. In the last Section 3.3 we discuss the computer implementation of the presented methodology.

3.1. Independent right-hand side

In the parametric case, as in the nonparametric one, we also have to assume knowledge of an interior point $\bar{x} \in \text{int } \Sigma_{tol}(A(p), [b], [p])$. By Theorem 1, the relation

$$A_0\bar{x} + \sum_{k=1}^{K} (A_k\bar{x})[p_k] \subseteq \text{int } [b],$$

presents a sufficient condition for a point $\bar{x} \in \mathbb{R}^n$ to be in the interior of $\Sigma_{tol}(A(p), [b], [p])$. It is still an open problem how to find an interior point $\bar{x}$. A natural trial point is an approximate midpoint solution or a point close to it if the condition (4) is not satisfied. However, the approach presented below cannot be applied if we cannot find (or prove) an interior point, see Example 3.

For a given $\bar{x} \in \text{int } \Sigma_{tol}(A(p), [b], [p])$, we compute the maximal nonnegative number $\eta$, such that

$$\eta \left( A_0[e] + \sum_{k=1}^{K} (A_k[e])[p_k] \right) \subseteq [b] - A_0\bar{x} \ominus \sum_{k=1}^{K} (A_k\bar{x})[p_k],$$

(5)
Theorem 2. For each \( \mathbf{A} \) and \( \eta \geq 0 \), such that (5) holds,
\[ \mathbf{x} + \eta[e] \subseteq \Sigma_{tol}(A(p), [b], [p]) \]

Proof. For each \( x \in \mathbf{x} + \eta[e] \) we have
\[
\mathbf{A}_0 \mathbf{x} + \sum_{k=1}^{K} (A_k x)[p_k] \subseteq \mathbf{A}_0(\mathbf{x} + \eta[e]) + \sum_{k=1}^{K} (A_k(\mathbf{x} + \eta[e])[p_k])
\]
sub distr. \[ \subseteq A_0 \mathbf{x} + \eta \mathbf{A}_0[e] + \sum_{k=1}^{K} (A_k \mathbf{x})[p_k] + \eta \sum_{k=1}^{K} (A_k[e])[p_k] \]
\[ (5) \]
\[ \subseteq A_0 \mathbf{x} + \sum_{k=1}^{K} (A_k \mathbf{x})[p_k] + [b] - A_0 \mathbf{x} \oplus \sum_{k=1}^{K} (A_k \mathbf{x})[p_k] = [b] \]

Thus, \( A_0 \mathbf{x} + \sum_{k=1}^{K} (A_k x)[p_k] \subseteq [b] \) for each \( x \in \mathbf{x} + \eta[e] \) implies \( \mathbf{x} + \eta[e] \subseteq \Sigma_{tol}(A(p), [b], [p]) \) by (3) of Corollary 1.

Example 1 ([12]). Find an inner estimation for the parametric tolerable solution set of the linear system
\[
\begin{pmatrix}
  p_1 & p_1 + 1/2 \\
  -2p_2 & p_2 + 1
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}
= \begin{pmatrix}
  [-1, 2] \\
  [-3, 3]
\end{pmatrix}, \quad p_1, p_2 \in [0, 1].
\]

For the solution \( x = (\frac{3}{7}, \frac{2}{7})^\top \) of \( A(p)x = \mathbf{b} \), the relation (4) holds true. Then from (5) we obtain \( \eta = \frac{16}{35} \) and the interval vector \( ([\frac{-1}{35}, 31/35], [-6/35, 26/35])^\top \), which is an inner estimation of the parametric tolerable solution set presented\(^2\) on Figure 1.

3.2. Parameter dependent right-hand side

If all \( p_\mathbf{e} \) are 1st class parameters,
\[
\Sigma(A(p_\mathbf{A}), b(p_\mathbf{e}), [p]) = \Sigma(A(p_\mathbf{A}), b([p_\mathbf{e}]), [p_\mathbf{A}]),
\]

\(^2\)The explicit description of a parametric AE-solution set by inequalities (see [11]) is used also for its visualization.
Figure 1: The parametric tolerable solution set for the system from Example 1 and its inner estimation found by the centered approach.

see [11], Thus, computing inner estimation of $\Sigma(A(p_A), b(p_E), [p])$ is equivalent to the same problem for $\Sigma_{tol}(A(p), [b], [p])$, where $[b] = b([p_E])$ and $[p] = [p_A]$.

