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On the Unbounded Parametric Tolerable Solution Set

Evgenija D. Popova

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Abstract We consider a linear algebraic system $A(p)x = b(q)$, where the elements of the matrix and the right-hand side vector are linear functions of uncertain parameters p, q varying within given intervals $\mathbf{p}$, $\mathbf{q}$. The so-called parametric tolerable solution set $\Sigma_{tol}(A(p), b(q), \mathbf{p}, \mathbf{q}) := \{x \in \mathbb{R}^n \mid (\forall p \in \mathbf{p})(\exists q \in \mathbf{q})(A(p)x = b(q))\}$ is studied for unboundedness. Basing on a characterization of the parametric tolerable solution set as a convex polyhedron, we present necessary and sufficient conditions (in both general and computable forms) for a nonempty parametric tolerable solution set to be unbounded. Every parametric tolerable solution set is represented as a sum of a linear subspace and a bounded convex polyhedron. The latter implies better estimations (outer and inner) for the unbounded parametric tolerable solution sets. Numerical examples illustrate the discussed methodology and the solution sets.

Keywords interval analysis · linear algebraic equations · dependent data · tolerable solution set · unboundedness

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1 Introduction

Consider the linear algebraic system

$$A(p)x = b(p), \tag{1}$$

E.D. Popova
Institute of Mathematics and Informatics, Bulgarian Academy of Sciences
Acad. G. Bonchev str., block 8, 1113 Sofia, Bulgaria
Tel.: +359-889662835
Fax: +359-2-9713649
E-mail: epopova@bio.bas.bg, epopova@math.bas.bg

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where

$$A(p) = A_0 + \sum_{k=1}^{K} p_k A_k, \quad b(p) = b_0 + \sum_{k=1}^{K} p_k b_k \tag{2}$$

for some numerical matrices $A_k \in \mathbb{R}^{m\times n}$ and numerical vectors $b_k \in \mathbb{R}^m$, $k = 0, 1, \ldots, K$, where $\mathbb{R}^n$, $\mathbb{R}^{m\times n}$ denote the set of real vectors with n components and the set of real $m \times n$ matrices, respectively. The parameters p_k, $k \in \mathcal{K} = \{1, \ldots, K\}$ are considered as uncertain and varying within given intervals

$$p \in \mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_K)^\top. \tag{3}$$

A real compact interval is $\mathbf{a} = [\underline{a}, \overline{a}] := \{a \in \mathbb{R} \mid \underline{a} \le a \le \overline{a}\}$. By $\mathbb{IR}^n, \mathbb{IR}^{m\times n}$ we denote the sets of interval n-vectors and interval $m \times n$ matrices, respectively. For $\mathbf{a} = [\underline{a}, \overline{a}]$, define mid-point $\check{a} := (\underline{a} + \overline{a})/2$ and radius $\hat{a} := (\overline{a} - \underline{a})/2$. These functionals are applied to interval vectors and matrices componentwise. Without loss of generality and in order to have a unique representation (2), we assume that $\hat{p}_k > 0$ for all $k \in \mathcal{K}$.

The parameter dependent linear system (1)–(3) presents a generalization of the nonparametric interval linear system $\mathbf{A}x = \mathbf{b}$, $\mathbf{A} \in \mathbb{IR}^{m\times n}$, $\mathbf{b} \in \mathbb{IR}^m$, which was mainly studied until recently. Thus, a nonparametric interval system $\mathbf{A}x = \mathbf{b}$, where the elements of the matrix and of the right-hand side vector are independent intervals, can be considered as a special case of a parametric system involving $mn + m$ interval parameters $a_{ij} \in \mathbf{a}_{ij}$, $b_i \in \mathbf{b}_i$, $1 \le i \le m$, $1 \le j \le n$. However, the practical problems, which require solving linear algebraic systems, are usually described by complicated dependencies between the uncertain model parameters. Therefore, the parameter dependent linear system (1)–(3) usually provides a more precise setting and more precise results for the real-life problems involving uncertainties, cf. [9].

The "solution" of a problem involving uncertain (interval) data can be defined in a vast variety of ways. Usually, the so-called united parametric solution set

$$\Sigma^p = \Sigma_{uni}\left(A(p), b(p), \mathbf{p}\right) := \{x \in \mathbb{R}^n \mid \exists p \in \mathbf{p}, A(p)x = b(p)\} \tag{4}$$

is considered and its interval estimations (outer and/or inner) are sought. However, various problems, e.g., in mathematical economy [2], [12], control engineering and parameter identification [19], [20], tolerance analysis [6], see also [14] for an interpretation and applications, consider the so-called (parametric) tolerable solution set. The parametric tolerable solution set is defined for two disjoint parameter vectors $p \in \mathbb{R}^{\mathcal{K}_p}$, $q \in \mathbb{R}^{\mathcal{K}_q}$, $\mathcal{K}_p \cup \mathcal{K}_q = \mathcal{K}$, by

$$\begin{aligned}\Sigma^p_{tol} &= \Sigma_{tol}(A(p), b(q), \mathbf{p}, \mathbf{q}) \\ &:= \{x \in \mathbb{R}^n \mid (\forall p \in \mathbf{p})(\exists q \in \mathbf{q})(A(p)x = b(q))\}.\end{aligned} \tag{5}$$

The nonparametric tolerable solution set and its interval estimations are studied by many authors. The notion of tolerance solutions is motivated by considerations concerning crane construction [7] and input-output planning with inexact data [12]. J. Rohn gives in [13] two equivalent descriptions of the nonparametric tolerable solution set. Various authors prove that $\Sigma_{tol}(\mathbf{A}, \mathbf{b})$ is a convex polyhedron[1] in different ways, e.g., [5], [13], [14]. Tolerance solutions are studied since by [6], [5], [14] and others, see the references in the latter work. The nonparametric tolerable solution set and its interval estimations are also studied as a special case of the more general class of nonparametric AE-solution sets. S. Shary develops a formal algebraic approach for finding inner/outer estimations of the nonparametric AE-solution sets, see [15] for a summary. This approach is further developed by A. Goldsztejn, e.g., [3]. O. Beaumont and B. Philippe apply linear programming techniques to solve the nonparametric interval tolerance problem [1]. I. Sharaya studies in [16, 17] some criteria for a nonparametric tolerable solution set $\Sigma_{tol}(\mathbf{A}, \mathbf{b})$ to be unbounded and proves that every tolerable solution set can be represented as a sum of a linear subspace and a bounded convex polyhedron.

