Arch Appl Mech
The final publication is available at Springer via
DOI 10.1007/s00419-016-1180-2

Improved Solution to the Generalized Galilei's Problem with Interval Loads

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Received: 2 February 2016 / Accepted: 8 September 2016

Abstract Model of a bar subjected to multiple axial external loads, where load magnitudes are considered as uncertain and represented by intervals, is recently considered by Elishakoff, Gabriele and Wang in this journal. Beside a high complexity of the proposed procedure for computing the intervals of the reaction and of the axial force distribution its assumptions are not always fulfilled and in these cases the obtained interval enclosures are far from the optimal (narrowest) ones. In this work we present a simple and efficient computational model for the reaction and for the axial force distribution which is based on the algebraic extension of classical interval arithmetic. It is proved that this model always yields the narrowest interval enclosure. The new interval model can be generalized and applied to other linear equilibrium equations in mechanics.

Keywords force equilibrium · interval arithmetic · proper and improper intervals · overestimation

1 Introduction

Model of a bar subject to multiple axial loads, where the magnitude of each load is considered as uncertain and represented by a real compact interval, is recently considered by Elishakoff, Gabriele and Wang in [3]. In deterministic setting, finding the reaction is based on the principle of static equilibrium, which is indispensable topic in the analysis of structures, strength of material, robotics and many others. A similar problem in the context of robotics was

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recently discussed in [11]. Therefore, it is important to build a suitable model that accounts for the uncertainties in the loads (and other model parameters), as well as for round-off errors in computations, which is consistent with the physical laws and the empirical relations of the deterministic model. This turned out to be a not trivial task.

It is shown in [3], [11] that an interval model based entirely on the classical interval arithmetic, in its set-theoretic interpretation as proposed by Moore [12], cannot provide a good estimation neither of the uncertain reaction nor of the load distribution. That is why, Elishakoff et al. proposed in [3] a new interval "operation" which is a hybrid between the classical interval subtraction and the subtraction between the so-called generalized (proper and improper) intervals — it cancels equal intervals that are subtracted and performs classical interval operation otherwise. The computational procedure requires combinatorial search of the intervals that may be canceled. This approach yields much narrower interval estimations and in some special cases even the exact interval for the reaction. However, the model, proposed in [3], and its computational procedure have two deficiencies: the model not always provides the exact interval for the reaction and for the force distribution, and the computational procedure has high (exponential) complexity.

In this work we propose a model of a beam loaded by multiple external loads, where the latter are considered as uncertain and varying within given intervals. We prove that basing on this interval model the exact intervals for the reaction and for the force distribution can be found in a unique and a simple way if we use the generalized interval arithmetic on proper and improper intervals [7]. We also show that, due to the relations between the three forms of generalized interval arithmetic: Kaucher interval arithmetic [7], modal interval arithmetic [20], and directed interval arithmetic [9,10], the unknown reaction and force distribution can be found also by using the classical interval arithmetic, as defined by the recent IEEE 1788$^{\mathrm{TM}}$-2015 standard for interval arithmetic [6]. Finally, we outline some generalizations of the proposed interval model to force equilibrium equations in 2 or 3 dimensions and other linear equilibrium equations in mechanics.

The structure of the paper is as follows. Section 2 presents some basic notions from classical interval arithmetic and its algebraic extensions and only those that will be used in the present paper. For more details and discussions we give some references. In section 3 we present the new interval model and prove that it always deliver the narrowest interval estimations of the force distribution and of the reaction. Numerical examples illustrate the new interval model and its advantages over that proposed in [3]. In section 3.3 we discuss how the proposed model can be implemented in a software environment supporting the classical interval arithmetic as defined by the recent IEEE 1788$^{\mathrm{TM}}$-2015 standard for interval arithmetic [6]. The applications of the proposed interval model to force equilibrium equations in more dimensions and the potential of the proposed approach for handling other linear equilibrium equations in mechanics are outlined in section 4. The article ends by some conclusions.

