Improved Enclosure for Some Parametric Solution Sets with Linear Shape

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Abstract

Consider linear algebraic systems where the elements of the matrix and the right-hand side vector depend linearly on a number of interval parameters. We prove some sufficient conditions for the united parametric solution set of such a system to have linear boundary. These conditions imply an equivalent representation of the parametric system where each parameter appears once in a diagonal matrix. The latter representation allows us to expand the scope of applicability of the best known so far interval method, developed by A. Neumaier and A. Pownuk, for enclosing the parametric solution set and to generalize the method for systems where the parameter dependencies connect the matrix and the right-hand side vector. Some examples demonstrate that: parametric solution sets with linear boundary appear in various application domains, the generalized method improves the solution enclosure and the proven sufficient conditions can be helpful for solving various other interval problems.

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1. Introduction

Denote by $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ the set of real vectors with n components and the set of real $m \times n$ matrices, respectively. A real compact interval is $[a] = [\underline{a}, \overline{a}] := \{a \in \mathbb{R} \mid \underline{a} \le a \le \overline{a}\}$. By $\mathbb{IR}^n, \mathbb{IR}^{m\times n}$ we denote the sets of interval n-vectors and interval $m \times n$ matrices, respectively. We consider linear algebraic systems having linear uncertainty structure

$$\begin{aligned} A(p)x &= b(p) \\ A(p) := A_0 + \sum_{k=1}^{K} p_k A_k, &\qquad b(p) := b_0 + \sum_{k=1}^{K} p_k b_k, \end{aligned} \tag{1}$$

where $A_k \in \mathbb{R}^{m\times n}$, $b_k \in \mathbb{R}^m$, $k = 0, \ldots, K$ and the parameters $p = (p_1, \ldots, p_K)^\top$ are considered to be uncertain and varying within given intervals $[p] = ([p_1], \ldots, [p_K])^\top$. For $[a] = [\underline{a}, \overline{a}]$, define mid-point $\check{a} := (\underline{a} + \overline{a})/2$, radius $\hat{a} := (\overline{a} - \underline{a})/2$ and absolute value (magnitude) $|[a]| := \max\{|\underline{a}|, |\overline{a}|\}$. These functions are applied to interval vectors and matrices componentwise. Without loss of generality and in order to have a unique representation (1), we assume that $\hat{p}_k > 0$ for all $1 \le k \le K$.

Parametric linear systems are common in applied mathematics where there are complicated dependencies between the model parameters which, due to various kinds of inexact knowledge, are considered as uncertain and varying within given intervals. A solution set of (1) can be defined in many different ways, cf. [1]. Usually, the set of solutions to (1) which is sought for is the united parametric solution set

$$\Sigma^p_{uni} = \Sigma\left(A(p), b(p), [p]\right) \quad := \quad \{x \in \mathbb{R}^n \mid \exists p \in [p], A(p)x = b(p)\}\,. \tag{2}$$

Performing worst-case analysis of uncertain systems, one is interested in finding an outer interval estimation $[u]$ of Σ^p_{uni}, that is finding an interval vector $[u] \in \mathbb{IR}^n$ such that

$$[u] \supseteq \square\Sigma^p_{uni} \supseteq \Sigma^p_{uni},$$

where $\square\Sigma^p_{uni}$ denotes the exact interval hull of a bounded solution set

$$\square\Sigma^p_{uni} := \bigcap\{[x] \in \mathbb{IR}^n \mid [x] \supseteq \Sigma^p_{uni}\}.$$

The goal is to develop efficient interval methods delivering interval vectors $[u]$ that are as close to $\square\Sigma^p_{uni}$ as possible. In what follows by solving a parametric linear system we will mean finding an interval vector that encloses the solution set and we will omit the adjective "united" when it does not cause a confusion.

Definition 1 ([2]). *A parametric matrix $A(p) \in \mathbb{R}^{n\times n}$ is strongly regular on $[p]$ if any of the following matrices is regular*

$$[B] := \square\{A^{-1}(\check{p})A(p) \mid p \in [p]\}, \qquad [B'] := \square\{A(p)A^{-1}(\check{p}) \mid p \in [p]\}.$$

Most of the interval methods for solving parametric linear systems, e.g., [3, 4, 2, 5], require or assume strong regularity of the parametric matrix. Since strong regularity is only a sufficient condition for regularity of a parametric matrix in a given interval box, these methods sometimes fail especially when the parameter intervals are large. In [6] Neumaier and Pownuk proposed an iterative method, the best known interval method so far for solving parametric linear systems. The method has the following advantages:

- it does not require strong regularity of $A(p)$ on $[p]$;
- the method is applicable for large parameter intervals;
- it usually provides rigorous solution estimation with a small overestimation factor;
- the method is scalable to comparatively high-dimensional systems involving many parameters. In [6] the merits of the method are demonstrated on linear systems with over 5000 variables and over 10000 parameters which appear in finite element (FE) analysis of uncertain truss structures.

Inspired by parametric linear systems that appear in FE analysis of linear elastic problems with interval parameters, the only structural assumption of the method is that the parametric linear system (1) has the form

$$(K + BDA)x \;=\; a + Fb, \tag{3}$$

with interval parameters isolated only in D and b.

In general, not every parametric system can be represented in the form (3). In FE analysis of truss structures form (3) appears naturally when deriving the element stiffness matrices. As we will show in Section 3.2, there are other domain-specific problems that can benefit from the method. However the corresponding parametric systems are in the general form (1) and some interval parameters appear in both the matrix and the right-hand side vector. Therefore, the goal of the present paper is two-fold:

g.1 to present a way for testing if a general parametric system can be transformed into a form similar to (3), where the parameters are involved in the diagonal matrix D, and to construct this form;

g.2 to generalize the method of Neumaier and Pownuk for suitable parametric systems involving parameter dependencies in both the matrix and the right-hand side vector.

The paper is organized as follows: Section 2 is devoted to parametric solution sets that have linear boundary with respect to some (or all) parameters. Some sufficient conditions for a parameter to exhibit this property are derived basing on an algorithm for explicit description of a parametric united solution set. The implications of these conditions for achieving the goal [g.1] above are presented at the end of Section 2.2 and for the more general class of parametric AE-solution sets are discussed in Section 2.3. In Section 3.1 the iterative method from [6] for finding outer estimation of Σ^p_{uni} is generalized to the case of dependencies between the matrix and the right-hand side vector. Section 3.2 contains some numerical examples showing that parametric solution sets with linear shape appear in various domain specific problems and demonstrating the advantages of the theory presented in the paper. Section 4 contains some conclusions.

2. Solution Sets with Linear Shape

In this section we present some properties of the parametric solution sets that are implied from the explicit description of the solution set.

2.1. Explicit description of parametric solution sets

A general procedure, called Fourier-Motzkin-like elimination of the parameters, that leads to a Oettli-Prager-type description of a parametric united solution set is proposed in [7] and improved in [8].