A recent progress in the explicit description of the more general class of parametric AE-solution sets [9] resulted in explicit description of the parametric tolerable solution set, which allowed proving various properties of the latter. In particular, it was proven in [9], see also [18], that every parametric tolerable solution set is a convex polyhedron. In [11] some numerical methods for outer interval estimation of parametric AE-solution sets are presented including special study of the methods for bounding the parametric tolerable solution set. Two methods for inner interval estimation of the nonempty parametric tolerable solution set are discussed in [10].

The present article aims to complete the initial investigations on the parametric tolerable solution set by studying the unbounded solution set. In Section 2 we define *null space* of the parametric matrix related to a tolerable solution set and prove necessary and sufficient conditions (in both general and computable forms) for a nonempty parametric tolerable solution set to be unbounded. In Section 3 we generalize the theory of I. Sharaya [16,17] about the structure of a nonparametric tolerable solution set as a sum of a linear subspace and a bounded convex polyhedron to the more general class of parametric tolerable solution sets. Then, on some numerical examples we illustrate the unbounded parametric tolerable solution sets and discuss how to obtain more accurate inner and/or outer interval estimations for such solution sets.

2 Unboundedness of the Parametric Tolerable Solution Set

The following definition is well-known from linear algebra.

Definition 1 For a matrix $A \in \mathbb{R}^{m\times n}$, the set $\mathcal{N}(A) := \{x \in \mathbb{R}^n \mid Ax = 0\} \subseteq \mathbb{R}^n$ is called the *null space*[2] of A.

[1] Analytically, a convex polyhedron is expressed as the solution set for a system of linear inequalities. A convex polyhedron may be empty, may coincides with the whole space $\mathbb{R}^n$, or may be a nonempty intersection of a finite number of closed half spaces. It may be also bounded or unbounded.

[2] Some authors call the null space *kernel* of the matrix.

Every null space is nonempty and trivially contains at least the zero vector.

Let us consider in more details the parametric tolerable solution set for a linear system involving a numerical vector $b \in \mathbb{R}^m$ in the right-hand side and an $m \times n$ matrix $A(p)$ whose elements depend linearly on a K-tuple of parameters p

$$\Sigma_{tol}(A(p), b, \mathbf{p}) := \{x \in \mathbb{R}^n \mid (\forall p \in \mathbf{p})(A(p)x = b)\}. \tag{6}$$

An explicit description of this solution set can be obtained, see [9], by successive elimination of the universally quantified parameters p_k, $k \in \mathcal{K}$, from the inequalities

$$A_0x - b + \sum_{k\in\mathcal{K}} p_k(A_kx) \ \leq 0 \leq \ A_0x - b + \sum_{k\in\mathcal{K}} p_k(A_kx) \tag{7}$$

$$\check{p}_k - \hat{p}_k \leq p_k \leq \check{p}_k + \hat{p}_k, \quad k \in \mathcal{K}, \tag{8}$$

where (7) is equivalent to $A(p)x = b$ and (8) is equivalent to $p_k \in \mathbf{p}_k$. In the elimination of each parameter we apply the distributivity of the universal quantifiers over conjunction and the following two relations

$$\lambda\check{p}_k - |\lambda|\hat{p}_k \leq \lambda p_k \leq \lambda\check{p}_k + |\lambda|\hat{p}_k, \qquad \lambda \in \mathbb{R} \tag{9}$$

$$(\forall t \in \mathbf{t} : v_1 \leq f(t) \leq v_2) \quad \Leftrightarrow \quad (v_1 \leq \min_{t\in\mathbf{t}} f(t)) \wedge (\max_{t\in\mathbf{t}} f(t) \leq v_2), \tag{10}$$

where $\mathbf{t} \in \mathbb{IR}$ and $f(t)$ is linear function of t. Thus, in the elimination of p_1 we apply relation (10), with $t = p_1$, to

$$A_0x - b + \sum_{k=2}^{K} p_k(A_kx) \leq -p_1(A_1x) \leq A_0x - b + \sum_{k=2}^{K} p_k(A_kx)$$

and obtain

$$-(A_1x)\check{p}_1 + |A_1x|\hat{p}_1 \leq A_0x - b + \textstyle\sum_{k=2}^{K} p_k(A_kx) \leq -(A_1x)\check{p}_1 - |A_1x|\hat{p}_1.$$

By rearranging the last inequalities and eliminating the next parameters we come to

$$\sum_{k\in\mathcal{K}} \hat{p}_k|A_kx| \leq A_0x + \textstyle\sum_{k\in\mathcal{K}} \check{p}_k(A_kx) - b \leq -\displaystyle\sum_{k\in\mathcal{K}} \hat{p}_k|A_kx|$$

that is

$$|A(\check{p})x - b| \leq -\sum_{k\in\mathcal{K}} \hat{p}_k|A_kx|. \tag{11}$$

The description (11) of $\Sigma_{tol}(A(p), b, \mathbf{p})$ is a special case of the description of $\Sigma_{tol}(A(p), \mathbf{b}, \mathbf{p})$ given by the following proposition.

