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INNER ESTIMATION OF LINEAR PARAMETRIC *AE*-SOLUTION SETS

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Abstract

We consider linear algebraic systems $A(p)x = b(p)$, where the elements of the matrix and the right-hand side vector are linear functions of uncertain parameters varying within given intervals. For the parametric AE-solution set defined for two disjoint index sets $\mathcal{A}, E$ by $\Sigma^p_{AE} := \{x \in \mathbb{R}^n \mid (\forall p_{\mathcal{A}} \in [p_{\mathcal{A}}]) (\exists p_{\mathcal{E}} \in [p_{\mathcal{E}}])(A(p)x = b(p))\}$, an inner interval estimation $[y] \subseteq \Sigma^p_{AE}$ is sought. In the special case when Σ^p_{AE} has linear shape, this paper gives parametric generalization of the so-called centred approach, which was developed so far only for tolerable solution sets.

Key words: interval linear equations, dependent data, AE-solution sets, inner inclusion

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1. Introduction. Consider linear algebraic systems having linear uncertainty structure

$$A(p)x \;=\; b(p),$$
$$(1) \qquad A(p) := A_0 + \sum_{k=1}^{K} p_k A_k, \qquad b(p) := b_0 + \sum_{k=1}^{K} p_k b_k,$$

where $A_k \in \mathbb{R}^{m\times n}$, $b_k \in \mathbb{R}^m$, $k = 0, \dots, K$ and the parameters $p = (p_1, \dots, p_K)^\top$ are considered to be uncertain and varying within given intervals $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_K)^\top$. Above, $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote the set of real vectors with n components and the set of real $m\times n$ matrices, respectively. A real compact interval is $\mathbf{a} = [\underline{a}, \overline{a}] := \{a \in \mathbb{R} \mid \underline{a} \le a \le \overline{a}\}$. By $\mathbb{IR}^n, \mathbb{IR}^{m\times n}$ we denote the sets of interval n-vectors and interval $m \times n$ matrices, respectively. For $\mathbf{a} = [\underline{a}, \overline{a}]$, define midpoint $\check{\mathbf{a}} := (\underline{a}+\overline{a})/2$, radius $\hat{\mathbf{a}} := (\overline{a} - \underline{a})/2$ and absolute value (magnitude) $|\mathbf{a}| := \max\{|\underline{a}|, |\overline{a}|\}$. These functions are applied to interval vectors and matrices componentwise. Without loss of generality and in order to have a unique representation (1), we assume that $\hat{\mathbf{p}}_k > 0$ for all $1 \leq k \leq K$.

We consider the parametric AE-solution sets of system (1), which are defined by

$$(2)\quad \Sigma^p_{AE} = \Sigma_{AE}(A(p), b(p), \mathbf{p}) \\ := \{x \in \mathbb{R}^n \mid (\forall p_{\mathcal{A}} \in \mathbf{p}_{\mathcal{A}})(\exists p_{\mathcal{E}} \in \mathbf{p}_{\mathcal{E}})(A(p)x = b(p))\},$$

where $\mathcal{A}$ and $\mathcal{E}$ are sets of indexes such that $\mathcal{A} \cup \mathcal{E} = \{1, \ldots, K\}$, $\mathcal{A} \cap \mathcal{E} = \varnothing$. For a given index set $\Pi = \{\pi_1, \ldots, \pi_k\}$, p_Π denotes $(p_{\pi_1}, \ldots, p_{\pi_k})$. Among the AE-solution sets most studied and of particular practical interest are: the (parametric) united solution set

$$\Sigma_{\mathrm{uni}}(A(p), b(p), \mathbf{p}) := \{x \in \mathbb{R}^n \mid (\exists p \in \mathbf{p})(A(p)x = b(p))\},$$

the (parametric) tolerable solution set

$$\Sigma(A(p_{\mathcal{A}}), b(p_{\mathcal{E}}), \mathbf{p}) := \{x \in \mathbb{R}^n \mid (\forall p_{\mathcal{A}} \in \mathbf{p}_{\mathcal{A}})(\exists p_{\mathcal{E}} \in \mathbf{p}_{\mathcal{E}})(A(p_{\mathcal{A}})x = b(p_{\mathcal{E}}))\}$$

and the (parametric) controllable solution set

$$\Sigma(A(p_{\mathcal{E}}), b(p_{\mathcal{A}}), \mathbf{p}) := \{x \in \mathbb{R}^n \mid (\forall p_{\mathcal{A}} \in \mathbf{p}_{\mathcal{A}})(\exists p_{\mathcal{E}} \in \mathbf{p}_{\mathcal{E}})(A(p_{\mathcal{E}})x = b(p_{\mathcal{A}}))\}.$$