In order to apply the approach, presented in the previous section, to parametric tolerable solution sets for systems where $b(p_E)$ involves 2nd class $E$-parameters, we need to obtain the explicit description of the parametric tolerable solution set by means of inequalities, resp. interval inclusions. Therefore, first, we have to eliminate all 2nd class $E$-parameters by the improved Fourier-Motzkin-like parameter elimination procedure presented in [10, 11]. If $b(p_E)$ involves 2nd class $E$-parameters, the description of $\Sigma(A(p_A), b(p_E), [p])$ involves more inequalities (resp. interval inclusions) than those presented in Corollary 1. Since all 2nd class $E$-parameters are involved only in the right-hand side of the parametric system, all solution set describing inequalities/inclusions are linear with respect to the unknowns. Then, we can apply the approach, presented in the previous section, to a system of interval inclusions

$$U_0 x + \sum_{k \in A} (U_k x)[p_k] \subseteq v([p_E])$$

which corresponds to the explicit description of the parametric tolerable solution set and the number $m$ of the inclusions is greater than the number $n$ of the unknowns. As Neumaier pointed out in [7], there is no essential difference between the square case ($m = n$) and the overdetermined case ($m > n$). The only change is that for $m > n$ one has to find a $\tilde{x}$ by approximately “solving” the overdetermined midpoint system $U(\tilde{p}_A)\tilde{x} = v(\tilde{p}_E)$ by least squares.

Even when $b(p_E)$ involves 2nd class $E$-parameters solving an overdeter-
mined point system is not always necessary. One can use the approximate point solution of the original square midpoint system \( A(\bar{p}_A) x = b(\bar{p}_E) \). However, verifying that a point \( \tilde{x} \) is an interior point of the parametric tolerable solution set has to be done by the relation

\[
U_0\tilde{x} + \sum_{k=1}^{K} (U_k\tilde{x})[p_k] \subseteq \text{int } v([p_E]),
\]

(8)
as well as the number \( \eta \) must be computed from the relation

\[
\eta \left( U_0[e] + \sum_{k=1}^{K} (U_k[e])[p_k] \right) \subseteq v([p_E]) - U_0\tilde{x} \ominus \sum_{k=1}^{K} (U_k\tilde{x})[p_k].
\]

(9)

**Theorem 3.** Assume that the inclusion relation (7) characterizes explicitly a parametric tolerable solution set \( \Sigma(A(p_A), b(p_E), [p]) \). For \( \tilde{x} \in \text{int } \Sigma(A(p_A), b(p_E), [p]) \) and \( \eta \geq 0 \), such that (9) holds,

\[
\tilde{x} + \eta[e] \subseteq \Sigma(A(p_A), b(p_E), [p]).
\]

The proof is similar to the proof of Theorem 2.

**Example 2.** Find an inner estimation for the parametric tolerable solution set of the linear system

\[
\begin{pmatrix}
p_1 & p_2 \\
-2p_1 & p_2 + 1/2
\end{pmatrix} x = \begin{pmatrix}
q_1 \\
q_2 - q_1
\end{pmatrix},
\]

where \( p_1 \in [0, 1], p_2 \in [1/2, 3/2], q_1, q_2 \in [-1, 2] \).