Proposition 1 ([9, Proposition 5.8]) *For a K-tuple of parameters p involved in $A(p) \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{IR}^m$*

$$\Sigma_{tol}(A(p), \mathbf{b}, \mathbf{p}) = \{x \in \mathbb{R}^n \mid |A(\check{p})x - \check{b}| \leq \hat{b} - \sum_{k\in\mathcal{K}} \hat{p}_k |A_k x|\}.$$

Proposition 2

$$\Sigma_{tol}(A(p), 0, \mathbf{p}) = \bigcap_{k\in\mathcal{K}\cup\{0\}} \mathcal{N}(A_k).$$

Proof Since the two sides of the inequality (11) have different signs, $\Sigma_{tol}(A(p), 0, \mathbf{p})$ is not the empty set if and only if

$$|A(\check{p})x| = 0 = \sum_{k\in\mathcal{K}} \hat{p}_k |A_k x|. \tag{12}$$

It is obvious that the trivial point $x = 0$ belongs to $\Sigma_{tol}(A(p), 0, \mathbf{p})$ and therefore the latter is not empty. The right-hand side equality in (12) holds true iff $A_k x = 0$ for all $k \in \mathcal{K}$. Then $|A(\check{p})x| = |A_0 x + \sum_{k\in\mathcal{K}} \check{p}_k(A_k x)| = 0$ holds true iff $A_0 x = 0$. The equality relation (12) implies also that

$$\Sigma_{tol}(A(p), 0, \mathbf{p}) \neq \{0\} \text{ iff } \bigcap_{k\in\mathcal{K}\cup\{0\}} \mathcal{N}(A_k) \neq \{0\}.$$

Definition 2 For an $m \times n$ parametric matrix $A(p) := A_0 + \sum_{k\in\mathcal{K}} p_k A_k$ involved in a parametric tolerable solution set, where $p \in \mathbf{p} \in \mathbb{IR}^K$, the set

$$\mathcal{N}(A(p), \mathbf{p}) := \{x \in \mathbb{R}^n \mid (\forall p \in \mathbf{p})(A(p)x = 0)\}$$

is called *null space of the parametric matrix.*

From the definition of a parametric tolerable solution set we have $\mathcal{N}(A(p), \mathbf{p}) = \Sigma_{tol}(A(p), 0, \mathbf{p})$ and Proposition 2 implies the following

Corollary 1 $\mathcal{N}(A(p), \mathbf{p}) \neq \{0\}$ *iff* $\bigcap_{k\in\mathcal{K}\cup\{0\}} \mathcal{N}(A_k) \neq \{0\}$.

Lemma 1 *Let $\Sigma_{tol}(A(p), b(q), \mathbf{p}, \mathbf{q}) \neq \emptyset$. If $\mathcal{N}(A(p), \mathbf{p}) \neq \{0\}$, then $\Sigma_{tol}(A(p), b(q), \mathbf{p}, \mathbf{q})$ is unbounded.*

Proof By Definition 2, for all $p \in \mathbf{p}$ there exists $c \in \mathbb{R}^n$ such that $A(p)c = 0$. This implies

$$\forall t \in \mathbb{R}, \forall p \in \mathbf{p}, \exists c \in \mathbb{R}^n, A(p)(tc) = 0. \tag{13}$$

Let $\tilde{x} \in \Sigma_{tol}(A(p), b(q), \mathbf{p}, \mathbf{q})$. The latter implies

$$\forall p \in \mathbf{p}, \exists q \in \mathbf{q}, A(p)\tilde{x} = b(q). \tag{14}$$

We add (13) and (14) and get

$$\forall t \in \mathbb{R}, \forall p \in \mathbf{p}, \exists q \in \mathbf{q}, \exists c \in \mathbb{R}^n, A(p)\tilde{x} + A(p)(tc) = b(q). \tag{15}$$

Since $A(p)\tilde{x} + A(p)(tc) = A(p)(\tilde{x} + tc)$, by the definition of $\Sigma_{tol}(A(p), b(q), \mathbf{p}, \mathbf{q})$, the line $\tilde{x} + \mathbb{R}c \in \Sigma_{tol}(A(p), b(q), \mathbf{p}, \mathbf{q})$, that is the parametric tolerable solution set is unbounded.

Lemma 2 *If $\Sigma_{tol}(A(p), b(q), \mathbf{p}, \mathbf{q})$ is unbounded, $\mathcal{N}(A(p), \mathbf{p}) \neq \{0\}$.*

Proof By [9, Theorem 5.11], $\Sigma_{tol}(A(p), b(q), \mathbf{p}, \mathbf{q})$ is a convex polyhedron. Every orthant Ort in $\mathbb{R}^n$, defined by

$$\delta_i x_i \geq 0,\ \delta_i \in \{-1, +1\},\ i = 1, \ldots, n,$$

is a convex polyhedron. The intersection of convex polyhedra is a convex polyhedron. Therefore, $\Sigma_{tol}(A(p), b(q), \mathbf{p}, \mathbf{q}) \bigcap Ort$ is a convex polyhedron. Since $\Sigma_{tol}(A(p), b(q), \mathbf{p}, \mathbf{q})$ is unbounded, there exists an orthant $\widetilde{Ort}$ such that the convex polyhedron $\Sigma_{tol}(A(p), b(q), \mathbf{p}, \mathbf{q}) \bigcap \widetilde{Ort}$ is also unbounded. Therefore, there exists a ray

$$\tilde{x} + tc \in \Sigma_{tol}(A(p), b(q), \mathbf{p}, \mathbf{q}) \bigcap \widetilde{Ort}, \quad \text{where } c \in \widetilde{Ort}, c \neq 0, t \in \mathbb{R}^+.$$

Since $\tilde{x} + tc \in \Sigma_{tol}(A(p), b(q), \mathbf{p}, \mathbf{q})$, by the definition of this solution set,

$$\forall t \in \mathbb{R}^+, \forall p \in \mathbf{p}, \exists q \in \mathbf{q}, \exists c \in \widetilde{Ort}, c \neq 0, A(p)(\tilde{x} + tc) = b(q).$$

The latter implies $A(p)\tilde{x} + tA(p)c = b(q)$ and also $t(A(p)c) = 0$. Since t may be arbitrary large,

$$\forall p \in \mathbf{p}, A(p)c = 0,$$

which means that $\mathcal{N}(A(p), \mathbf{p}) \neq \{0\}$.