Although the nonparametric AE-solution sets are studied to some extent (see, e.g., [2,13] and the references given therein), there are only a few results on the more general case of linear parameter dependency. The so-called "centred" approach for inner estimation of a solution set is developed by Neumaier in [5] for the nonparametric united solution set and for the nonparametric tolerable solution set. In [9] this approach is generalized for the parametric tolerable solution set. In this paper we expand the parametric centred approach for parametric AE-solution sets (in particular the most used: united and controllable solution sets) which have linear shape (boundary). Parametric solution sets with linear shape appear often in the applied domains, see, e.g., [6], ([10], Examples 5.2, 5.3).

2. Theoretical background.

Definition 2.1 ([12])**.** A parameter p_k, $1 \leq k \leq K$, is of 1st class if it occurs in only one equation of the system (1).

Definition 2.2 ([12])**.** A parameter p_k, $1 \leq k \leq K$, is of 2nd class if it is involved in more than one equation of the system (1).

The parametric centred approach is based on the explicit description of the parametric AE-solution set. How to obtain it is discussed in [8]. The description is explicit if the system involves only 1st class existentially quantified parameters, called $\mathcal{E}$-parameters for short, and in some other special cases. In the general case, the description can be obtained by an algorithmic procedure. One has first to eliminate all 2nd class $\mathcal{E}$-parameters by the Fourier–Motzkin like parameter elimination procedure. The elimination of every 2nd class $\mathcal{E}$-parameter introduces, in general, new characterizing inequalities/inclusions, besides those given in ([12], Theorems 4.1, 4.2). The degree of polynomials involved in the characterizing inequalities/inclusions may be arbitrary high and depends on the 2nd class $\mathcal{E}$-parameters involved in the parametric matrix.

Definition 2.3. A parametric AE-solution set is called *linear*, in other words its shape is linear, if the boundary of the solution set consists of parts of hyperplanes.

The inequalities/inclusions describing a *linear* parametric AE-solution set involve only linear functions on the coordinate variables.

Theorem 2.1. *Let Σ^p_{AE} have linear shape. A point $x \in \mathbb{R}^n$ belongs to Σ^p_{AE}, if and only if*

$$|U(\check{\mathbf{p}})x - v(\check{\mathbf{p}})| \le \sum_{k=1}^{K} \delta_k \hat{\mathbf{p}}_k |U_k x - v_k| \tag{3}$$

equivalently

$$\sum_{k\in\mathcal{A}} \mathbf{p}_k(U_k x - v_k) \subseteq v_0 - U_0 x + \sum_{k\in\mathcal{E}} \mathbf{p}_k(v_k - U_k x), \tag{4}$$

where $U_k \in \mathbb{R}^{q\times n}$, $v_k \in \mathbb{R}^q$, $k = 0,\dots,K$, $U(p) = U_0 + \sum_{k=1}^{K} p_k U_k$, $v(p) = v_0 + \sum_{k=1}^{K} p_k v_k$, $q \ge m$, correspond to the explicit description of Σ^p_{AE} and $\delta_k := \{1 \text{ if } k \in \mathcal{E},\ -1 \text{ if } k \in \mathcal{A}\}$.

Proof. Inequalities (3) represent the explicit description of a Σ^p_{AE} with linear shape and follow from the Fourier–Motzkin like parameter elimination procedure. The equivalence between (3) and (4) is proven similarly to ([12], Theorem 3.3). □

We have $U_i(p) = A_i(p)$ and $v_i(p) = b_i(p)$ for $i = 1,\dots,m$. In the special case when the definition of Σ^p_{AE} involves only 1st class $\mathcal{E}$-parameters, $U(p) = A(p)$, $v(p) = b(p)$. The same is true for $\Sigma_{\mathrm{tol}}(A(p), \mathbf{b}, \mathbf{p})$. Any $\Sigma(A(p_{\mathcal{A}}), b(p_{\mathcal{E}}), \mathbf{p})$ also has linear shape independent of the number of 2nd class $\mathcal{E}$-parameters involved in the right-hand side vector. Some general sufficient conditions for a parametric AE-solution set to have linear shape will be given in a separate paper.