2 Interval arithmetic and its algebraic extensions

2.1 Classical interval arithmetic

A real compact interval is $\mathbf{a} = [a^-, a^+] := \{a \in \mathbb{R} \mid a^- \leq a \leq a^+; a^-, a^+ \in \mathbb{R}\}$. By $\mathbb{IR}$ we denote the set of all real compact intervals. For $\lambda \in \mathbb{R}$ define $\mathrm{sgn}(\lambda) := \{+ \text{ if } \lambda \geq 0, - \text{ otherwise}\}$. Define width of an interval $\mathbf{a} \in \mathbb{IR}$ by $\omega(\mathbf{a}) := a^+ - a^-$. For $\lambda \in \mathbb{R}$ and $\mathbf{a} \in \mathbb{IR}$

$$\lambda\mathbf{a} = \begin{cases} [\lambda a^-, \lambda a^+] & \text{if } \lambda \geq 0, \\ [\lambda a^+, \lambda a^-] & \text{if } \lambda < 0. \end{cases}$$

The operation addition is defined by

$$\mathbf{a} + \mathbf{b} = [a^- + b^-, a^+ + b^+]$$

and the subtraction is defined as $\mathbf{a} - \mathbf{b} := \mathbf{a} + (-1)\mathbf{b}$. There are two order relations

$$\mathbf{a} \subseteq \mathbf{b} \Longleftrightarrow b^- \leq a^- \text{ and } a^+ \leq b^+,$$
$$\mathbf{a} \leq \mathbf{b} \Longleftrightarrow a^- \leq b^- \text{ and } a^+ \leq b^+.$$

More details about classical interval arithmetic can be found, e.g., in [12].

We have $\mathbf{a} \leq 0 \Leftrightarrow a^- \leq a^+ \leq 0$ and $0 \leq \mathbf{a} \Leftrightarrow 0 \leq a^- \leq a^+$. For $\lambda \in \mathbb{R}$ and $\mathbf{a} \in \mathbb{IR}$, we say that λ belongs to the interior of $\mathbf{a}$, and denote $\lambda \in \mathrm{int}(\mathbf{a})$ if and only if $a^- < \lambda < a^+$. For $\mathbf{a} \in \mathbb{IR}$ such that $0 \notin \mathrm{int}(\mathbf{a})$, define

$$\mathrm{Abs}(\mathbf{a}) := \begin{cases} \mathbf{a} & \text{if } 0 \leq \mathbf{a} \\ -\mathbf{a} & \text{otherwise.} \end{cases}$$

It is shown in [10] that the interval structure $(\mathbb{IR}, +, -)$ possesses the following trichotomy property: for $\mathbf{a}, \mathbf{b} \in \mathbb{IR}$,

- T1 $\omega(\mathbf{a}) < \omega(\mathbf{b})$ means that $\mathbf{b}$ can be represented as $\mathbf{b} = \mathbf{a} + \mathbf{x}$, $x^- < x^+$, and the unique $\mathbf{x}$ satisfying $\mathbf{a} + \mathbf{x} = \mathbf{b}$ is represented by $\mathbf{x} = [b^- - a^-, b^+ - a^+]$.
- T2 $\omega(\mathbf{a}) > \omega(\mathbf{b})$ means that $\mathbf{a}$ can be represented as $\mathbf{a} = \mathbf{b} + \mathbf{y}$, $y^- < y^+$, and the unique $\mathbf{y}$ satisfying $\mathbf{b} + \mathbf{y} = \mathbf{a}$ is represented by $\mathbf{y} = [a^- - b^-, a^+ - b^+]$.
- T3 $\omega(\mathbf{a}) = \omega(\mathbf{b})$ means that both: i) $\mathbf{a} + \mathbf{x} = \mathbf{b}$ has a solution $[x, x]$, $x \in \mathbb{R}$, and ii) $\mathbf{b} + \mathbf{y} = \mathbf{a}$ has a solution $[y, y]$, $y \in \mathbb{R}$, and we have $\mathbf{y} = -\mathbf{x} = [a^- - b^-, a^+ - b^+]$.