The elimination of the 2nd class \( E \)-parameter \( q_1 \) results in one additional equality

\[
p_1 x_1 - (2p_2 + 1/2)x_2 = -q_2,
\]

which after the elimination of \( q_2 \) and \( p_1, p_2 \) turns into the interval inclusion

\[
x_1[p_1] - 2x_2[p_2] \subseteq \frac{1}{2}x_2 - [-q_2].
\]

Thus, the set of interval inclusions describing the parametric tolerable solution set is

\[
[p_1]x_1 + [p_2]x_2 \subseteq [q_1]
\]

\[
\frac{1}{2}x_2 - 2[p_1]x_1 + [p_2]x_2 \subseteq [q_2] - [q_1]
\]

(10)

\[-\frac{1}{2}x_2 + [p_1]x_1 - 2[p_2]x_2 \subseteq -[q_2].
\]

The solution of the midpoint system \( A(\bar{p}) x = b(\bar{q}) \) is \( x = (3/7, 2/7)^\top \), which satisfies the inclusions (10). One can easily check that for the points
\((-1,0)^\top, (1,0)^\top, (0,-2/7)^\top, (0,4/7)^\top, (-2/15,8/15)^\top, \) etc., which belong to the boundary of the parametric tolerable solution set (see Figure 2), the inclusion in the interior (8) does not hold. The computation of \(\eta\) by (9) gives \(2/9\), and thus the interval vector \(([13/63,41/63],[4/63,32/63])^\top\) presents an inner estimation of the parametric tolerable solution set, as seen on Figure 2.

![Figure 2: The parametric tolerable solution set for the system from Example 2 and its inner estimation (dotted line) found by the centered approach.](image)

**Example 3.** Consider the parametric linear system

\[
\begin{pmatrix}
p & 0 \\
0 & p
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
= \begin{pmatrix} q \\ q \end{pmatrix}.
\]

The set of interval inclusions describing the parametric tolerable solution set for this system is

\[
[p] \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) \subseteq \begin{pmatrix} q \\ q \\ 0 \end{pmatrix}.
\]

For \(p \in [-1,1], q \in [-1,1]\), resolving the above inclusion (or the equivalent system of inequalities), we obtain the degenerate parametric tolerable solution set \(\{-1 \leq x_1 \leq 1, x_2 = x_1\}\). Since finding an interior point is not possible for any degenerate solution set, the centered approach is not applicable for such solution sets, although they are not empty. For \(p \in [-1,1], q \in [1,2]\), the parametric tolerable solution set is the empty set.

3.3. Numerical issues

In computational terms, from (5)

\[
\eta = \min_{1 \leq i \leq m} \frac{\min\{|r_i^2|, |r_i|\}}{d_i},
\]
where, with the notation \( A_k := (a_{k,ij}) \in \mathbb{R}^{m \times n} \) for \( k = 0, \ldots, K \), \( d_i = \sum_{j=1}^n |a_{0,ij}| + \sum_{k=1}^K \left( \sum_{j=1}^n |a_{k,ij}| \right) \max \{ |p_k|, |\tilde{p}_k| \} \), \( d_i \neq 0 \) and

\[
[r] := [b] \oplus \left( A_0 \tilde{x} + \sum_{k=1}^K (A_k \tilde{x})[p_k] \right).
\]

The formula for inner interval subtraction implies

\[
x = b - A(\bar{p}) \tilde{x} + \sum_{k=1}^K |A_k \tilde{x}| \bar{p}_k, \quad \bar{r} = b - A(\bar{p}) \tilde{x} - \sum_{k=1}^K |A_k \tilde{x}| \bar{p}_k.
\]

On another side, for \( \tilde{x} \in \Sigma_\text{tol}(A(p), [b], [p]), 0 \in [r] \), that is \( x \leq 0 \leq \bar{r} \). Thus, \( \min(|\bar{r}_i|, |\tilde{r}_i|) = \min(-\bar{r}_i, \tilde{r}_i) \). For the latter and \( [t] = A_0 \tilde{x} + \sum_{k=1}^K (A_k \tilde{x})[p_k] \) we have

\[
\min \{ -\bar{r}_i, \tilde{r}_i \} = \min \left\{ \hat{b}_i - (\bar{b}_i - \bar{t}_i), \hat{b}_i - (\tilde{t}_i - \tilde{b}_i) \right\} = \hat{b}_i - \max \left\{ \bar{b}_i - \bar{t}_i, \tilde{t}_i - \tilde{b}_i \right\} = \hat{b}_i - |\bar{b}_i - |t_i||,
\]

which implies

\[
\eta = \min_{1 \leq i \leq m} \frac{\hat{b}_i - |\bar{b}_i - (A_{0,i} \tilde{x} + \sum_{k=1}^K (A_{k,i} \tilde{x})[p_k])|}{d_i}.
\]

In the presence of round-off errors, the resulting inner estimation is valid if \( [r] = [b] \oplus \left( A_0 \tilde{x} + \sum_{k=1}^K (A_k \tilde{x})[p_k] \right) \) is rounded inward, \( d_i \) are rounded upwards, \( \eta \) is rounded downwards, and \( \tilde{x} + \eta[e] \) is rounded inward.