Since the methodology we consider requires knowledge of a point which belongs to the interior of the solution set, in what follows we assume that Σ^p_{AE} is not empty and non-degenerate (having a nonempty interior). Denote by int $\mathcal{X}$ the topological interior of the set $\mathcal{X}$ in $\mathbb{R}^n$ with the standard topology.

We need also the so-called inner interval subtraction, defined for $\mathbf{a}, \mathbf{b} \in \mathbb{IR}$ by

$$\mathbf{a} \ominus \mathbf{b} := [\underline{a} - \underline{b}, \overline{a} - \overline{b}],$$

where $\mathbf{a} \ominus \mathbf{b} = \varnothing$ if $\underline{a} - \underline{b} > \overline{a} - \overline{b}$. Inner interval subtraction can be represented by the arithmetic operations and functions in the generalized interval space of proper and improper intervals known as KAUCHER interval arithmetic [4] or modal interval arithmetic [1]. In order to facilitate the derivations in the next section we will use some properties of the arithmetic on proper and improper intervals.

The set of *proper* intervals $\mathbb{IR}$ is extended in [4] by the set $\{[\underline{a}, \overline{a}] \mid \underline{a}, \overline{a} \in \mathbb{R}, \underline{a} \geq \overline{a}\}$ of *improper* intervals obtaining thus the set $\mathbb{I}^*\mathbb{R} = \{[\underline{a}, \overline{a}] \mid \underline{a}, \overline{a} \in \mathbb{R}\}$ of generalized intervals. The conventional interval arithmetic and lattice operations, order relations and other functions are isomorphically extended onto the whole set $\mathbb{I}^*\mathbb{R}$, [4]. Modal interval analysis [1] imposes a logical-semantic background on generalized intervals (considered there as modal intervals) and allows giving a logical meaning to the interval results. The conventional interval arithmetic can be obtained as a projection of the generalized interval arithmetic on $\mathbb{IR}$. An element-to-element symmetry between proper and improper intervals is expressed by the "`dual`" operator $\mathtt{dual}(\mathbf{a}) := [\overline{a}, \underline{a}]$ for $\mathbf{a} = [\underline{a}, \overline{a}] \in \mathbb{I}^*\mathbb{R}$. `dual` is applied componentwise to vectors and matrices. For $\mathbf{a}, \mathbf{b} \in \mathbb{I}^*\mathbb{R}$

$$\mathtt{dual}(\mathtt{dual}(\mathbf{a})) = \mathbf{a}, \qquad \mathtt{dual}(\mathbf{a} \circ \mathbf{b}) = \mathtt{dual}(\mathbf{a}) \circ \mathtt{dual}(\mathbf{b}),\ \circ \in \{+, -, \times, /\}.$$

The generalized interval arithmetic structure possesses group properties with respect to addition and multiplication operations. Thus, $\mathbf{a} \ominus \mathbf{b} = \mathbf{a} - \mathtt{dual}(\mathbf{b})$.

We will use the following sub-distributive property, proven in [1],

$$\text{impr}(\mathbf{a})\mathbf{b} + \mathbf{ac} \ \subseteq\ \mathbf{a}(\mathbf{b} + \mathbf{c}), \tag{5}$$

wherein $\text{impr}(\mathbf{a}) := \{\mathbf{a} \text{ if } \underline{a} \geq \overline{a},\ \mathtt{dual}(\mathbf{a}) \text{ if } \underline{a} \leq \overline{a}\}$ and $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{I}^*\mathbb{R}$.