Starting from the following trivial description of Σ^p_{uni},

$$\Sigma^p_{uni} = \{x \in \mathbb{R}^n \mid \exists p_k \in \mathbb{R}, k = 1, \ldots, K : (4)\text{–}(5) \text{ hold}\},$$

where

$$A_0x - b_0 + \sum_{k=1}^{K} p_k\,(A_kx - b_k) \leq 0 \leq A_0x - b_0 + \sum_{k=1}^{K} p_k\,(A_kx - b_k) \qquad (4)$$

$$\check{p}_k - \hat{p}_k \leq \; p_k \; \leq \check{p}_k + \hat{p}_k, \qquad k = 1, \ldots, K, \qquad (5)$$

Theorem 1 below shows how the parameters in the trivial set of inequalities can be eliminated successively in order to obtain a new description not involving p_k, $k = 1, \ldots, K$.

Theorem 1 ([8]). *Let $g_\lambda(x), f_{\lambda\nu,1}(x), f_{\lambda\nu,2}(x), f_{\lambda\mu}(x)$, $\lambda = 1, \ldots, m(\geq n)$, $\nu = 1, \ldots, k_1 - 1$, $k_1 \geq 1$, be real-valued functions of $x = (x_1, \ldots, x_n)^\top$ on some subset $D \subseteq \mathbb{R}^n$. Assume that there exists a nonempty set $\mathcal{T} \subseteq \{1, \ldots, m\}$ such that $f_{\lambda k_1}(x) \not\equiv 0$ for all $\lambda \in \mathcal{T}$. For the parameters p_μ, $\mu = k_1, \ldots, K$ varying in $\mathbb{R}$ and for x varying in D define the sets S_1, S_2 by*

$$
\begin{aligned}
S_1 &:= \{x \in D \mid \exists p_k \in \mathbb{R}, k = k_1, \ldots, K : (6), (7) \text{ hold}\},\\
S_2 &:= \{x \in D \mid \exists p_k \in \mathbb{R}, k = k_1 + 1, \ldots, K : (8), (9), (10) \text{ hold}\},
\end{aligned}
$$

where inequalities (6), (7) and (8), (9), (10), respectively, are given by

$$
\begin{aligned}
g_\lambda(x) + \sum_{\nu=1}^{k_1-1} f_{\lambda\nu,1}(x)\check{p}_\nu \mp \sum_{\nu=1}^{k_1-1} f_{\lambda\nu,2}(x)\hat{p}_\nu + \\
\sum_{\mu=k_1+1}^{K} f_{\lambda\mu}(x)p_\mu \leq -f_{\lambda k_1}(x)p_{k_1} \leq \cdots \qquad \lambda = 1, \ldots, m
\end{aligned}
\tag{6}
$$

$$
\check{p}_\mu - \hat{p}_\mu \leq p_\mu \leq \check{p}_\mu + \hat{p}_\mu, \qquad \mu = k_1, \ldots, K,
\tag{7}
$$

$$
\begin{aligned}
g_\lambda(x) + \sum_{\nu=1}^{k_1-1} f_{\lambda\nu,1}(x)\check{p}_\nu \mp \sum_{\nu=1}^{k_1-1} f_{\lambda\nu,2}(x)\hat{p}_\nu + f_{\lambda k_1}(x)\check{p}_{k_1} \mp |f_{\lambda k_1}(x)|\hat{p}_{k_1} + \\
\sum_{\mu=k_1+1}^{K} f_{\lambda\mu}(x)p_\mu \leq 0 \leq \cdots, \qquad \lambda = 1, \ldots, m
\end{aligned}
\tag{8}
$$

and for $\alpha, \beta \in \mathcal{T}$, $\alpha < \beta$

$$
\begin{aligned}
g_\alpha(x)f_{\beta k_1}(x) - g_\beta(x)f_{\alpha k_1}(x) + \sum_{\nu=1}^{k_1-1} \left(f_{\beta k_1}(x)f_{\alpha\nu,1}(x) - f_{\alpha k_1}(x)f_{\beta\nu,1}(x)\right)\check{p}_\nu \mp \\
\sum_{\nu=1}^{k_1-1} \left(|f_{\beta k_1}(x)|f_{\alpha\nu,2}(x) + |f_{\alpha k_1}(x)|f_{\beta\nu,2}(x)\right)\hat{p}_\nu + \\
\sum_{\mu=k_1+1}^{K} \left(f_{\alpha\mu}(x)f_{\beta k_1}(x) - f_{\beta\mu}(x)f_{\alpha k_1}(x)\right)p_\mu \leq 0 \leq \cdots,
\end{aligned}
\tag{9}
$$

$$\check{p}_\mu - \hat{p}_\mu \leq \ p_\mu \ \leq \check{p}_\mu + \hat{p}_\mu, \qquad \mu = k_1+1, \ldots, K. \tag{10}$$

The "$\cdots$" in the right-hand side inequalities denotes the left side expression in the left inequality with the bottom sign (+) in front of the terms involving a parameter radius, while "$\mp$" in the left inequality should be read "$-$"[1]*. (Trivial inequalities which are true for any $x \in \mathbb{R}^n$ can be omitted.) Then $S_1 = S_2$.*

The inequalities (8) are called *end-point inequalities* because they are obtained by combining (6) with (7). The inequalities (9) are called *cross inequality pairs* because they are obtained by combining two inequality pairs (6). Note that the resulting inequalities (8) and (9) have the form (6) which allows the elimination process to continue with the next parameters. The following corollary is indispensable in the implementation of Theorem 1 and keeps the degree of the polynomials involved in the cross inequalities (9) minimal.

Corollary 1 ([7]). *With the notations and the assumptions of Theorem 1 for α, $\beta \in \mathcal{T}$, let $f_{k_1}(x)$, $\tilde{f}_{\alpha k_1}(x)$, $\tilde{f}_{\beta k_1}(x)$ be real-valued functions such that*

$$f_{\alpha k_1}(x) = f_{k_1}(x)\tilde{f}_{\alpha k_1}(x), \qquad f_{\beta k_1}(x) = f_{k_1}(x)\tilde{f}_{\beta k_1}(x).$$

Then the assertion of Theorem 1 remains true if $f_{\alpha k_1}(x)$, $f_{\beta k_1}(x)$ are replaced in (9) by $\tilde{f}_{\alpha k_1}(x)$ and $\tilde{f}_{\beta k_1}(x)$ respectively.

In this paper we apply Corollary 1 when $f_{k_1}(x)$ is a common factor for all $\alpha, \beta \in \mathcal{T}$ and the corresponding $\tilde{f}_{\alpha k_1}(x)$, $\tilde{f}_{\beta k_1}(x)$ are numerical constants.