One can use a correct computer implementation of the arithmetic on proper and improper intervals, called Kaucher interval arithmetic [6] or modal interval arithmetic [2], and its properties in order to provide the necessary rounding directions by outwardly rounded interval operations, cf. [2]. The following computational chain provides a guaranteed inner estimation \([z]\) of the parametric tolerable solution set.

\[
[d] = A_0[e] + \sum_{k=1}^K (A_k[e])[p_k]
\]

\[
[r] = \text{dual}(b) - A_0 \tilde{x} - \sum_{k=1}^K (A_k \tilde{x})[p_k]
\]

\[
\eta = \min_{1 \leq i \leq m} \{ \min(|\bar{r}_i|, |\tilde{r}_i|) / d_i \}
\]

\[
[z] = \text{dual}(\tilde{x} + \eta \text{dual}(e))
\]
where all arithmetic operations (except the division on the third line, which is floating-point division rounded to near) are outwardly rounded floating-point interval operations in Kaucher arithmetic. For a Kaucher interval \([a, \overline{a}]\), such that either \(a \leq \underbar{a}\) or \(a \geq \overline{a}\), \(\text{dual}([a, \overline{a}]) := [\overline{a}, a]\). If \(\overline{a}_i > \underbar{a}_i\) for some \(i = 1, \ldots, m\), then the computed inner estimation is the empty set.

Similarly, verifying the condition (4) in floating-point arithmetic we have to compute the left-hand side with outward rounding and \(b([pE])\) with inward rounding.

The computational complexity of the centered approach is \(O(mn)\). However, if \(b([pE])\) involves 2nd class \(E\)-parameters, obtaining the explicit representation of the parametric solution set \(\Sigma(A(pA), b(pE), [p])\) will require some additional effort.

### 4. Vertex Approach for Inner Estimation of \(\Sigma^p_{\text{tol}}\)

It was proven in [12, Theorem 1]'\(^3\) that

\[
\Sigma(A(pA), b(pE), [p]) = \bigcap_{\tilde{p}_A \in \mathcal{B}_A} \Sigma(A(\tilde{p}_A), b(pE), [pE]),
\]

where \(\mathcal{B}_A\) is the set of all vertices of the interval box \([pA]\). Thus,

\[
\bigcap_{\tilde{p}_A \in \mathcal{B}_A} [y(\tilde{p}_A)] \subseteq \Sigma(A(pA), b(pE), [p]),
\]

where \([y(\tilde{p}_A)] \subseteq \Sigma(A(\tilde{p}_A), b(pE), [pE])\) for \(\tilde{p}_A \in \mathcal{B}_A\). Computing \(\bigcap_{\tilde{p}_A \in \mathcal{B}_A} [y(\tilde{p}_A)]\) will be called the “vertex” (or endpoint) approach for finding an inner estimation of a parametric tolerable solution set.

The vertex approach can be applied in two ways, namely, for each \(\tilde{p}_A \in \mathcal{B}_A\) compute \([v(\tilde{p}_A)] \subseteq \Sigma(A(pA), b(pE), [p])\) by applying the centered approach for:

1. a fixed approximate point \(\tilde{x} \in \text{int } \Sigma(A(pA), b(pE), [p]),\) usually \(\tilde{x}\) is the solution of \(A(\tilde{p}_A)\tilde{x} = b(\tilde{p}_E);\)

2. the corresponding approximate points \(\tilde{x}(\tilde{p}_A) \in \text{int } \Sigma(A(pA), b(pE), [p])\) which are solutions of the corresponding point systems \(A(\tilde{p}_A)\tilde{x} = b(\tilde{p}_E), \tilde{p}_A \in \mathcal{B}_A.\)