2.2 Kaucher interval extension

The set of compact intervals $\mathbb{IR}$, called *proper* intervals, is extended in [7] by the set $\overline{\mathbb{IR}} := \{[a^-, a^+] \mid a^-, a^+ \in \mathbb{R}, a^- \geq a^+\}$ of *improper* intervals obtaining thus the set $\mathbb{KR} = \mathbb{IR} \bigcup \overline{\mathbb{IR}} = \{[a^-, a^+] \mid a^-, a^+ \in \mathbb{R}\}$ of all ordered couples of real numbers called also *generalized intervals*. The conventional interval arithmetic and lattice operations, order relations and other interval functions are isomorphically extended onto the whole set $\mathbb{KR}$, [7]. For example, operations multiplication by a scalar, addition and subtraction for generalized intervals retain the same formulae as for proper intervals in $\mathbb{IR}$. For every[1] $[a] = [a^-, a^+] \in \mathbb{KR}$, define a binary variable *direction* by

$$\tau([a]) := \operatorname{sgn}(a^+ - a^-) = \begin{cases} + & \text{if } a^- \leq a^+, \\ - & \text{if } a^- > a^+. \end{cases}$$

All elements of $\mathbb{KR}$ with positive direction are called proper intervals and the elements with negative direction are called improper intervals. An element-to-element symmetry between proper and improper intervals is expressed by the "Dual" operator. For $[a] = [a^-, a^+] \in \mathbb{KR}$, $\operatorname{Dual}([a]) := [a^+, a^-]$. For $[a], [b] \in \mathbb{KR}$,

$$\operatorname{Dual}(\operatorname{Dual}([a])) = [a], \tag{1}$$

$$\operatorname{Dual}([a] \circ [b]) = \operatorname{Dual}([a]) \circ \operatorname{Dual}([b]), \ \circ \in \{+, -, \times, /\}. \tag{2}$$

Define proper projection of a generalized interval $[a]$ by

$$\operatorname{pro}([a]) := \begin{cases} [a] & \text{if } \tau([a]) = +, \\ \operatorname{Dual}([a]) & \text{if } \tau([a]) = -. \end{cases}$$

We have $[a] \leq 0$ iff $\operatorname{pro}([a]) \leq 0$ and $0 \leq [a]$ iff $0 \leq \operatorname{pro}([a])$. The generalized interval arithmetic structure possesses group properties with respect to the operation addition and with respect to the operation multiplication of intervals that do not involve zero. For $[a], [b] \in \mathbb{KR}$, $0 \notin \operatorname{pro}([b])$,

$$[a] - \operatorname{Dual}([a]) = 0, \qquad [b]/\operatorname{Dual}([b]) = 1. \tag{3}$$

Lattice operations are closed with respect to the inclusion relation; handling of norm and metric are very similar to norm and metric in linear spaces, [7]. Addition operation in $\mathbb{KR}$ is commutative and associative; associativity does not hold true in (interval) floating point arithmetic.

2.3 Other isomorphic interval extensions

Modal interval analysis imposes a logical-semantic background on generalized intervals (considered there as modal intervals) and allows giving a logical meaning to the interval results, see [20] for more details. For computing, modal interval analysis uses the same algebraic structure $(\mathbb{KR}, +, \times)$ and further develops its properties, e.g., how to use it for inner and/or outer interval approximation in floating point.

[1] For the sake of better understanding, we denote intervals from $\mathbb{KR}$ by $[a]$. Of course, $\mathbf{a} \in \mathbb{IR} \subset \mathbb{KR}$, and thus $[a] = \mathbf{a} \in \mathbb{KR}$ is a correct assignment.