Definition 2 ([8]). *A parameter p_k, $1 \leq k \leq K$, is of 1st class if it occurs in only one equation of the system (1).*

Definition 3 ([8]). *A parameter p_k, $1 \leq k \leq K$, is of 2nd class if it is involved in more than one equation of the system (1).*

---

[1] For example, the expanded (6) is

$$g_\lambda(x) + \sum_{\nu=1}^{k_1-1} f_{\lambda\nu,1}(x)\check{p}_\nu - \sum_{\nu=1}^{k_1-1} f_{\lambda\nu,2}(x)\hat{p}_\nu \ + \sum_{\mu=k_1+1}^{K} f_{\lambda\mu}(x)p_\mu \ \leq -f_{\lambda k_1}(x)p_{k_1} \leq$$
$$g_\lambda(x) + \sum_{\nu=1}^{k_1-1} f_{\lambda\nu,1}(x)\check{p}_\nu + \sum_{\nu=1}^{k_1-1} f_{\lambda\nu,2}(x)\hat{p}_\nu \ + \sum_{\mu=k_1+1}^{K} f_{\lambda\mu}(x)p_\mu.$$

It does not matter how many times a 1st class parameter appears within an equation. A parameter p_k is of 1st class iff the vector $b_k - A_k x$ has only one nonzero component. A parameter p_k is of 2nd class iff the vector $b_k - A_k x$ has more than one nonzero components.

It was proven in [8] that, independent of the order of parameter elimination, the elimination of 1st class parameters does not introduce any cross inequalities. Cross inequalities are generated only by the elimination of 2nd class parameters. The number of cross inequalities and the degree of the polynomials involved in them may increase with each eliminated 2nd class parameter. Thus, we can estimate the shape of a parametric solution set, i.e., the maximal degree of the polynomial equations describing the solution set boundary. An important consequence for the 1st class parameters is given by the following corollary.

Corollary 2 ([1]). *The infimum/supremum of a parametric solution set is attained at particular end-points of the intervals for the 1st class parameters.*

2.2. Main Results

Definition 4. *A parametric solution set is called* linear*, in other words its shape is linear, if the boundary of the solution set consists of parts of hyperplanes.*

The inequalities describing a *linear* parametric solution set involve only linear functions on the coordinate variables.

Theorem 2. *Let for some parameter p_k, $k = 1, \ldots, K$, $g_k(x)$ be the polynomial greatest common divisor (GCD) of the elements of $A_k x - b_k$, where x is the vector of coordinate variables. If $g_k(x) = \mathrm{const}$ for $A_k = 0$ and $g_k(x) \neq \mathrm{const}$ otherwise, then every cross inequality generated in the elimination of this parameter by Theorem 1 involves polynomials whose degree is not greater than the maximal degree of the polynomials involved in the two inequalities that were combined.*

Proof. For 1st class parameters p_k the proof is given in [8]. Let p_k be a 2nd class parameter. With the notations of Theorem 1 let the first $k_1 - 1$ eliminated parameters be of 1st class and next we eliminate the parameters p_k, $k = k_1, \ldots, k_2$ that satisfy the assumption of the theorem. Since $A_k x - b_k = g_k(x) f_k$, $f_k \in \mathbb{R}^n$, $f_k = (f_{1k}, \ldots, f_{nk})^\top$ for $k = k_1, \ldots, k_2$, by Corollary

1 the cross inequalities (9) generated in the elimination of p_{k_1} read as follows

$$f_{\beta k_1} g_\alpha(x) - f_{\alpha k_1} g_\beta(x) + \sum_{\nu=1}^{k_1-1} \left(f_{\beta k_1} f_{\alpha\nu,1}(x) - f_{\alpha k_1} f_{\beta\nu,1}(x)\right) \check{p}_\nu \quad \mp$$

$$\sum_{\nu=1}^{k_1-1} \left(|f_{\beta k_1}| f_{\alpha\nu,2}(x) + |f_{\alpha k_1}| f_{\beta\nu,2}(x)\right) \hat{p}_\nu + \sum_{\mu=k_1+1}^{k_2} g_\mu(x) \left(f_{\beta k_1} f_{\alpha\mu} - f_{\alpha k_1} f_{\beta\mu}\right) p_\mu +$$

$$\sum_{\mu=k_2+1}^{K} \left(f_{\beta k_1} f_{\alpha\mu}(x) - f_{\alpha k_1} f_{\beta\mu}(x)\right) p_\mu \ \le 0 \le \ \cdots . \quad (11)$$

It is obvious that all polynomials involved in the terms of (11) are of first degree. In the elimination of the next parameters, the inequalities (11) will be combined with the parameter inequalities (10). Thus, at the end of the parameter elimination process the solution set characterizing inequalities (11) will involve only polynomials of first degree. In general, the parameters p_μ, $\mu = k_1+1, \ldots, k_2$, are involved in both the initial inequalities (8) and the cross inequalities (11). Thus, the vector of coefficients for a currently eliminated parameter p_μ, $\mu = k_1+1, \ldots, k_2$, will consist of the vector $A_\mu x - b_\mu = g_\mu(x) f_\mu$ augmented by $g_\mu(x)\tilde{f}_\mu(\alpha, \beta)$ for $\alpha, \beta \in \mathcal{T}$, where $\mathcal{T}$ is the indexes of the hitherto generated cross inequalities involving p_μ. For $\mu = k_1+1$, $\tilde{f}_{k_1+1}(\alpha, \beta) = (f_{\beta k_1} f_{\alpha\mu} - f_{\alpha k_1} f_{\beta\mu})$. This implies that the elimination of all remaining parameters p_μ, $\mu = k_1+1, \ldots, k_2$, may increase the number of the characterizing cross inequalities. However, it will not increase the degree of the polynomials involved in the generated new cross inequalities. The number of the cross inequalities generated in the elimination of the parameters p_μ, $\mu = k_2+1, \ldots, K$, will (in general) increase and the degree of the polynomials involved in these inequalities will (in general) increase too.

The theorem can be proven also if we do not fix the above order of parameter elimination. However, the proof will be more cumbersome. □

Lemma 1. *For a given parameter p_k, $k = 1, \ldots, K$, the following conditions are equivalent:*

(i) the nonzero elements of $A_k x - b_k$ are linearly dependent

(ii) $g_k(x)$ =const if $A_k = 0$ and $g_k(x)$ is a nonconstant polynomial of $x_1, \ldots, x_n$ otherwise, where $g_k(x)$ is the polynomial GCD of the elements of $A_k x - b_k$

(iii) *rank*$((A_k|b_k)) = 1$*, where* $(A_k|b_k) \in \mathbb{R}^{n\times(n+1)}$ *is the matrix obtained by augmenting the columns of* A_k *with the vector* b_k*.*

Proof. The proof is trivial. □

While verifying that a parameter contributes linearly to the boundary of a parametric solution set, based on the condition of Lemma 1 (ii), can be done in a programming environment supporting symbolic algebraic computations, verifying the equivalent condition of Lemma 1 (iii) can be based on a pure numeric representation. However, in both cases one should avoid using inexact representations of the numerical values and has to use rational numbers.

For any parameter that satisfies the conditions of Theorem 2 (and therefore Lemma 1 conditions), we will call that it contributes linearly to the boundary of Σ^p_{uni}. In other words, the boundary of Σ^p_{uni} is linear with respect to this parameter.