By Lemma 1 and Lemma 2 we prove the following theorem.

Theorem 1 *Let a parametric tolerable solution set $\Sigma_{tol}(A(p), b(q), \mathbf{p}, \mathbf{q})$ be nonempty. This solution set is unbounded if and only if $\mathcal{N}(A(p), \mathbf{p}) \neq \{0\}$.*

Theorem 1 and Corollary 1 give the following computable criterion for a nonempty parametric tolerable solution set to be unbounded.

Theorem 2 *Let a parametric tolerable solution set $\Sigma_{tol}(A(p), b(q), \mathbf{p}, \mathbf{q})$ be nonempty. This solution set is unbounded if and only if $\bigcap_{k \in \mathcal{K}_p \cup \{0\}} \mathcal{N}(A_k) \neq \{0\}$, where $\mathcal{K}_p \bigcap \mathcal{K}_q = \emptyset$, $\mathcal{K}_p \bigcup \mathcal{K}_q = \mathcal{K}$.*

3 Structure of the Parametric Tolerable Solution Set

Lemma 3 *$\mathcal{N}(A(p), \mathbf{p})$ is a linear subspace of $\mathbb{R}^n$.*

Proof The proof is trivial via the three properties of a subspace.

It is well-known that every linear subspace of $\mathbb{R}^n$ is a convex polyhedron.

Theorem 3 *The parametric tolerable solution set can be represented as*

$$\Sigma_{tol}(A(p), b(q), \mathbf{p}, \mathbf{q}) \ = \ \mathcal{N}(A(p), \mathbf{p}) + V(p, q), \tag{16}$$

where $V(p, q) := \Sigma_{tol}(A(p), b(q), \mathbf{p}, \mathbf{q}) \bigcap \mathcal{N}'$ is a polytope[3] and $\mathcal{N}'$ is a complementary subspace to $\mathcal{N}(A(p), \mathbf{p})$ in $\mathbb{R}^n$.

[3] A convex polytope is a bounded convex polyhedron.

Proof The proof follows that one given in [16]. We present it here for completeness.

For arbitrary vectors $x \in \mathbb{R}^n$ and $c \in \mathcal{N}(A(p), \mathbf{p})$ we have

$$\forall p \in \mathbf{p}, \qquad A(p)(x + c) = A(p)x + A(p)c = A(p)x.$$

Therefore $x \in \Sigma^p_{tol} := \Sigma_{tol}(A(p), b(q), \mathbf{p}, \mathbf{q})$ implies that $(x + c) \in \Sigma^p_{tol}$. The latter means that $\Sigma^p_{tol} + \mathcal{N}(A(p), \mathbf{p}) \subseteq \Sigma^p_{tol}$. Since $0 \in \mathcal{N}(A(p), \mathbf{p})$, we have $\Sigma^p_{tol} \subseteq \Sigma^p_{tol} + \mathcal{N}(A(p), \mathbf{p})$. Therefore, $\Sigma^p_{tol} = \Sigma^p_{tol} + \mathcal{N}(A(p), \mathbf{p})$.

Let $\mathcal{N}'$ be an arbitrary linear subspace which is complementary to $\mathcal{N}(A(p), \mathbf{p})$ in $\mathbb{R}^n$. Then $\Sigma^p_{tol} = \mathcal{N}(A(p), \mathbf{p}) + (\Sigma^p_{tol} \cap \mathcal{N}')$. Denote $V(p, q) := \Sigma^p_{tol} \cap \mathcal{N}'$. Since both Σ^p_{tol} and $\mathcal{N}'$ are convex polyhedrons, $\Sigma^p_{tol} \cap \mathcal{N}'$ is also a convex polyhedron. We will prove that $V(p, q)$ is bounded.

Assume that $V(p, q)$ is unbounded. Then $V(p, q)$ contains a ray. Let c denote its direction. The intersection of the ray with the orthant containing c is again a ray from $V(p, q) := \Sigma^p_{tol} \cap \mathcal{N}'$ having the same direction and another origin. Denote by x the origin of the new ray. The new ray $(x + R_+c) \in \Sigma^p_{tol}$ and therefore

$$\forall p \in \mathbf{p},\ \exists q \in \mathbf{q},\ A(p)(x + R_+c) = b(q).$$

Since $A(p)(x + R_+c) = A(p)x + A(p)(R_+c) = A(p)x + R_+(A(p)c)$,

$$\forall p \in \mathbf{p},\ \exists q \in \mathbf{q},\ A(p)x + R_+(A(p)c) = b(q)$$

implies $A(p)c = 0$, which means that $c \in \mathcal{N}(A(p), \mathbf{p})$. The latter contradicts the choice $(x + R_+c) \in \mathcal{N}'$. Therefore, $V(p, q)$ is bounded.

Theorem 3 describes the parametric tolerable solution set $\Sigma_{tol}(A(p), b(q), \mathbf{p}, \mathbf{q})$ more precisely than the description of $\Sigma_{tol}(A(p), b(q), \mathbf{p}, \mathbf{q})$ as a convex polyhedron. This is because some convex polyhedrons cannot be represented as a sum of a linear subspace and a convex polytope. Examples of 3-dimensional convex polyhedrons which cannot have the latter representation are given in [16,17]. In the examples discussed in the next section Theorem 3 is used for graphical visualization of nonempty unbounded parametric tolerable solution sets.