In [9], S. Markov considers the system $\mathcal{K} = (\mathbb{KR}, +, \times)$ using two forms for presenting the elements of $\mathbb{KR} = \mathbb{IR} \bigcup \overline{\mathbb{IR}}$: component-wise (used by Kaucher [7], modal interval analysis [20] and others) and *normal* form. The latter considers generalized (directed) intervals $[a] \in \mathbb{KR}$ as couples $(\mathrm{pro}([a]), \lambda)$ of a proper interval $\mathrm{pro}([a])$ and a binary variable $\lambda \in \Lambda = \{+, -\}$, $\lambda = \tau([a])$ is called direction, which implies the name "directed interval". It is demonstrated that the normal form, which is equivalent to the direct product $\mathbb{IR} \bigotimes \Lambda$, generates two types of operations between proper intervals — the first type are the usual operations between proper intervals, which are also called "outer" interval operations, and the second type operations are the "inner" (or "non-standard") interval operations, the latter denoted by $\circ^-$, $\circ \in \{+, -, \times, /\}$. The set of normal (proper) intervals together with the set of outer and inner interval operations $\mathcal{M} = (\mathbb{IR}, +, +^-, \times, \times^-)$ presents an algebraic completion of the set-theoretic interval arithmetic $(\mathbb{IR}, +, -, \times, /)$. Opposite to $\mathcal{K}$, $\mathcal{M}$ makes no use of improper intervals, [8], [9], [10]. It is shown in [2], [9], [10] that the two interval systems $\mathcal{M} = (\mathbb{IR}, +, +^-, \times, \times^-)$ and $\mathcal{K} = (\mathbb{KR}, +, \times)$ are closely related and that the understanding the relations between them can greatly increase the scope of applications of interval arithmetic. While the system $\mathcal{K}$ plays an important role in the formal algebraic manipulations, the system $\mathcal{M}$ can be used for the interpretation of the results in terms of proper intervals.

Here we present only some basic relations that will be used latter in the paper. For more details consult [2], [9], [10] and the references given therein. For $(\mathbf{a}, \alpha), (\mathbf{b}, \beta)$, $\alpha, \beta \in \Lambda$,

$$(\mathbf{a}, \alpha) + (\mathbf{b}, \beta) = \left(\mathbf{a} +^{\alpha\beta} \mathbf{b},\ \tau((\mathbf{a}, \alpha) +^{\alpha\beta} (\mathbf{b}, \beta))\right), \tag{4}$$

wherein $++ = -- = +$, $+- = -+ = -$, for $\alpha\beta \in \Lambda$, $+^+ = +$,

$$\mathbf{a} +^- \mathbf{b} := \begin{cases} [a^- + b^+, a^+ + b^-] & \text{if } \omega(\mathbf{a}) \geq \omega(\mathbf{b}) \\ [b^- + a^+, b^+ + a^-] & \text{if } \omega(\mathbf{a}) \leq \omega(\mathbf{b}), \end{cases} \tag{5}$$

$$\tau((\mathbf{a}, \alpha) +^{\alpha\beta} (\mathbf{b}, \beta)) = \begin{cases} \alpha & \text{if } \alpha = \beta, \\ \alpha & \text{if } \alpha = -\beta, \omega(\mathbf{a}) > \omega(\mathbf{b}), \\ \beta & \text{if } \alpha = -\beta, \omega(\mathbf{a}) < \omega(\mathbf{b}), \\ + & \text{if } \alpha = -\beta, \omega(\mathbf{a}) = \omega(\mathbf{b}). \end{cases} \tag{6}$$

3 Interval model of a multiply loaded bar

Consider a bar loaded by k axial forces[2], $\overrightarrow{p}_1, \ldots, \overrightarrow{p}_k$ distributed (without loss of generality) uniformly along the beam axis, see Fig. 1. Forces may act in opposite directions. The magnitude p_i of each force $\overrightarrow{p}_i$ is considered as uncertain and varying within a real compact interval, $p_i \in \mathbf{p}_i$. We shall assume

[2] In the text of this work forces (and other vector quantities) are denoted by drawing a short arrow above the letter used to represent it. This is necessary in order to distinguish vectors from the interval-valued scalars, which are denoted by bold-face letters, and other real-valued scalars. The magnitude of a vector will be denoted by the corresponding italic-face letter.