---

\(^3\)Theorem 1 in [12] concerns the general case of an arbitrary parametric \(AE\)-solution set.
Applying the second version of the vertex approach one may face several problems. The matrix $A(\tilde{p}_A)$ may be singular when $\Sigma(A(\tilde{p}_A), b(p_E), [p_E])$ is unbounded for some $\tilde{p}_A \in B_A$. If so, $A(\tilde{p}_A)\tilde{x} = b(\tilde{p}_E)$ can be solved by least squares in order to obtain an approximate point solution. In case when a particular approximate point solution does not belong to the interior of $\Sigma(A(p_A), b(p_E), [p])$, it can be replaced by the solution of $A(\tilde{p}_A)\tilde{x} = b(\tilde{p}_E)$, or another point from the interior of the solution set. The special case when $\Sigma(A(p_A), b(p_E), [p])$ is unbounded will be considered in a separate paper.

Both the vertex approach and the centered approach are implemented in the environment of Mathematica® utilizing a Mathematica package [13] for Kaucher interval arithmetic.

The next examples demonstrate that the inner estimations of a parametric tolerable solution set, obtained by the parametrized centered approach and by the two versions of the vertex approach, can be different.

**Example 4.** For the parametric system defined in Example 1, the inner estimation obtained by the vertex approach around the midpoint solution of the original system coincides with the inner estimation obtained by the parametrized Neumaier’s approach. However, the inner estimation, obtained by the vertex approach at the approximate solutions of each vertex system, is $([-1/10, 4/5], [-11/28, 3/5])^\top$, see Figure 3 a). Note, that the matrix at the endpoint $(0, 0)^\top$ is singular and the approximate solution in this case is computed by the least squares. Besides, the approximate solution of the linear system at the vertex $(0, 1)^\top$ does not satisfy the condition (4). Therefore for the inner estimation of the solution set of the system at this vertex we used the approximate midpoint solution of the original system.

![Figure 3](image-url) - Figure 3: a) $\Sigma_{tol}(A(p), [b], [p])$ for the system in Example 1 and its inner estimations obtained by: the centered approach (dash line) and the combined vertex approach (solid line). b) $\Sigma_{tol}(A([p]), [b])$ in gray, where $A(p)$, $[b]$ and $[p]$ are defined in Example 1, and its inner estimation obtained by the vertex approach together with a).
A corresponding nonparametric tolerable solution set \( \Sigma_{tol}(A([p]), [b]) \) is a subset of the parametric tolerable solution set \( \Sigma_{tol}(A(p), [b], [p]) \) for any parameter vector \( p \). Since the end-point approach provides the best inner estimation for a nonparametric tolerable solution set, see also Section 5.1, it may happen that the inner estimation (obtained by the end-point approach) of the corresponding nonparametric tolerable solution set has bigger volume than the obtained inner estimations of the parametric tolerable solution set. This is shown on Figure 3 b) for the present example.

**Example 5.** Consider the linear parametric system

\[
\begin{pmatrix}
-2p_1 & p_2 + 1/2 \\
p_1 & -2p_2 - 1/2
\end{pmatrix} x = \begin{pmatrix}
-1, 2 \\
-3, 3
\end{pmatrix}, \quad p_1 \in [0, 1], p_2 \in [1/2, 3/2].
\]

The solution of the midpoint system is \((-5/7, -1/7)^T\) and belongs to the interior of the parametric tolerable solution set. The inner estimation obtained by the vertex approach at the midpoint solution is \((-13/14, -1/2], [-5/14, 1/14])^T\) (presented on Figure 4 by solid line), while the inner estimation obtained by the parametrized Neumaier’s approach is \((-25/28, -15/28], [-9/28, 1/28])^T\) (presented on Figure 4 by dotted line). The second version of the vertex approach gives an inner estimation \((-73/130, 1/40], [-19/40, 1/4])^T\). Applying this approach the matrices at the vertices \((0, 1/2)^T\) and \((0, 3/2)^T\) are singular and the corresponding approximate solutions are obtained by least squares.