Note, that Theorems 1 and 2, which define conditions for $\Sigma_{tol}(A(p), b(q), \mathbf{p}, \mathbf{q})$ to be unbounded, assume that the solution set is not empty. Verifying that a parametric tolerable solution set is empty or not can be done by first finding explicit description of the solution set by means of inequalities (cf. [9]), or interval inclusions, and then solving these inequalities/inclusions. However, this might be a difficult task. On another hand, we may apply Theorem 3 in combination with Theorems 1 and 2 to obtain some computable sufficient conditions for $\Sigma_{tol}(A(p), b(q), \mathbf{p}, \mathbf{q})$ to be bounded. Since $\mathcal{N}(A(p), \mathbf{p})$ is always nonempty and $\mathcal{N}(A(p), \mathbf{p})$ is bounded iff $\mathcal{N}(A(p), \mathbf{p}) = \{0\}$, from the representation (16) and Theorem 3, which says that $V(p, q)$ is always bounded, we obtain

$$(\Sigma_{tol}(A(p), b(q), \mathbf{p}, \mathbf{q}) \text{ is bounded}) \iff \\ ((\mathcal{N}(A(p), \mathbf{p}) = \{0\}) \text{ or } (V(p, q) = \emptyset)). \quad (17)$$

Thus, we proved

Proposition 3 *If $\mathcal{N}(A(p), \mathbf{p}) = \{0\}$, respectively if $\bigcap_{k=0}^{K_p} \mathcal{N}(A_k) = \{0\}$, then $\Sigma_{tol}(A(p), b(q), \mathbf{p}, \mathbf{q})$ is bounded (in particular, $\Sigma_{tol}(A(p), b(q), \mathbf{p}, \mathbf{q})$ might be empty).*

The logical negation of the expression (17) gives

$$(\Sigma_{tol}(A(p), b(q), \mathbf{p}, \mathbf{q}) \text{ is unbounded}) \Longleftrightarrow \\ ((\mathcal{N}(A(p), \mathbf{p}) \neq \{0\}) \text{ and } (V(p,q) \neq \emptyset)) . \quad (18)$$

Since $\Sigma_{tol}(A(p), b(q), \mathbf{p}, \mathbf{q}) = \emptyset$ in Theorem 3 is equivalent to $V(p,q) = \emptyset$, we obtain

Proposition 4 *If $\mathcal{N}(A(p), \mathbf{p}) \neq \{0\}$, respectively if $\bigcap_{k=0}^{K_p} \mathcal{N}(A_k) \neq \{0\}$, then $\Sigma_{tol}(A(p), b(q), \mathbf{p}, \mathbf{q})$ is unbounded or empty.*

Since the interval vectors are bounded sets, interval estimations (outer and/or inner) are suitable for bounded nonempty solution sets. Theorem 3 allows also a way for obtaining more accurate outer and/or inner estimations for a nonempty unbounded parametric tolerable solution set. In the representation (16) $V(p,q)$ is bounded and describes a parametric tolerable solution set. Therefore, an outer or inner interval estimation $\mathbf{z}$ of $V(p,q)$ can be found by methods for nonempty and bounded parametric tolerable solution sets. For example, [10] discusses the so-called parametric centered (Neumaier's) approach and the vertex approach for inner estimation of the parametric tolerable solution set, [11] presents a parametric single step method, the vertex approach and a linear programming approach for outer estimation of the bounded parametric tolerable solution set. Then, the corresponding estimation of $\Sigma_{tol}(A(p), b(q), \mathbf{p}, \mathbf{q})$ has the form $\mathcal{N}(A(p), \mathbf{p}) + \mathbf{z}$.

4 Numerical Examples

Example 1 Consider the parametric linear system $A(p)x = \mathbf{b}$, where

$$A(p) := \begin{pmatrix} 1+p_1 & 2-3p_2 & 1+p_1 \\ 2-p_1+p_2 & 4 & 2-p_1+p_2 \\ 3+p_2 & 6+7p_2 & 3+p_2 \end{pmatrix}, \quad \mathbf{b} := \begin{pmatrix} [-\frac{1}{2}, \frac{3}{2}] \\ [-\frac{1}{4}, \frac{3}{4}] \\ [-\frac{3}{2}, \frac{5}{2}] \end{pmatrix} \quad (19)$$

and $p_1 \in [\frac{1}{2}, \frac{3}{2}]$, $p_2 \in [\frac{1}{2}, \frac{5}{2}]$.

First, we are interested to check if $\Sigma_{tol}(A(p), \mathbf{b}, \mathbf{p})$ is bounded or unbounded. By the methods of linear algebra we find

$$\begin{aligned} \mathcal{N}(A_0) &= \{u(-1,0,1)^\top,\ v(-2,1,0)^\top \mid u, v \in \mathbb{R}\} \\ \mathcal{N}(A_1) &= \{u(-1,0,1)^\top,\ v(0,1,0)^\top \mid u, v \in \mathbb{R}\} \\ \mathcal{N}(A_2) &= \{u(-1,0,1)^\top \mid u \in \mathbb{R}\}. \end{aligned}$$

Thus, $\mathcal{N}(A(p), \mathbf{p}) := \bigcap_{k \in \{0,1,2\}} \mathcal{N}(A_k) = \{u(-1,0,1)^\top \mid u \in \mathbb{R}\}$. Since $\mathcal{N}(A(p), \mathbf{p}) \neq \{0\}$, if nonempty, $\Sigma_{tol}(A(p), \mathbf{b}, \mathbf{p})$ is unbounded. As a subspace in $\mathbb{R}^3$ complementary to $\mathcal{N}(A(p), \mathbf{p})$ we take $\mathcal{N}' = \{(x_1, x_2, x_3)^\top \mid x_3 = 0\}$. Then the polytope $V(p, b) = \Sigma_{tol}(A(p), \mathbf{b}, \mathbf{p}) \cap \mathcal{N}'$ is the parametric tolerable solution set for the linear system with the same right-hand side vector and a new parametric matrix $B(p) = (A_{\bullet 1}(p); A_{\bullet 2}(p))$, where $A_{\bullet j}$ denotes the j-th column of a matrix $A \in \mathbb{R}^{m\times n}$, that is $V(p, b) := \Sigma_{tol}(B(p), \mathbf{b}, \mathbf{p})$ for the system

$$\begin{pmatrix} 1+p_1 & 2-3p_2 \\ 2-p_1+p_2 & 4 \\ 3+p_2 & 6+7p_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} [-\frac{1}{2}, \frac{3}{2}] \\ [-\frac{1}{4}, \frac{3}{4}] \\ [-\frac{3}{2}, \frac{5}{2}] \end{pmatrix} \tag{20}$$

with the same values for the parameters p.