Corollary 3. *The infimum/supremum of a parametric solution set is attained at particular end-points of the intervals for the parameters that satisfy the conditions of Theorem 2 (and therefore Lemma 1 conditions).*

The next example demonstrates that the conditions of Theorem 2 are only sufficient for a parametric solution set to have linear shape. The parameters in a linear system may not satisfy the conditions of Theorem 2 while the solution set has linear shape.

Example 1. *Consider the parametric linear system* $A(p)x = b(p)$*, where*

$$A(p) = \begin{pmatrix} p_1 & -p_2 \\ -p_2 & p_1 \end{pmatrix}, \quad b(p) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad p_1 \in [\frac{1}{2}, \frac{3}{2}], p_2 \in [-\frac{9}{20}, -\frac{1}{20}].$$

Neither of the 2nd class parameters p_1, p_2 *satisfies the requirements of Theorem 2. Nevertheless, the parametric united solution set* $\Sigma_{uni}(A(p), b(p), [p])$ *has linear shape. The solution set is described by the following inequalities*

$$|A(\check{p})x - b(\check{p})| \leq \sum_{k=1}^{2} \hat{p}_k |A_k x - b_k|, \tag{12}$$

$$\left| x_1 - \frac{x_1^2}{4} - x_2 + \frac{x_2^2}{4} \right| \leq \frac{1}{5}|x_1^2 - x_2^2|, \tag{13}$$

$$|x_1 - x_1^2 - x_2 + x_2^2| \leq \frac{1}{2}|-x_1^2 + x_2^2|. \tag{14}$$

These can be obtained either by applying Theorem 1 and removing superfluous/redundant inequalities, or by applying Theorem 4.1 in [8], where superfluous/redundant inequalities are removed. Both cross inequalities (13)–(14) have a nonconstant common factor $|x_1 - x_2|$ *which reduces (12)–(14) to*

$$\begin{aligned}
|A(\check{p})x - b(\check{p})| &\leq \sum_{k=1}^{2} \hat{p}_k |A_k x - b_k|, \\
|x_1 - x_2| &\geq 0, \\
\left|1 - \frac{x_1}{4} - \frac{x_2}{4}\right| &\leq \frac{1}{5}|x_1 + x_2|, \\
|1 - x_1 - x_2| &\leq \frac{1}{2}|x_1 + x_2|.
\end{aligned}$$

Solving these inequalities gives the line segment $\frac{20}{39} \leq x_1 \leq \frac{20}{11} \ \wedge \ x_2 = x_1$.

As seen in the above example, Theorem 4.1 in [8] implies more general conditions for a two-dimensional parametric solution set to have linear shape. It requires some further research to determine if a 2nd class parameter (in the general case) contributes linearly to the boundary of a parametric solution set. We formulate here some general conditions which, unfortunately, are not constructive.

Proposition 1. *Let a 2nd class parameter be involved first in the parameter elimination process. If for every generated cross inequality it holds exactly one of (i), (ii) , where*

(i) this cross inequality does not form the boundary of the solution set

(ii) this cross inequality forms the boundary of the solution set and the inequality is equivalent to inequalities involving only linear polynomials,

then this parameter contributes linearly to the boundary of the parametric united solution set.

All parameters that appear only on the right-hand side of a parametric system contribute linearly to the boundary of the solution set. In what follows we consider a slightly modified (and equivalent) representation of the parametric system (1) by renaming the parameters that appear only on the right-hand side of the system. Namely, we consider the parametric linear algebraic system in the form

$$\begin{aligned}
A(p)x &= b(p,q) \\
A(p) := A_0 + \sum_{k=1}^{k_1} p_k A_k, &\qquad b(p,q) := b_0 + \sum_{k=1}^{k_1} p_k b_k + \sum_{k=k_1+1}^{K} q_k b_k.
\end{aligned} \tag{15}$$

Theorem 3. *Consider the parametric system (15). Denote by $g_k(x)$ the GCD of the elements of A_kx, $k = 1,\dots,k_1$. Let $g_k(x)$ be a nonconstant polynomial for every $k = 1,\dots,k_1$. Define*

$$L := (l_1|\dots|l_{k_1}) \in \mathbb{R}^{n\times k_1}, \quad \textit{where} \quad l_k := A_kx/g_k(x) \in \mathbb{R}^n$$
$$R := (r_1|\dots|r_{k_1})^\top \in \mathbb{R}^{k_1\times n}, \quad \textit{where} \quad r_k := (\frac{\partial g_k(x)}{\partial x_1},\dots,\frac{\partial g_k(x)}{\partial x_n})^\top \in \mathbb{R}^n.$$

If there exists $t_k \in \mathbb{R}$ such that $t_kl_k = b_k := \partial b(p,q)/\partial p_k$ for every $k = 1,\dots,k_1$, then

(i) $\Sigma_{uni}(A(p), b(p,q), [p], [q])$ has linear shape and

(ii) the parametric system (15) is equivalently represented as

$$(A_0 + LDR)x \;=\; b_0 + LDt + Fq, \tag{16}$$

where $F := (b_{k_1+1}|\dots|b_K) \in \mathbb{R}^{n\times(K-k_1)}$, $t = (t_1,\dots,t_{k_1})^\top$ and $D = Diag(p)$.

Proof. (i) All parameters p, q satisfy the conditions of Theorem 2. In particular, $A_kx - b_k = g_k(x)l_k - t_kl_k = (g_k(x) - t_k)l_k$ for $k = 1,\dots,k_1$. (ii) follows trivially by the construction. □

Theorem 3 shows when[2] and how a linear system in general parametric form (1) can be represented in an equivalent form (16), where the parameters that appear in the matrix $A(p)$ are isolated in a diagonal matrix. The latter representation allows a more efficient handling of the parameter dependencies by the iteration method presented in Section 3.

2.3. Implications for the Parametric AE-Solution Sets

The parametric united solution set (2) is only a special case of the huge class of parametric AE-solution sets defined by

$$\Sigma_{AE}(A(p), b(p), [p]) :=$$
$$\{x \in \mathbb{R}^n \mid \forall p_{\mathcal{A}} \in [p_{\mathcal{A}}], \exists p_{\mathcal{E}} \in [p_{\mathcal{E}}], A(p_{\mathcal{A}}, p_{\mathcal{E}})x = b(p_{\mathcal{A}}, p_{\mathcal{E}})\}$$

for any two disjoint index sets $\mathcal{A} \cup \mathcal{E} = \{1,\dots,K\}$. Deriving the explicit description of the parametric AE-solution sets in [1], it was shown that

[2] if there is a particular structure of the parameter dependencies defined by the sufficient conditions of the theorem (and, therefore, Lemma 1 conditions)

the elimination of universally quantified parameters does not generate any cross inequalities. In other words, the shape of a parametric AE-solution set is determined only by the existentially quantified parameters. Therefore Theorem 2 is valid also for the existentially quantified parameters of this most general case. Respectively, Corollary 3 generalizes to

Corollary 4. *The infimum/supremum of a parametric AE-solution set is attained at particular endpoints of the intervals for the universally quantified parameters and for the existentially quantified parameters that satisfy the conditions of Theorem 2 or the conditions of Proposition 1.*

The most important application of Corollaries 3, 4 is for sharpening the bounds of the solution set enclosures. Thus, the above results expand the scope of applicability of the considerations in [9, Section 3] about sharpening the outer bounds of a parametric AE-solution set.