Let $\mathbb{F} \subset \mathbb{R}$ be the set of floating-point numbers and $\mathbb{IF}, \mathbb{I}^*\mathbb{F}$ be the corresponding interval sets. Denote by $\Diamond$ the outward and by $\bigcirc$ the inward rounding, $\Diamond, \bigcirc : \mathbb{I}^*\mathbb{R} \longrightarrow \mathbb{I}^*\mathbb{F}$. The following properties (cf. [1]) show that inner numerical approximations can be obtained at no additional cost only by outward directed rounding and the `dual` operator in $\mathbb{I}^*\mathbb{F}$. For $\mathbf{a}, \mathbf{b} \in \mathbb{I}^*\mathbb{F}$, $\circ \in \{+, -, \times, /\}$,

$$\bigcirc(\mathbf{a}) = \mathtt{dual}(\Diamond(\mathtt{dual}\mathbf{a})); \qquad \mathbf{a} \odot \mathbf{b} = \mathtt{dual}(\mathtt{dual}\mathbf{a} \mathbin{\diamond\!\!\!\circ} \mathtt{dual}\mathbf{b}). \tag{6}$$

3. Inner estimation of linear $\boldsymbol{\Sigma^p_{AE}}$. We consider linear systems (1) and their parametric AE-solution sets which have linear shape. In this case the complete characterization of Σ^p_{AE} is given by a system of linear interval inclusions (4). We assume knowledge of an interior point $\tilde{x}$ for Σ^p_{AE}. By Theorem 2.1, the relation

$$\sum_{k \in \mathcal{A}} \mathbf{p}_k (U_k \tilde{x} - v_k) \ \subseteq\ \text{int} \left(v_0 - U_0 \tilde{x} + \sum_{k \in \mathcal{E}} \mathbf{p}_k (v_k - U_k \tilde{x}) \right) \tag{7}$$

presents a sufficient condition for a point $\tilde{x} \in \mathbb{R}^n$ to be in the interior of Σ^p_{AE}. It is an open problem how to find a suitable interior point $\tilde{x}$. A first natural trial point is the approximate midpoint solution. It can be found either by solving the quadratic system $A(\check{\mathbf{p}})x = b(\check{\mathbf{p}})$ if $m = n$, or by solving the overdetermined system $U(\check{\mathbf{p}})x = v(\check{\mathbf{p}})$ by least squares. It has to be verified by relation (7) that the chosen $\tilde{x}$ is an interior point for Σ^p_{AE}.

For a given $\tilde{x} \in \text{int } \Sigma^p_{AE}$, we compute the maximal nonnegative number η, such that

$$(8)\quad \eta \left(U_0\mathbf{e} + \sum_{k=1}^{K} \mathbf{p}_k(U_k\mathbf{e}) \right) \subseteq v_0 - U_0\tilde{x} + \sum_{\mu\in\mathcal{E}} \mathbf{p}_\mu(v_\mu - U_\mu\tilde{x}) \ominus \sum_{\nu\in\mathcal{A}} \mathbf{p}_\nu(U_\nu\tilde{x} - v_\nu),$$

where $\mathbf{e} = ([-1,1],\ldots,[-1,1])^\top$. Note that $\mathbf{u} \ominus \mathbf{v} = \varnothing$ if $\hat{\mathbf{v}} > \hat{\mathbf{u}}$. However, for $\tilde{x} \in \text{int } \Sigma^p_{AE}$, (4), respectively (7), implies that the right-hand side of (8) will not be an empty set.

Theorem 3.1. *For $\tilde{x} \in int\ \Sigma^p_{AE}$ and $\eta \geq 0$, such that* (8) *holds,*

$$\tilde{x} + \eta\mathbf{e} \subseteq \Sigma^p_{AE}.$$

Proof. For each $x \in \tilde{x} + \eta\mathbf{e}$ we have

$$\begin{aligned} U_0x + \sum_{\nu\in\mathcal{A}} \mathbf{p}_\nu(U_\nu x - v_\nu) &\subseteq U_0(\tilde{x} + \eta\mathbf{e}) + \sum_{\nu\in\mathcal{A}} \mathbf{p}_\nu(U_\nu(\tilde{x} + \eta\mathbf{e}) - v_\nu) \\ &\overset{\text{sub distr.}}{\subseteq} U_0\tilde{x} + \eta U_0\mathbf{e} + \sum_{\nu\in\mathcal{A}} \mathbf{p}_\nu(U_\nu\tilde{x} - v_\nu) + \eta \sum_{\nu\in\mathcal{A}} \mathbf{p}_\nu(U_\nu\mathbf{e}). \end{aligned}$$