Applying Proposition 1 we find an explicit description of $V(p, b)$ by the inequalities

$$\begin{aligned} \left|-\frac{1}{2} + 2x_1 - \frac{5}{2}x_2\right| &<= 1 - \frac{|x_1|}{2} - 3|x_2| \\ \left|-\frac{1}{4} + \frac{5}{2}x_1 + 4x_2\right| &<= \frac{1}{2} - \frac{3}{2}|x_1| \\ \left|-\frac{1}{2} + \frac{9}{2}x_1 + \frac{33}{2}x_2\right| &<= 2 - |x_1 + 7x_2| \end{aligned}$$

If we find the explicit description of $\Sigma_{tol}(A(p), \mathbf{b}, \mathbf{p})$ by Proposition 1 and replace x_3 by zero, we will get the same inequalities. The equivalent description of $V(p, b)$ by interval inclusions is

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \mathbf{p}_1 \left(\begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) + \mathbf{p}_2 \left(\begin{pmatrix} 0 & -3 \\ 1 & 0 \\ 1 & 7 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) \subseteq \begin{pmatrix} [-\frac{1}{2}, \frac{3}{2}] \\ [-\frac{1}{4}, \frac{3}{4}] \\ [-\frac{3}{2}, \frac{5}{2}] \end{pmatrix}.$$

In order to compute inner or outer estimations of the parametric tolerable solution sets that are considered below, we apply the so-called vertex approach which is defined by the following proposition.

Proposition 5 (special case of [11, Theorem 1])

$$\Sigma_{tol}(A(p), \mathbf{b}, \mathbf{p}) = \bigcap_{\tilde{p} \in \mathcal{B}} \Sigma_{uni}(A(\tilde{p}), \mathbf{b}),$$

where $\mathcal{B}$ *is the set of all vertices of the interval box* $\mathbf{p}$, *that is* $\mathcal{B} := \{\check{p}_1 + \delta_1\hat{p}_1, \ldots, \check{p}_{K_p} + \delta_{K_p}\hat{p}_{K_p} \mid \delta_1, \ldots, \delta_{K_p} \in \{-1, +1\}\}$, K_p *is the dimension of* p.

In the special case of interval systems $Ax = \mathbf{b}$ with point matrices $A \in \mathbb{R}^{m\times n}$, the Neumaier's centered approach [6] yields $\mathbf{x}_{in} \subseteq \Sigma_{uni}(A, \mathbf{b})$ using

$$\begin{aligned} &\mathbf{x}_{in} := \tilde{x} + \eta\, \mathbf{e}, \quad \text{where} \\ &\eta \in \mathbb{R}, \quad \eta := \min_{1\le i\le m} \frac{\min\{|\underline{b}_i - A_{i\bullet}\tilde{x}|, |\overline{b}_i - A_{i\bullet}\tilde{x}|\}}{|A_{i\bullet}|\mathbf{1}}, \end{aligned} \tag{21}$$

$\tilde{x}$ is a given interior point for $\Sigma_{uni}(A, \mathbf{b})$[4], usually $\tilde{x} = A^{-1}\check{b}$, $\mathbf{e} := ([-1,1], \ldots, [-1,1])^\top \in \mathbb{IR}^n$, $\mathbf{1} := (1,\ldots,1)^\top \in \mathbb{R}^n$, $A_{i\bullet}$ is the i-th row of A, the absolute value is applied to vectors/matrices componentwise.

First, we look for an inner estimation of $\Sigma_{tol}(B(p), \mathbf{b}, \mathbf{p})$ and apply the vertex approach to the system (20). For every vertex $\tilde{p} \in \mathcal{B}$ we compute (if possible) $\mathbf{x}_{in}(\tilde{p}) \subseteq \Sigma_{uni}(B(\tilde{p}), \mathbf{b})$ following the above methodology. Every $\mathbf{x}_{in}(\tilde{p})$ is computed using a fixed $\tilde{x} = B^{-1}(\check{p})\check{b}$ obtained by least squares. Then $\bigcap_{\tilde{p}\in\mathcal{B}} \mathbf{x}_{in}(\tilde{p})$ yields the following inner estimation of $V(p, b) := \Sigma_{tol}(B(p), \mathbf{b}, \mathbf{p})$

$$([0.177,\ 0.199], [-0.034,\ -0.012])^\top,$$

where (and below) the exact interval values are rounded inward to the third place after the decimal point. Note that other choices of $\tilde{x}$ in (21), for example $\tilde{x}(\tilde{p}) = B^{-1}(\tilde{p})\check{b}$ if it is an interior point for the corresponding solution set or a combination of the two choices, usually yield different inner estimations, see a discussion in [10]. A combination of the two choices of $\tilde{x}$ yields a larger inner estimation

$$([0.116,\ 0.164], [-0.048,\ 0.022])^\top.$$

Thus, an inner estimation of $\Sigma_{tol}(A(p), \mathbf{b}, \mathbf{p})$ is the unbounded interval vector

$$\left\{ u \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} [0.116,\ 0.164] \\ [-0.048,\ 0.022] \\ 0 \end{pmatrix} \mid u \in \mathbb{R} \right\}$$

and its edges parallel to $\mathcal{N}(A(p), \mathbf{p}) = \{u(-1, 0, 1)^\top \mid u \in \mathbb{R}\}$ are drawn in Fig. 1 b). For comparison, the vertex approach for inner estimation of a parametric tolerable solution set applied to the original system (19), with the same choice $\tilde{x} = A^{-1}(\check{p})\check{b}$ for every $\tilde{p} \in \mathcal{B}$, gives the *bounded* interval vector $([0.087,\ 0.101], [-0.031,\ -0.016], [0.087,\ 0.101])^\top$, while a combined choice of $\tilde{x}$ yields the interval vector

$$([0.041,\ 0.088], [-0.036,\ 0.010], [0.041,\ 0.088])^\top, \tag{22}$$

which is presented also in Fig. 1 b) for comparison.