In the special case of a parametric united solution set let L be the index set of the parameters that satisfy the conditions of Theorem 2 and for a given index set $I = \{i_1, \ldots, i_k\}$, $\mathcal{B}_I := \{(\check{p}_{i_1} + \delta_{i_1}\hat{p}_{i_1}, \ldots, \check{p}_{i_k} + \delta_{i_k}\hat{p}_{i_k}) \mid \delta_1, \ldots, \delta_k \in \{\pm 1\}\}$ be the set of all interval endpoints and $\mathcal{K} = \{1, \ldots, K\}$. Then $\square\Sigma^p_{uni}$ can be obtained either by the combinatorial approach as

$$\square\Sigma^p_{uni} = \square \bigcup_{\tilde{p}_L \in \mathcal{B}_L} \square\Sigma_{uni}(A(\tilde{p}_L, p_{\mathcal{K}\setminus L}), b(\tilde{p}_L, p_{\mathcal{K}\setminus L}), [p_{\mathcal{K}\setminus L}]),$$

or following the approach presented in [10, 11] to prove numerically which endpoint of $[p_k]$, $k \in L$, forms the lower, respectively the upper, bound for every component of $\square\Sigma^p_{uni}$. The latter approach is illustrated in Example 3. Even if $L \subset \mathcal{K}$ (only some of the parameters satisfy the conditions of Theorem 2), this approach can sharpen considerably the enclosure $[u]$ of the solutions set. That is $[\underline{u}_i, \overline{u}_i] \subseteq [v_i]$, $1 = 1, \ldots, n$, where $[v]$ is an enclosure of $\Sigma_{uni}(A(p), b(p), [p])$, $[\underline{u}_i, t_l]$ is an enclosure of $\Sigma_{uni}(A(\tilde{p}_{L_l}, p_{\mathcal{K}\setminus L}), b(\tilde{p}_{L_l}, p_{\mathcal{K}\setminus L}), [p_{\mathcal{K}\setminus L}])$, $[t_u, \overline{u}_i]$ is an enclosure of $\Sigma_{uni}(A(\tilde{p}_{L_u}, p_{\mathcal{K}\setminus L}), b(\tilde{p}_{L_u}, p_{\mathcal{K}\setminus L}), [p_{\mathcal{K}\setminus L}])$ and $\tilde{p}_{L_l}, \tilde{p}_{L_u}$ are appropriate elements of $\mathcal{B}_L$.

In [6, Theorem 5.1] Neumaier and Pownuk proved that if the system $(A_0 + L\mathrm{Diag}(p)R)x(p, b) = Fb$, $p \in [p]$, $b \in [b]$, is uniquely solvable for all $p \in [p]$ then the extremal values of any component of the solution set are attained at vertices of the box $[p] \times [b]$. Corollary 3 has a bigger scope of applicability. It can be applied either to all parameters of a system if they all satisfy the conditions of Theorem 2, or only to those parameters that satisfy these conditions. While [6, Theorem 5.1] proves that the solution set is monotone on the parameters, Corollary 3 does not imply monotonicity.

The solution set may not be monotone on a given parameter and the interval hull of the solution set can be attained at the endpoints of the interval for this parameter. This is demonstrated in Example 3.

Theorem 2 can facilitate the solution of other interval problems related to parametric solution sets. An example is finding an inner interval estimation of the parametric AE-solution set with linear shape [12].

3. Iterative Solution Enclosure

In this section we generalize the method developed by Neumaier and Pownuk in [6].

3.1. The Method

Consider the parametric system (16).

Theorem 4. *Let $D_0 \in \mathbb{R}^{n\times n}$ be such that $A_0 + LD_0R$ is invertible and put*

$$C := (A_0 + LD_0R)^{-1}.$$

(i) the solution $x = x(p,q)$ of (16) is related to $y = Rx(p,q)$ by the equations

$$\begin{aligned} x &= Cb_0 + CFq + CL(D_0t + d), && (17)\\ y &= RCb_0 + RCFq + RCL(D_0t + d), && (18) \end{aligned}$$

where

$$d = (D_0 - D)(y - t). \tag{19}$$

(ii) If there are vectors $w \geq 0$, $w' > 0$ and w'' such that

$$\begin{aligned} &w' \leq w - |D_0 - D|\,|RCL|w\\ &w'' \geq |D_0 - D||RCb_0 + RCFq + RCLD_0t - t|, \end{aligned} \tag{20}$$

then

$$d \in [d] := [-\alpha w, \alpha w], \quad \alpha = \max_i \frac{w''_i}{w'_i}. \tag{21}$$

Proof. The proof follows the one given in [6]. For completeness we present it below.

(i) Equation (17) follows from

$$\begin{aligned} CLd &= CL(D_0 - D)(Rx - t)\\ &= C(A_0 + LD_0R)x - C(A_0 + LDR)x - CL(D_0 - D)t\\ &= x - C(b_0 + Fq) - CLD_0t, \end{aligned}$$

and the multiplication with R gives (18).

(ii) We put

$$\beta = \max_i |d_i|/w_i$$

and note that $|d| \le \beta w$, with equality in some component i. The definition of α implies $w'' \le \alpha w'$. Hence by (18)–(20),

$$\begin{aligned}
|d| &= |(D_0 - D)(RCb_0 + RCFq + RCL(D_0 t - d) - t)| \\
&\le |D_0 - D||RCb_0 + RCFq + RCLD_0 t - t| + |D_0 - D||RCL|\beta w \\
&\le w'' + \beta(w - w') \le \alpha w' + \beta(w - w').
\end{aligned}$$

Thus $\beta w_i = |d_i| \le \alpha w' + \beta(w_i - w_i')$, hence $\beta w_i' \le \alpha w_i'$. Since $w' > 0$, we conclude that $\beta \le \alpha$, and (21) follows. □

We now assume $p \in [p]$, $q \in [q]$ as interval bounds for the parameter uncertainties. Since (20) implies that $w' > 0$ if $w > 0$ and D_0 is close enough to D, we take D_0 as the midpoint of $[D]$, and w, e.g., as the vector with all entries one. Then (20) is satisfied with

$$\begin{aligned}
w' &:= w - |D_0 - [D]||RCL|w \\
w'' &:= |D_0 - [D]||RCb_0 + RCF[q] + RCLD_0 t - t|.
\end{aligned}$$

If $w' > 0$ then the enclosure (21) is valid. If this is not the case, as proposed in [6], we may compute the largest eigenvalue ϱ (= the spectral radius) of the matrix

$$M := |D_0 - [D]||RCL|.$$

If $\varrho < 1$, any $w > 0$ sufficiently close to an associated eigenvector makes $w' > 0$.