By (8) the last expression to the right is contained in

$$\begin{aligned} &U_0\tilde{x} + \sum_{\nu\in\mathcal{A}} \mathbf{p}_\nu(U_\nu\tilde{x} - v_\nu) \ominus \eta \sum_{\mu\in\mathcal{E}} \mathbf{p}_\mu(U_\mu\mathbf{e}) + v_0 \\ &\qquad - U_0\tilde{x} + \sum_{\mu\in\mathcal{E}} \mathbf{p}_\mu(v_\mu - U_\mu\tilde{x}) \ominus \sum_{\nu\in\mathcal{A}} \mathbf{p}_\nu(U_\nu\tilde{x} - v_\nu), \end{aligned}$$

which due to $\mathbf{a} \ominus \mathbf{a} = 0$ is equivalent to

$$\begin{aligned} v_0 + \sum_{\mu\in\mathcal{E}} \mathbf{p}_\mu(v_\mu - U_\mu\tilde{x}) \ominus \eta \sum_{\mu\in\mathcal{E}} \mathbf{p}_\mu(U_\mu\mathbf{e}) &\overset{(5)}{\subseteq} v_0 + \sum_{\mu\in\mathcal{E}} \mathbf{p}_\mu(v_\mu - U_\mu\tilde{x} - \eta\,\texttt{dual}(U_\mu\mathbf{e})) \\ &\subseteq v_0 + \sum_{\mu\in\mathcal{E}} \mathbf{p}_\mu(v_\mu - U_\mu(\tilde{x} + \eta\mathbf{e})) \\ &= v_0 + \sum_{\mu\in\mathcal{E}} \mathbf{p}_\mu(v_\mu - U_\mu x). \end{aligned}$$

The last inclusion is due to $\mathrm{impr}(\mathbf{a}) \subseteq \mathbf{a}$ and the equality is due to the initial assumption $x \in \tilde{x} + \eta\mathbf{e}$.

Thus $U_0x + \sum_{\nu\in\mathcal{A}} \mathbf{p}_\nu(U_\nu x - v_\nu) \subseteq v_0 + \sum_{\mu\in\mathcal{E}} \mathbf{p}_\mu(v_\mu - U_\mu x)$ for each $x \in \tilde{x} + \eta\mathbf{e}$ implies $\tilde{x} + \eta\mathbf{e} \subseteq \Sigma^p_{AE}$ by (4) of Theorem 2.1. □

In computational terms, from (8)

$$\eta = \min_{1\le i\le q} \frac{\min\{|\underline{r}_i|, |\overline{r}_i|\}}{\overline{d}_i},$$

where $\mathbf{r} := v_0 - U_0\tilde{x} + \sum_{\mu\in\mathcal{E}} \mathbf{p}_\mu(v_\mu - U_\mu\tilde{x}) \ominus \sum_{\nu\in\mathcal{A}} \mathbf{p}_\nu(U_\nu\tilde{x} - v_\nu)$ and $\mathbf{d} := U_0\mathbf{e} + \sum_{k=1}^K \mathbf{p}_k(U_k\mathbf{e})$.

The formula for inner interval subtraction implies

$$\underline{r} = v(\check{p}) - U(\check{p})\tilde{x} - t,\ \overline{r} = v(\check{p}) - U(\check{p})\tilde{x} + t,\ t = \sum_{\mu\in\mathcal{E}} \hat{p}_\mu|v_\mu - U_\mu\tilde{x}| - \sum_{\nu\in\mathcal{A}} \hat{p}_\nu|v_\nu - U_\nu\tilde{x}|.$$

With the notation $U_k := (u_{k,ij}) \in \mathbb{R}^{q\times n}$ for $k = 0, \ldots, K$, we have

$$\overline{d}_i = \sum_{j=1}^n |u_{0,ij}| + \sum_{k=1}^K \left(\sum_{j=1}^n |u_{k,ij}|\right) \max\{|\underline{p}_k|, |\overline{p}_k|\}, \quad \overline{d}_i \neq 0.$$

In the presence of round-off errors, the resulting inner estimation is valid if $\mathbf{r}$ is rounded inward, $\overline{d}_i$ are rounded upwards, η is rounded downwards, and $\tilde{x} + \eta\mathbf{e}$ is rounded inward.