Since we have found an inner estimation of $\Sigma_{tol}(B(p), \mathbf{b}, \mathbf{p})$ we are sure that the latter and resp. $\Sigma_{tol}(A(p), \mathbf{b}, \mathbf{p})$ are not empty. In order to find an outer interval estimation of $\Sigma_{tol}(B(p), \mathbf{b}, \mathbf{p})$ we may follow the methodology presented in [8] and take the first two components of the outer estimation for the parametric *tolerable* solution set of the square augmented system

$$\begin{pmatrix} B(p) & -I \\ 0 & B^\top(p) \end{pmatrix} z \;=\; \begin{pmatrix} \mathbf{b} \\ 0 \end{pmatrix}.$$

Outer estimation of the tolerable solution set to a square parametric system can be found by the parametric single step method presented in [11]. However, the above augmented parametric matrix is not strongly regular on $\mathbf{p}$

[4] Sufficient conditions for a point to be in the topological interior of a solution set are presented in [6], [10].

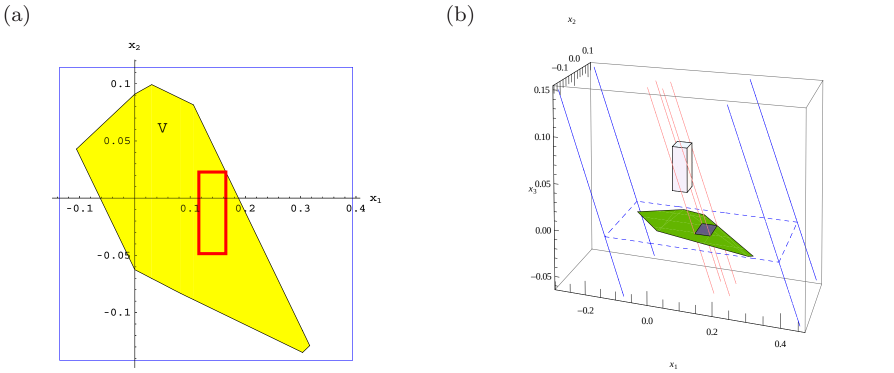

Fig. 1 (a) $\Sigma_{tol}(B(p), \mathbf{b}, \mathbf{p})$ for the system (20) from Example 1 and its inner and outer interval estimations. (b) The edges of the unbounded inner and outer interval estimations (that correspond to (a)) of $\Sigma_{tol}(A(p), \mathbf{b}, \mathbf{p})$ together with the bounded inner estimation (22).

and the parametric single step method fails. That is why, we apply again the vertex approach either to the above square augmented system or to the overdetermined system $B(p)x = \mathbf{b}$. For each $\tilde{p} \in \mathcal{B}$, $\mathbf{x}_{out}(\tilde{p}) \supseteq \Sigma_{uni}(B(\tilde{p}), \mathbf{b})$ is obtained as $\mathbf{x}_{out}(\tilde{p}) = B^{-1}(\tilde{p})\mathbf{b}$, where $B^{-1}(\tilde{p})$ is the Moore-Penrose pseudoinverse of $B(\tilde{p})$, cf. [4]. Then, $\bigcap_{\tilde{p}\in\mathcal{B}} \mathbf{x}_{out}(\tilde{p})$ yields an outer interval estimation of $V(p, b) := \Sigma_{tol}(B(p), \mathbf{b}, \mathbf{p})$, which gives the unbounded interval vector[5]

$$\left\{ u \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} [-0.137,\ 0.395] \\ [-0.142,\ 0.115] \\ 0 \end{pmatrix} \mid u \in \mathbb{R} \right\}$$

as an outer estimation for $\Sigma_{tol}(A(p), \mathbf{b}, \mathbf{p})$; its edges parallel to $\mathcal{N}(A(p), \mathbf{p}) = \{u(-1, 0, 1)^\top \mid u \in \mathbb{R}\}$ are drawn in Fig. 1 b).

Note that the available methods for inner (or outer) estimation of a parametric tolerable solution set may fail in obtaining the corresponding estimation. However, this does not imply that the parametric tolerable solution set is empty. The following example gives an illustration.

Example 2 Consider the parametric system from Example 1, where the parameter p_1 is set to zero and $b_3 = \frac{1}{4}$. The null space of the parametric matrix is nontrivial and it is the same as that in Example 1. However, since $b_3 = \frac{1}{4}$ the parametric tolerable solution set will be degenerate if it is nonempty. In this case, neither the approaches from [10] for inner estimation work, nor we can find a point which belongs to the solution set. The polytope $V(p_2, b)$ which corresponds to the parametric tolerable solution set of the system

$$\begin{pmatrix} 1 & 2-3p_2 \\ 2+p_2 & 4 \\ 3+p_2 & 6+7p_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} [-\frac{1}{2}, \frac{3}{2}] \\ [-\frac{1}{4}, \frac{3}{4}] \\ \frac{1}{4} \end{pmatrix}$$

[5] Here, the exact numerical intervals are rounded outward to the third place after the decimal point.

will be also degenerate if it is not empty. We may try to find an outer estimation of $V(p_2, b)$ if the latter is not empty. The parametric Bauer-Skeel method from [11] applied to the corresponding augmented system does not work since the corresponding parametric system is not strongly regular. Fortunately, the vertex approach gives the outer interval enclosure

$$([-0.115,\ 0.341], [-0.080,\ 0.040])^\top,$$

which means that the parametric tolerable solution set is not empty. Then, the outer estimation of the original unbounded parametric tolerable solution set is

$$\left\{ u \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} [-0.115,\ 0.341] \\ [-0.080,\ 0.040] \\ 0 \end{pmatrix} \mid u \in \mathbb{R} \right\}.$$

Finding explicit description of the initial parametric tolerable solution set and solving the inequalities we obtain the line $\{(u, -\frac{1}{60}, \frac{7}{60} - u) \mid u \in \mathbb{R}\}$, which is presented in Figure 2 together with its outer estimation.