Fig. 1 Bar loaded by a number of axial forces $\overrightarrow{p}_1, \ldots, \overrightarrow{p}_k$; $\overrightarrow{r}$ is the unknown reaction

that the relative uncertainty of the magnitude is not greater than 100% of its nominal value, which implies that $0 \leq \mathbf{p}_i$, $i = 1, \ldots, k$. The goal is to find an estimation of the unknown reaction $\overrightarrow{r}$, that is, an interval for the magnitude of $\overrightarrow{r}$ and its direction. The interval model should be consistent with the deterministic force equilibrium equation $\overrightarrow{r} + \sum_{i=1}^{k} \overrightarrow{p}_i = 0$. Since the problem is one-dimensional, the last vector equilibrium equation is reduced to the algebraic magnitude equilibrium equation

$$r + \sum_{i=1}^{k} p_k = 0.$$

An intermediate task is to construct the diagram of interval axial forces finding the resultant $\overrightarrow{N}_j(x)$ of j forces to the right of a cut at arbitrary cross section x, see Fig. 2.

Fig. 2 Section cut that determines the resultant axial force $\overrightarrow{N}_j(x)$.

In the deterministic case we have $\overrightarrow{N}_j(x) = \sum_{i=1}^{j} \overrightarrow{p}_i$, which reduces to the algebraic magnitude equation

$$N_j(x) = \sum_{i=1}^{j} p_i.$$

In what follows we will omit the argument x of the resultant axial force $\overrightarrow{N}_j(x)$.

3.1 Interval model in Kaucher arithmetic

Fig. 3 presents the free body diagram of a bar loaded with a single axial force $\overrightarrow{p}$.

Fig. 3 Free body diagram of a bar loaded with a single force $\overrightarrow{p}$, $\overrightarrow{r}$ is the reaction

In deterministic setting, finding the reaction of the only load is modeled by finding the solution ($r = -p$) of the equilibrium equation $p + r = 0$.

If the load magnitude is modeled by an interval variable $\mathbf{p}$ to describe the uncertainty in its value, then the magnitude of the reaction will be uncertain too and we are interested to find the direction of the reaction and an interval estimation of the reaction magnitude. In this, we have to preserve the physical interpretation of the reaction. Contrary to classical interval arithmetic, generalized interval arithmetic structure $(\mathbb{KR}, +)$ is a group with respect to addition operation and the algebraic equation $[a]+[x] = 0$ has always an algebraic solution $[x] = -\mathrm{Dual}([a])$, see (3). This is exactly what we need in order to model the interval equilibrium equation and to compute the interval estimation of the uncertain reaction. Therefore, we fix a basic direction, which without loss of generality coincides with the positive coordinate of the chosen coordinate system Ox in Fig. 3, and represent the axial forces that are applied in the same direction as the basic one by nonnegative proper intervals $\mathbf{p} \in \mathbb{IR}$, $\mathbf{p} \geq 0$. To compute the interval value of the corresponding reaction we use the generalized interval structure $(\mathbb{KR}, +)$ and obtain $[r] = -\mathrm{Dual}(\mathbf{p}) = [-p^-, -p^+]$. The generalized interval $[r]$ is negative, $[r] \leq 0$, and its direction is also negative, $\tau([r]) = -p^+ - (-p^-) = -(p^+ - p^-) \leq 0$. Now, we have to return back to our modeling space $\mathbb{IR}$ and to ***interpret*** there the result obtained in $(\mathbb{KR}, +)$. Namely, the direction of the reaction is determined by the sign of the generalized interval $[r]$ (it is opposite to the chosen basic direction) and the magnitude of the reaction varies in the proper interval $\mathbf{r} = \mathrm{Abs}(\mathrm{pro}([r])) = \mathbf{p}$. The latter completely corresponds to our physical model.