One can use a correct computer implementation of the arithmetic on proper and improper intervals, and the property (6) for a computer implementation of the parametric centred approach in order to provide safe bounds for the inner estimation. Based on (6), the following computational chain provides a guaranteed inner estimation $\mathbf{z}$ of the parametric AE-solution set.

$$\begin{aligned}
\mathbf{d} &= U_0\mathbf{e} + \sum_{k=1}^K \mathbf{p}_k(U_k\mathbf{e}),\\
\mathbf{r} &= \mathtt{dual}\left(v_0 - U_0\tilde{x} + \sum_{\mu\in\mathcal{E}} \mathtt{dual}(\mathbf{p}_\mu)(v_\mu - U_\mu\tilde{x}) - \sum_{\nu\in\mathcal{A}} \mathbf{p}_\nu(v_\nu - U_\nu\tilde{x})\right),\\
\eta &= \min_{1\le i\le q}\{\min\{|\underline{r}_i|, |\overline{r}_i|\}/\overline{d}_i\},\\
\mathbf{z} &= \mathtt{dual}(\tilde{x} + \eta\,\mathtt{dual}(\mathbf{e})),
\end{aligned}$$

where all arithmetic operations (except the division on the third line, which is a floating-point division rounded to near) are outwardly rounded floating-point interval operations in $\mathbb{I}^*\mathbb{F}$. Similarly, verifying the condition (7) in floating-point arithmetic we have to compute the left-hand side with outward rounding and the right-hand side with inward rounding.

The computational complexity of the centred approach is $O(qn)$. However, obtaining the explicit representation of the parametric AE-solution set will require some additional effort. The parametric centred method is implemented in the environment of *Mathematica*®. Another implementation environment suitable for large sparse data is the C-XSC class library, [3,11].

Example 3.1. We look for an inner estimation of the united solution set Σ^g_{uni} to the parametric system modelling a resistive electrical network[1], which is considered for visualization in [10], Example 5.3. The explicit description of Σ^g_{uni} by Fourier–Motzkin-like elimination of the parameters leads to

$$U(g)=\begin{pmatrix} g_1+g_6 & -g_6 & 0 & 0 & 0\\ -g_6 & g_2+g_6+g_7 & -g_7 & 0 & 0\\ 0 & -g_7 & g_3+g_7+g_8 & -g_8 & 0\\ 0 & 0 & -g_8 & g_4+g_8+g_9 & -g_9\\ 0 & 0 & 0 & -g_9 & g_5+g_9\\ -g_1 & -g_2-g_7 & g_7 & 0 & 0\\ g_6 & -g_2-g_6 & -g_3-g_8 & g_8 & 0\\ g_1 & g_2 & g_3+g_8 & -g_8 & 0\\ 0 & g_7 & -g_3-g_7 & -g_4-g_9 & g_9\\ -g_6 & g_2+g_6 & g_3 & g_4+g_9 & -g_9\\ -g_1 & -g_2 & -g_3-2g_8 & g_4+2g_8+g_9 & -g_9\\ 0 & 0 & g_8 & -g_4-g_8 & -g_5\\ 0 & -g_7 & g_3+g_7 & g_4 & g_5\\ g_6 & -g_2-g_6 & -g_3 & -g_4-2g_9 & g_5+2g_9\\ g_1 & g_2 & g_3+2g_8 & -g_4-2g_8-2g_9 & g_5+2g_9 \end{pmatrix},\quad v(g)=\begin{pmatrix} 10\\ 0\\ 10\\ 0\\ 0\\ -10\\ -10\\ 20\\ -10\\ 10\\ -20\\ 0\\ 10\\ -10\\ 20 \end{pmatrix}.$$

Note that $U_i(g) = A_i(g)$ and $v_i(g) = b_i(g)$ for $i = 1, \ldots, 5$. The parameters vary within $g_i \in [0.99, 1.01]$, $i = 1, \ldots, 9$, as in [10]. The approximate midpoint solution is an interior point for the parametric united solution set. Then, the parametric centred approach gives the following (rounded inward) guaranteed inner estimation

$$([7.084, 7.098], [4.175, 4.189], [5.448, 5.461], [2.175, 2.189], [1.084, 1.098])^\top.$$

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