Figure 4: The parametric tolerable solution set for the system from Example 5 and its inner estimations obtained by the centered approach (dotted line) and by the two vertex approaches (solid line and dash line).
5. Applications

5.1. Nonparametric Tolerable Problems

The computational complexity of the vertex approach applied to a nonparametric tolerable solution set $\Sigma_{\text{tol}}([A], [b])$, which is considered as a parametric one involving $mn$ independent parameters in the matrix $[A] \in \mathbb{R}^{m \times n}$, is $O(2^{mn})$. In [17] S. Shary proposed a complicated search-like algorithm by which he reduced the computational complexity of the nonparametric vertex approach to $O(m^2n)$. This better computational complexity can be achieved in another simpler way by considering a suitable equivalent parametric tolerable solution set. Namely, we consider an equivalent parametric tolerable problem $\Sigma_{\text{tol}}(A(p), [b], [p])$, where $A(p) = A_0 + \sum_{k=1}^{n} A_k p_k$ is defined by

$$A_0 = \text{mid}([A])$$

$$A_{k, \nu} = \begin{cases} \text{rad}([A]_{*,k}) & \text{if } \nu = k, \\ 0 & \text{if } \nu \neq k, \end{cases} \quad \nu = 1, \ldots, n$$

$$p_k \in [-1, 1], \quad k = 1, \ldots, n.$$ (13)

For a matrix $A \in \mathbb{R}^{m \times n}$, $A_{*,j}$ denotes the $j$-th column of $A$.

The relation

$$\Sigma_{\text{tol}}([A], [b]) = \Sigma_{\text{tol}}(A(p), [b], [p])$$ (14)

follows from the inclusion relations between parametric tolerable solution sets, proven in [11]. Then, an inner estimation of $\Sigma_{\text{tol}}([A], [b])$ can be obtained with computational complexity $O(2^n)$ by applying the vertex approach for computing the inner estimation of $\Sigma_{\text{tol}}(A(p), [b], [p])$, where $A(p)$ is the row-independent parametric matrix defined by (11)–(13).

Definition 3 ([11], [16]). A parametric matrix $A(p)$, defined by (1b), is called row-independent if for all $k \in \{1, \ldots, K\}$ and all $i \in \{1, \ldots, m\}$, $\text{Card}(\mathcal{J}) < 2$, where $\mathcal{J} := \{j \mid 1 \leq j \leq n, a_{k,i,j} \neq 0\}$.

Since the relation (14) holds true for any row-independent parametric matrix $A(p)$ such that $A([p]) = [A]$, instead of the parametric matrix (11)–(13) one can consider any other row-independent parametric matrix $A(p)$ such that $A([p]) = [A]$. Examples 2 and 5 above illustrate this property. Considering an equivalent parametric tolerable solution set instead of a nonparametric one may reduce the computational complexity of any other problem (besides for the inner estimation) related to the nonparametric tolerable solution set.
Example 6 (Example 6.3 in [3]). Consider the nonparametric tolerable solution set $\Sigma_{\text{tol}}([A], [b])$, where $[A] = A([p])$ and

$$A([p]) := \begin{pmatrix} p_1 & p_2 & 0 & 0 & 0 & 0 \\ p_1 & p_2 & p_3 & 0 & 0 & 0 \\ 0 & p_2 & p_3 & p_4 & 0 & 0 \\ 0 & 0 & p_3 & p_4 & p_5 & 0 \\ 0 & 0 & 0 & p_4 & p_5 & p_6 \\ 0 & 0 & 0 & 0 & p_5 & p_6 \end{pmatrix}, \quad [b] = \begin{pmatrix} [0.9, 1.1] \\ [-1.1, -0.9] \\ [0.9, 1.1] \\ [-1.1, -0.9] \\ [0.9, 1.1] \\ [-1.1, -0.9] \end{pmatrix},$$