With this or another initial interval enclosure $[d]$ for d (needed when w' is not strictly positive), one can use interval arithmetic in the three formulas (17)–(19) to get enclosures $[x]$ for x, $[y]$ for y and a generally improved enclosure for d. The enclosures can be further improved by iterating this, and by intersecting with the previously computed enclosures. It is sufficient to iterate the enclosures for y and d, and compute the enclosures for x when the intersected results no longer improve significantly. Thus we iterate

$$\begin{aligned}
[y] &= \{(RCb_0) + (RCF)[q] + (RCL)(D_0 t + [d])\} \cap [y], \\
[d] &= \{(D_0 - [D])([y] - t)\} \cap [d]
\end{aligned}$$

until some stopping test holds, and then get the enclosure

$$[x] := (Cb_0) + (CF)[q] + (CL)(D_0 t + [d])$$

for all x satisfying (16) and for some $p \in [p]$, $q \in [q]$. The stopping criterion could be same as that one used in [6], when the sum of widths of the components of $[d]$ does not improve by a factor of 0.999 but after at most 10 iterations, or another one.

3.2. Numerical Examples

The parametric method, proposed by Neumaier and Pownuk [6], is implemented in C-XSC [14]. Both the original method and its present generalization for dependencies in both the matrix and the right-hand side vector are implemented in the environment of *Mathematica*®.

Our first example demonstrates the advantages of the proposed generalized method for efficient solving of parametric systems that involve dependencies in both the matrix and the right-hand side vector and having solution set with linear shape.

Example 2. *Consider the system $A(p)x = b(p)$ where*

$$\begin{pmatrix} p_1 + p_2 & p_1 - p_2 \\ p_1 - p_2 & p_1 + p_2 \end{pmatrix} x \;=\; \begin{pmatrix} 2p_2 + 3p_1 - 1 \\ -2p_2 + 3p_1 + 3 \end{pmatrix}, \qquad p \in \begin{pmatrix} [\frac{1}{2}, \frac{3}{2}] \\ [\frac{1}{2}, \frac{3}{2}] \end{pmatrix}.$$

The original method from [6] gives $[x_{NP}] = ([-2.51, 6.51], [-2.51, 6.51])^\top$ for the nonparametric right-hand side vector $b([p])$, while all methods, that require or assume strong regularity of the parametric matrix, e.g., [4, 2, 5, 3], fail because the parametric matrix is not strongly regular for the given parameter values. The equivalent parametric system in form (16) reads

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} x \;=\; \begin{pmatrix} -1 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

The presented generalized method gives $[x_{gNP}] = ([0.5, 3.51], [0.5, 3.51])^\top$. One can easily check that $[x_{NP}]$ overestimates[3] $[x_{gNP}]$ by 66.6%.

In [6] the method is illustrated on parametric systems stemming from FE analysis of truss structures. As concluded therein, "the method is significantly more accurate and significantly more faster than the element-by-element method [13]". Our next example demonstrates that other practical problems may require solving parametric linear systems that have linear shape of the solution set and involve dependencies in both the matrix and the right-hand side vector.

[3] For two intervals $[a], [b] \in \mathbb{IR}$, $[a] \subseteq [b]$, the percentage by which $[b]$ overestimates $[a]$ is given by $100(1 - \hat{a}/\hat{b})$.

Example 3. *Consider the resistive circuit (direct current) presented in Fig. 1 after [15, 16]. It consists of five nodes and eleven branches. The electric parameters have the following nominal values:*

$$e_1 = e_2 = 100V,\ e_5 = e_7 = 10V,\ R_j = 100\Omega,\ j = 1, \ldots, 11.$$

We will consider the resistances R_j, $j = 1, \ldots, 11$, of the branch elements as unknown but bounded to vary within 10% tolerance intervals. The tolerance analysis problem is to find bounds for all node voltages V_i, $i = 1, \ldots, 5$.

The method of loop analysis is used to set up the following system of parametric equations[4]

$$\begin{pmatrix} \frac{1}{R_1}+\frac{1}{R_3}+\frac{1}{R_6} & -\frac{1}{R_3} & 0 & 0 & -\frac{1}{R_6} \\ -\frac{1}{R_3} & \frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4}+\frac{1}{R_5} & -\frac{1}{R_4}-\frac{1}{R_5} & 0 & 0 \\ 0 & -\frac{1}{R_4}-\frac{1}{R_5} & \frac{1}{R_4}+\frac{1}{R_5}+\frac{1}{R_7}+\frac{1}{R_{10}} & -\frac{1}{R_7} & 0 \\ 0 & 0 & -\frac{1}{R_7} & \frac{1}{R_7}+\frac{1}{R_8}+\frac{1}{R_9} & -\frac{1}{R_9} \\ -\frac{1}{R_6} & 0 & 0 & -\frac{1}{R_9} & \frac{1}{R_6}+\frac{1}{R_9}+\frac{1}{R_{11}} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{pmatrix} = \begin{pmatrix} \frac{e_1}{R_1} \\ \frac{e_2}{R_2}-\frac{e_5}{R_5} \\ \frac{e_5}{R_5}+\frac{e_7}{R_7} \\ -\frac{e_7}{R_7} \\ 0 \end{pmatrix}$$

Without loss of generality we substitute in the system $p_i = \frac{1}{R_i}$, $i = 1, \ldots, 11$.

Since all the parameters p in the final system satisfy the conditions of Theorem 2, the parametric united solution set has linear shape. By Corollary 3 the exact interval hull of the solution set is attained at particular endpoints of the parameter intervals. For small number of parameters one can use the combinatorial approach over the endpoints of the parameter intervals. For large number of parameters one can use a more cumbersome approach based on proving global and/or "local" monotonicity of the solution components with respect to the parameters, cf. [11, 10]. This way one can determine which parameter endpoints generate the infimum (resp. the supremum) of a given solution component. We applied this approach in order to

[4] The same circuit, however with alternating current (which leads to complex-valued parameters), has been considered in [16]. The parametric system solved in [16] has a bug in the elements $a_{23} = a_{32}$ of the matrix.

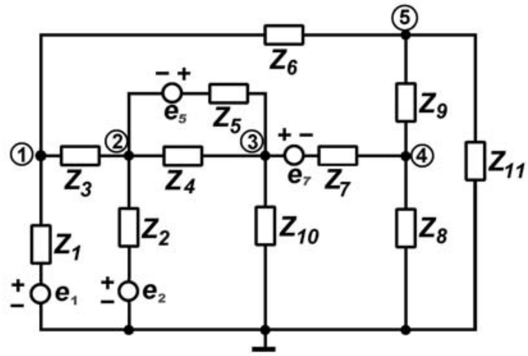

Figure 1: A resistive electrical circuit after [15].

demonstrate the advantages of Corollary 3. By solving several derivative parametric systems we found that all components of the parametric solution set are globally monotone with respect to $p_1 \uparrow$, $p_2 \uparrow$, $p_8 \downarrow$, $p_{10} \downarrow$, $p_{11} \downarrow$, *where* $\uparrow$ *denotes monotonically increasing and* $\downarrow$ *denotes monotonically decreasing. Depending on the monotonicity, we fix these parameters at the corresponding endpoints of their intervals and obtain two new parametric systems (one for the lower bound and one for the upper bound of the solution set) involving the remaining parameters. Then we solve derivative parametric interval systems of the latter new parametric systems in order to prove local monotonicity of the remaining parameters at each solution component. Repeat this process iteratively until obtaining all data presented in Table 1. More details are given in [11, 10].*

sol.comp.	p_3	p_4	p_5	p_6	p_7	p_9
1	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
2	$\uparrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\uparrow$
3	$\uparrow$	$\uparrow$	$\uparrow$	$\downarrow$	$\downarrow$	$\uparrow$
4	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
5	$\downarrow$	$\downarrow,\uparrow$	$\downarrow,\uparrow$	$\uparrow$	$\uparrow$	$\downarrow$

Table 1: Monotonic dependence of the solution components for the system from Example 3 on some parameters.