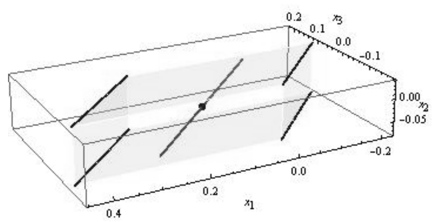

Fig. 2 The unbounded parametric tolerable solution set (the line in the middle) for the system from Example 2 and its outer estimation.

Examples of nonempty and bounded 2-dimensional parametric tolerable solution sets can be found in [9]–[11].

In the special case of nonparametric interval systems, the unboundedness of the tolerable solution set is expressed in [16,17] by the existence of linearly dependent degenerate (point) columns in the matrix. Despite of this simple result, the parametric approach presented here can be also applied to a nonparametric tolerable solution set if we consider an equivalent parametric tolerable solution set, where the matrix involves column dependencies, as presented in [10, Section 5.1].

5 Conclusion

Given are necessary and sufficient conditions (in both general and computable forms) for a nonempty parametric tolerable solution set to be unbounded. The representation of every parametric tolerable solution set as a sum of a linear subspace $L(A(p), \mathbf{p})$ and a bounded convex polyhedron $V(p, q)$ allows obtaining of suitable inner and outer estimations for the unbounded parametric tolerable solution sets. Although the methodology for inner estimations [10] can be applied to the original parametric system, the obtained *bounded* interval vector is much smaller than the *unbounded* tolerable solution set. On the other hand, unbounded solution sets cannot be estimated outwardly by bounded interval vectors. That is why, the unbounded estimation $L(A(p), \mathbf{p})+\mathbf{z}$, where $\mathbf{z}$ is the corresponding inner/outer interval estimation of the bounded parametric tolerable solution set $V(p, q)$, is more accurate. The numerical examples, given in Section 4, illustrate how to apply the discussed methodology.

Developing a suitable methodology that proves the unboundedness of a general parametric AE-solution set (in particular, for the united and the controllable solution sets) is an interesting open problem.

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References

1. Beaumont, O. and Philippe, B.: Linear interval tolerance problem and linear programming techniques. Reliable Computing 7, 433-447 (2001)
2. Dymowa, L.: Soft Computing in Economics and Finance. Intelligent Systems Reference Library, Vol. 6, Springer, Heidelberg (2011)
3. Goldsztejn, A.: A right-preconditioning process for the formal-algebraic approach to inner and outer estimation of AE-solution sets. Reliable Computing 11, 443–478 (2005)
4. Golub, G. H. and Van Loan, C. F.: Matrix computations (3rd ed.). Johns Hopkins, Baltimore (1996)
5. Kelling, B. and Oelschlägel, D.: Zur Lösung von linearen Toleranzproblemen, Wiss. Zeitschrift TH Leuna-Merseburg 33(1), 121-131 (1991)
6. Neumaier, A.: Tolerance analysis with interval arithmetic. In: Freiburger Intervall-Berichte 86/9, pp. 5–19, Albert-Ludwigs Univ., Freiburg (1986)
7. Nüding, E. and Wilhelm, W.: Über Gleichungen und über Lösungen. ZAMM 52, T188-T190 (1972)
8. Popova, E.D.: Improved solution enclosures for over- and underdetermined interval linear systems. In: I. Lirkov, S. Margenov, J. Waśniewski (eds.), LNCS 3743, pp. 305–312, Springer, Heidelberg (2006)
9. Popova, E.D.: Explicit description of AE solution sets for parametric linear systems. SIAM J. Matrix Anal. Appl. 33, 1172–1189 (2012)
10. Popova, E.D.: Inner estimation of the parametric tolerable solution set. Computers and Mathematics with Applications 66, 1655–1665 (2013)
11. Popova, E.D., Hladík, M.: Outer enclosures to the parametric AE solution set. Soft Computing 17, 1403-1414 (2013)

12. Rohn, J.: Input-output model with interval data. Econometrica 48, 767–769 (1980)
13. Rohn, J.: Inner solutions of linear interval systems. In: K. Nickel (ed.), Interval Mathematics 1985, Lecture Notes on Computer Science 212, pp. 157-158, Springer Verlag, New York (1986)
14. Shary, S.P.: Solving the linear interval tolerance problem. Mathematics and Computers in Simulation 39, 53–85 (1995)
15. Shary, S.P.: A new technique in systems analysis under interval uncertainty and ambiguity. Reliable Computing 8, 321–418 (2002)
16. Sharaya, I.A.: Structure of the tolerable solution set of an interval linear system. Computational Technologies 10, 103–119 (2005) in Russian
17. Sharaya, I.A.: On unbounded tolerable solution sets. Reliable Computing 11, 425–432 (2005)
18. Sharaya, I.A., Shary, S.P.: Tolerable solution set for interval linear systems with constraints on coefficients. Reliable Computing 15, 345–357 (2011)
19. Sokolova, S.P., Kuzmina, E.A.: Dynamic properties of interval systems. SPIIRAS Proceedings, issue 7, pp. 215–221. Nauka (2008) in Russian
20. Wang, Y.: Semantic tolerance modeling with generalized intervals. ASME Journal of Mechanical Design 130(8):081701 (2008)