$p_1, \ldots, p_6 \in [0.999, 1.001]$. Since $\Sigma_{\text{tol}}([A], [b]) = \Sigma_{\text{tol}}(A([p]), [b], [p])$, we follow the parametric approach, presented above in this section. Thus, the parametric centered approach for $\Sigma_{\text{tol}}(A([p]), [b], [p])$ gives the following interval vector

$$(( [-0.0316, 0.0316], [0.9684, 1.0316], [-2.0316, -1.9684], [1.9684, 2.0316], [-1.0316, -0.9684], [0.0316, 0.0316])^\top,$$

which is contained in $\Sigma_{\text{tol}}([A], [b])$. An inner estimation of $\Sigma_{\text{tol}}([A], [b])$ in the form of a parallelepiped was found in [3], while a straightforward application of the formal algebraic approach considered in [19] fails. After several trials to squeeze the components of $[b]$, as recommended by [19, Theorem 6.4], one may come to a system whose formal solution is an inner estimation. For $b_1, b_3, b_5 \in [0.95, 1.1]$, S. Shary has found

$$(( [-0.0499, 0.0499], [1.001, 1.0489], [-2.0479, -2.002], [2.002, 2.0479], [-1.0489, -1.001], [-0.0499, 0.0499])^\top.$$

5.2. Controllability of Linear Dynamical Systems

Consider a time-invariant, continuous-time dynamical system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t), \end{cases}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$ and the vector-valued functions $u, x, y$ are called the input, the state, and the output of the system, respectively. In case when the matrices $A$ and $B$ are uncertain and varying within given interval matrices $A \in [A], B \in [B]$, we have a family of dynamical systems. It was shown by Sokolova and Kuzmina in [20] that
if $[A]$ is asymptotically stable, the interval object is completely controllable if and only if
\[
\text{rank}[V] = n, \quad \text{where } [V] \subseteq \Sigma_{\text{tol}}([A], [B]) \quad \text{and}
\]
\[
\Sigma_{\text{tol}}([A], [B]) := \{ V \in \mathbb{R}^{n \times n} \mid (\forall A \in [A])(\exists B \in [B])(AV + VA^T = -BB^T) \}.
\]
Thus, the controllability analysis reduces to finding inner estimation of the tolerable solution set for the interval Lyapunov matrix equation
\[
AV + VA^T = -BB^T.
\]
It is a common approach to transform a matrix equation into a system of linear equations

\[ Px = f, \]

where $P = I_n \otimes A + A \otimes I_n$, $\otimes$ denotes the Kronecker product (see [5], e.g.),
\[ x = \text{vec}(V), \quad f = \text{vec}(-BB^T), \quad \text{vec} \text{ is the operation of stacking the columns of a matrix in order to obtain one long vector.} \]

It is obvious that the matrix $P$ involves complicated linear dependencies between the elements $a_{ij} \in [a_{ij}]$ of $A \in [A]$ and we need parametric methods for finding inner estimation for the parametric tolerable solution set of the linear system

\[ P(a_{ij})x = f(b_{ij}). \]

**Example 7.** In order to illustrate the parameter dependencies in the matrix $P$, we consider an interval Lyapunov matrix equation of small size, where

\[
\text{mid}([A]) = \begin{pmatrix} -1 & -1 & 2 \\ 3 & -2 & -5 \\ -2 & 1 & -5 \end{pmatrix}, \quad \text{rad}([a_{ij}]) = 3/100,
\]

\[
[B] = ([31/4, 41/4], [-37/4, -27/4], [103/4, 113/4])^T. \quad \text{The matrix of the corresponding linear system has the form}
\]

\[
\begin{pmatrix}
2a_{11} & a_{12} & a_{13} & a_{12} & 0 & 0 & a_{13} & 0 & 0 \\
a_{21} & a_{11} + a_{22} & a_{23} & 0 & a_{12} & 0 & 0 & a_{13} & 0 \\
a_{31} & a_{32} & a_{11} + a_{33} & 0 & 0 & a_{12} & 0 & 0 & a_{13} \\
a_{21} & 0 & 0 & a_{11} + a_{22} & a_{12} & a_{13} & a_{23} & 0 & 0 \\
0 & a_{21} & 0 & a_{21} & 2a_{22} & a_{23} & 0 & a_{23} & 0 \\
0 & 0 & a_{21} & a_{31} & a_{32} & a_{22} + a_{33} & 0 & 0 & a_{23} \\
a_{31} & 0 & 0 & a_{32} & 0 & 0 & a_{11} + a_{33} & a_{12} & a_{13} \\
0 & a_{31} & 0 & 0 & a_{32} & 0 & a_{21} & a_{22} + a_{33} & a_{23} \\
0 & 0 & a_{31} & 0 & 0 & a_{32} & a_{31} & a_{32} & 2a_{33}
\end{pmatrix}.
\]