Note that the fifth solution component is not monotone on the parameters p_4 *and* p_5*. The arrows for these parameters in the corresponding row of Table 1 denote the local monotonicity for the lower, resp. the upper bound* *of the interval hull. Thus, both the lower and the upper endpoints of the fifth component of the solution set hull is computed in the upper endpoints of p_4 and p_5.*

The exact interval hull of the parametric solution set is

$$([\frac{1075850}{19223},\frac{1194650}{18143}],\ [\frac{8839900}{171471},\frac{1389700}{22541}],\ [\frac{6048320}{181869},\frac{2724960}{63757}],\ [\frac{519228}{35897},\frac{78012}{3521}],\ [\frac{14016630}{629927},\frac{53229110}{1725359}])^\top.$$

Since the parametric united solution set has linear shape, by Theorem 3 the parametric system can be represented in the form (16), where the set of q-parameters is empty. However, the parameters p_5 and p_7 appear in both the matrix $A(p)$ and the right-hand side vector $b(p)$ of the system. Therefore, we apply the generalized parametric method from Theorem 4 and obtain the following solution enclosure (rounded outwardly at the fifth digit after the decimal point)

$$([55.10036, 66.93172],\ [50.65584, 62.71315],\ [32.17210, 43.54983],\ [13.34860, 22.80113],\ [21.18106, 31.54621])^\top.$$

The obtained solution enclosure overestimates the exact interval hull componentwise by

$$(16.5\%, 16.2\%, 16.6\%, 18.6\%, 17.0\%)^\top.$$

The parametric single step (Bauer-Skeel) method[5] *[5, 3], performed in rational arithmetic, overestimates the solution enclosure obtained by the present method by*

$$(9.4\%, 10.3\%, 10.7\%, 11.2\%, 9.6\%)^\top.$$

In this example we study further the applicability of the method from Theorem 4 by expanding the tolerance intervals for all the parameters. At 30% tolerance intervals some components of the solution enclosure by the present method become intervals involving zero which is meaningless for the practice, while the single step method gives such an enclosure at 21% tolerance intervals. At 40% tolerance intervals the parametric matrix is no more strongly regular and the single step method fails. At 53% tolerance intervals the vector w' is not positive, however, computing an enclosure is still possible. At 57% tolerance intervals the regularity of the parametric matrix over the interval box cannot be proven and the present method fails.

[5] This method yields solution enclosures of comparable quality to the self-verified parametric fixed-point iteration [2].

Example 4. *In [17] the authors present a detailed analysis of the monotonicity properties of a linear dynamical system*

$$\begin{aligned}
\dot{Q}_1(t) &= -(k_{1,2}+k_{1,3})Q_1(t)+k_{2,1}Q_2(t)+k_{3,1}Q_3(t)+EGP\\
\dot{Q}_2(t) &= k_{1,2}Q_1(t)-(k_{2,1}+k_{2,0})Q_2(t)\\
\dot{Q}_3(t) &= k_{1,3}Q_1(t)-(k_{3,1}+k_{3,0})Q_3(t)\\
G(t) &= Q_1(t)/V_l
\end{aligned}$$

where all the parameters are uncertain and vary within given intervals[6]*, modeling the insulin secretion after Cobelli [18] and propose to change the variables: $S_1 = Q_1$, $S_2 = Q_1 + Q_2$, $S_3 = Q_1 + Q_3$, so that the new system on S, having the same output, is monotone with respect to all the states and parameters of the model.*

A finite difference scheme applied to both the original and the transformed systems results in linear algebraic systems with interval parameters. For the original system, the algebraic system is

$$\begin{pmatrix} 1+h(k_{1,2}+k_{1,3}) & -hk_{2,1} & -hk_{3,1}\\ -hk_{1,2} & 1+h(k_{2,0}+k_{2,1}) & 0\\ -hk_{1,3} & 0 & 1+h(k_{3,0}+k_{3,1}) \end{pmatrix} \begin{pmatrix} Q_1^{(k+1)}\\ Q_2^{(k+1)}\\ Q_3^{(k+1)} \end{pmatrix} = \begin{pmatrix} Q_1^{(k)}+hEGP\\ Q_2^{(k)}\\ Q_3^{(k)} \end{pmatrix}$$

and for the transformed problem the algebraic system is

$$\begin{pmatrix} 1+h(k_{1,2}+k_{1,3}+k_{2,1}+k_{3,1}) & -hk_{2,1} & -hk_{3,1}\\ h(k_{1,3}-k_{2,0}+k_{3,1}) & 1+hk_{2,0} & -hk_{3,1}\\ h(k_{1,2}+k_{2,1}-k_{3,0}) & -hk_{2,1} & 1+hk_{3,0} \end{pmatrix} \begin{pmatrix} S_1^{(k+1)}\\ S_2^{(k+1)}\\ S_3^{(k+1)} \end{pmatrix} = \begin{pmatrix} S_1^{(k)}+hEGP\\ S_2^{(k)}+hEGP\\ S_3^{(k)}+hEGP \end{pmatrix}$$

The parameters of both algebraic systems satisfy the conditions of Theorem 2. Studying the monotonicity properties of the two parametric solution sets

[6] the meaning and the values of the model parameters can be found in [18, 17]

as in Example 3 we find which endpoints of the parameters form the lower, resp. the upper, bounding model for the system output $Q_1(t)/V_l$. The results we obtain coincide with those presented in [17].

The method from [6] is known as the best parametric method so far. Our next example shows that in some special cases this may be not so.

Example 5. *Consider*

$$\begin{pmatrix} p_1 & \frac{1}{2} - p_2 \\ 1 + p_1 & p_2 \end{pmatrix} x = \begin{pmatrix} 6 \\ 6 \end{pmatrix}, \qquad p \in \begin{pmatrix} [\frac{1}{2}, \frac{3}{2}] \\ [\frac{1}{2}, \frac{3}{2}] \end{pmatrix}.$$

The parametric single step method and the self-verified parametric fixed-point iteration method yield best solution enclosures if the parameter dependencies are only in the columns of the matrix (as in the considered system) and in $b(p)$. These methods give the solution enclosure $([-3, 12], [-24, 18])^\top$, while the method from [6] gives the enclosure

$$([-5.8722, 14.8721], [-34.2288, 28.2288])^\top.$$

4. Conclusion

In this paper we presented some sufficient conditions for a parametric united solution set to have linear boundary with respect to a given parameter. These conditions, presented in Lemma 1, are simple and easy verifiable. The examples, provided in Section 3.2 and [6] show that interval linear systems, where the parameters satisfy Theorem 2, appear in various application domains. The method from Theorem 4 can be applied to linear systems that resulted from discretization of linear elliptic partial differential equations with interval-valued parameters in any application domain.