For simplicity we find inner estimation of $\Sigma_{\text{tol}}(P(a_{ij}), f([b_{ij}]), [A])$. For the approximate midpoint solution

\[
\tilde{x} \approx (110.142, -16.3874, 26.2365, -16.3874, 93.4944, -40.674, 26.2365, -40.674, 54.4268)^T
\]
the parametric centered approach gives the symmetric interval matrix

\[
\begin{bmatrix}
92.951, 94.0378 & -41.2174, -40.1306 & 53.8834, 54.9702
\end{bmatrix}.
\]

The vertex approach at the above \( \tilde{x} \) gives a symmetric interval matrix which is slightly wider than the above. However, applying the second version of the vertex approach, the intersection of the inner estimations at the vertices of \([A]\) is an empty set for some solution components.

The same parametric approach can be applied if the matrices involved in the dynamical system depend linearly on a number of interval parameters.

6. Conclusion

This paper presents a first attempt to find inner estimation of the parametric tolerable solution set. For this solution set a parametric generalization of the so-called centered approach is developed and two versions of the so-called vertex approach are discussed. In general, the three kinds of inner estimations vary in their properties. Therefore, the estimations obtained by the considered approaches are compared on some numerical examples which illustrate the properties of the estimations. The examples show that neither of the considered approaches gives inner estimation which is optimal with respect to inclusion. Note that the inner estimations obtained by the centered approach and by the vertex approach at a fixed \( \tilde{x} \) depend much on the choice of \( \tilde{x} \). Sometimes the second version of the vertex approach provides an inner estimation which has a bigger volume than those provided by the other approaches. This is not strange in view that this approach provides the best outer enclosure (sometimes the interval hull) for the parametric tolerable solution set, cf. [12]. However, for some other problems (see, e.g. Example 7), having a nonempty parametric tolerable solution set, the second version of the vertex approach may fail if the intersection of the inner estimations for the vertex problems is an empty set for some solution components.

Finding a point which belongs to the interior of the parametric tolerable solution set is a necessary condition for the application of the centered approach. Finding an interior point proves that the solution set is not empty. As the Example 3 shows, we cannot find any interior point for the nonempty degenerate (parametric tolerable) solution sets. In these cases finding a point which belongs to the solution set itself proves that the latter is not empty.
Otherwise, proving that a solution set is nonempty can be done by solving the solution set describing inequalities.

For the special case of nonparametric $AE$- (in particular tolerable) solution sets $\Sigma_{AE}([A], [b])$, there is a third approach for finding their inner estimation. The so-called formal algebraic approach, see [19], seeks an inner estimation $[y] \subseteq \Sigma_{AE}([A], [b])$ as a formal algebraic solution to an appropriate interval linear system in Kaucher interval arithmetic. Although it is proven in [15] that the formal algebraic approach provides inclusion maximal inner estimation, this approach is not always feasible (if the formal solution does not exists or does not provide the required inner estimation, this does not mean that the corresponding $\Sigma_{AE}([A], [b])$ is empty). A right-preconditioning process for the formal algebraic approach, developed by Goldsztejn in [3], is usually feasible and provides inner estimations in the form of parallelepipeds which are more precise than the estimates in form of interval vectors. That is why, it might be useful to develop and study a parametric generalization of the formal algebraic approach and the right-preconditioning process for parametric $AE$-solution sets.

Since the centered approach has polynomial computational complexity, it is quite suitable for large scale problems as those aiming controllability analysis of linear dynamical systems.

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(http://cose.math.bas.bg/webMathematica/webComputing/ParametricAESSet.jsp)