For parametric solution sets having linear boundary with respect to some parameters we can improve their outer interval estimations in two ways:

(i) Fix the values of the parameters that satisfy Theorem 2 at appropriate endpoints of their intervals and find sharper bounds for the respective parametric systems involving the remaining less number of interval parameters.

(ii) By Theorem 3 one can represent a general parametric system in a form (16) where the interval parameters are isolated in a diagonal matrix. This *conversion* theorem allows expanding the scope of applicability of the method from [6] to other problem domains than truss structures. Theorem 4 generalizes the method from [6] to account for the dependencies between the matrix and the right-hand side vector. The generalized method expands the scope of applicability and increases the sharpness of the solution enclosure retaining all good properties of the original method that are mentioned in the Introduction and demonstrated in [6].

Finally, the conditions for a parameter to contribute linearly to the boundary of a solution set are applicable in the more general context of parametric AE-solution sets and to other tasks regarding a parametric solution set. These applications are discussed in Section 2.3 of this paper.

References

[1] E.D. Popova, Explicit description of AE solution sets for parametric linear systems, SIAM J. Matrix Anal. Appl. 33 (2012) 1172–1189.

[2] E.D. Popova, Generalizing the parametric fixed-point iteration, Proceedings in Applied Mathematics & Mechanics 4 (2004) 680–681.

[3] M. Hladík, Enclosures for the solution set of parametric interval linear systems, Int. J. Appl. Math. Comp. Sci. 22 (3) (2012) 561–574.

[4] L. Kolev, A Method for Outer Interval Solution of Parametric Systems, Reliable Computing 10 (2004) 227–239.

[5] I. Skalna, A method for outer interval solution of systems of linear equations depending linearly on interval parameters, Reliable Computing 12 (2) (2006) 107–120.

[6] A. Neumaier, and A. Pownuk, Linear systems with large uncertainties, with applications to truss structures, Reliable Computing 13 (2007) 149-172.

[7] G. Alefeld, V. Kreinovich, G. Mayer, On the solution sets of particular classes of linear interval systems, J. Comput. Appl. Math. 152 (2003) 1–15.

[8] E.D. Popova, Explicit description of 2D parametric solution sets, BIT Numerical Mathematics 52 (2012) 179-200.

[9] E.D. Popova, M. Hladík, Outer enclosures to the parametric AE solution set, Soft Computing 17 (2013) 1403-1414.

[10] L. Kolev, Componentwise determination of the interval hull solution for linear interval parameter systems, Reliable Computing 20 (2014) 1–24.

[11] E.D. Popova, Computer-assisted proofs in solving linear parametric problems, in Post-proceedings of 12th GAMM–IMACS Int. Symposium on Scientific Computing, Computer Arithmetic and Validated Numerics, Duisburg, Germany, IEEE Computer Society Press, 2006.

[12] E.D. Popova, Inner estimation of linear parametric AE-solution sets, C. R. Acad. Bulgare Sci. 67(1) (2014) 13–20.

[13] R.L. Muhanna, R.L. Mullen, H. Zhang, Interval Finite Element as a Basis for Generalized Models of Uncertainty in Engineering Mechanics, in R.L.Muhanna, R.L.Mullen (Eds), Proceedings of the NSF Workshop on Reliable Engineering Computing, Svannah, Georgia USA, Sept. 15-17, 2004, 353–370.

[14] M. Zimmer, Software zur hocheffizienten Loesung von Intervallgleichungssystemen mit C-XSC, PhD Thesis, Bergische Universität Wuppertal, 2013.

[15] L. Kolev, Interval Methods for Circuit Analysis, World Scientific, 1993.

[16] E.D. Popova, L. Kolev, W. Krämer, A solver for complex-valued parametric linear systems, Serdica Journal of Computing 4 (2010) 123-132.

[17] D. de Pereda, S. Romero-Vivo, B. Ricarte, and J. Bondia, Guaranteed computation methods for compartmental in-series models under uncertainty, Computers and Mathematics with Applications 66 (9) (2013) 1595–1605.

[18] C. Cobelli, G. Toffolo, E. Ferrannini, A model of glucose kinetics and their control by insulin, compartmental and noncompartmental approaches, Mathematical Biosciences 72 (2) (1984) 291-315.

Erratum to:

Improved enclosure for some parametric solution sets with linear shape

(*Computers and Mathematics with Applications* 68(9):994–1005, 2014)

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Example 5 in the above mentioned paper [4] reports that the method of Neumaier and Pownuk [2] yields a worse enclosure of the united solution set to the considered parametric interval linear system than the single step parametric Bauer-Skeel method [5, 1] and the self-verified parametric fixed-point method (parametric Krawczyk iteration) [3]. This is not true and resulted from a bug in the software. All three methods yield interval vectors of similar quality for the considered example. Namely, the parametric Bauer-Skeel method executed in exact arithmetic gives the solution enclosure

$$([-3, 12], [-24, 18])^\top. \tag{1}$$

The parametric Krawczyk iteration expanded by iterative refinement with stopping criterion

$$\texttt{dist}(\mathbf{x}^{\texttt{new}}, \mathbf{x}^{\text{old}}) <= \delta, \tag{2}$$

where δ specifies a desired accuracy (say $\delta = 10^{-5}$), delivers enclosure of (1) with accuracy δ, that is

$$([-3.0000125, 12, 0000125], [-24.00004428, 18.00004428])^\top.$$

The method of Neumaier and Pownuk, implemented with a the same stopping criterion above, yields the enclosure

$$([-3.00002019, 12, 00002019], [-24.00007190, 18.00007190])^\top.$$

It was also noticed that the formula in line -6 on page 1000 of the printed paper [4] also contains a bug and should be read as

$$|d| \quad = \quad |(D_0 - D)(RCb_0 + RCFq + RCL(D_0t + d) - t)|.$$

April 7, 2017

References

[1] M. Hladík, Enclosures for the solution set of parametric interval linear systems, Int. J. Appl. Math. Comput. Sci. 22 (2012) (3) 561–574.

[2] A. Neumaier, and A. Pownuk, Linear systems with large uncertainties, with applications to truss structures, Reliable Computing 13 (2007) 149-172.

[3] E.D. Popova, Generalizing the parametric fixed-point iteration, Proceedings in Applied Mathematics & Mechanics 4 (2004) 680–681.

[4] E.D. Popova, Improved enclosure for some parametric solution sets with linear shape, Computers and Mathematics with Applications 68 (2014) 9, 994–1005.

[5] I. Skalna, A method for outer interval solution of systems of linear equations depending linearly on interval parameters, Reliable Computing 12 (2) (2006) 107–120.