The above consideration implies that if a force $\overrightarrow{p}$ is applied in a direction opposite to the chosen basic direction, the magnitude $\mathbf{p}$ of the former has to be represented by a nonpositive improper interval $[p] = -\mathrm{Dual}([p]) = [-p^-, -p^+]$. Then, the reaction magnitude is computed similarly as above, $[r] = -\mathrm{Dual}([p]) \stackrel{(1)}{=} [p] = \mathbf{p}$, and its interpretation is that the reaction acts in the same direction as the basic one (opposite to the direction of the applied load) with magnitude equal to that of the applied load $\mathbf{p}$.

Let us now consider a bar loaded by two axial forces $\overrightarrow{p}_1, \overrightarrow{p}_2$ which are applied in two opposite directions and have uncertain interval magnitude $p_1 \in \mathbf{p}_1, p_2 \in \mathbf{p}_2$, $\mathbf{p}_1, \mathbf{p}_2 \geq 0$. The corresponding free body diagram is presented in Fig. 4.

We want to compute the resultant of the two axial forces and then the reaction. As in the previous consideration, first we fix a basic direction. Without loss of generality, let the basic direction coincides with the positive direction

Fig. 4 Bar loaded with two forces $\overrightarrow{p}_1$, $\overrightarrow{p}_2$ in opposite directions; $\overrightarrow{r}$ is the unknown reaction

of the coordinate system Ox in Fig. 4. Next, we represent the applied loads whose directions coincide with the basic direction by nonnegative proper intervals, namely $[p_1] = \mathbf{p}_1$, and we represent the applied loads whose directions are opposite to the basic direction by nonpositive improper intervals $[p_2] = -\mathrm{Dual}(\mathbf{p}_2)$. Thus, we compute in $(\mathbb{KR}, +)$ the uncertain resultant magnitude $[N_2]$ by

$$\begin{aligned}[N_2] &= [p_1] + [p_2] = [p_1] - \mathrm{Dual}(\mathbf{p}_2) \\ &= [p_1^-, p_1^+] + [-p_2^-, -p_2^+] \\ &= [p_1^- - p_2^-, p_1^+ - p_2^+]. \end{aligned} \tag{7}$$

In the interpretation, the sign of $[N_2]$ determines the direction of the resultant force $\overrightarrow{N}_2$ and $\mathrm{Abs}(\mathrm{pro}([N_2]))$ is the magnitude of $\overrightarrow{N}_2$. The sign of $[N_2]$ depends on the configuration between the endpoints of the two interval arguments as follows:

$$\begin{aligned} [N_2] = 0 &\quad \text{if } \mathbf{p}_1 = \mathbf{p}_2, \\ \mathrm{pro}([N_2]) \ni 0 &\quad \text{if } \mathbf{p}_1 \subseteq \mathbf{p}_2 \text{ or } \mathbf{p}_2 \subseteq \mathbf{p}_1, \\ \mathrm{sign}([N_2]) \le 0 &\quad \text{if } \mathbf{p}_1 \le \mathbf{p}_2, \\ \mathrm{sign}([N_2]) \ge 0 &\quad \text{if } \mathbf{p}_2 \le \mathbf{p}_1. \end{aligned}$$

Note here that the direction of $[N_2]$ depends on the relations between the widths of $\mathbf{p}_1$, $\mathbf{p}_2$, and it should not be confused with the direction of the resultant vector $\overrightarrow{N}_2$ which is the sign of $[N_2]$.

For a better understanding of our interval model we present here another derivation of $[N_2]$, which is aligned to some extent with the reasoning in [3] (canceling equal proper intervals that are subtracted). We have

$$[N_2] = [p_1] + [p_2] = [p_1] - \mathrm{Dual}(\mathbf{p}_2). \tag{8}$$

If $\mathbf{p}_2 = \mathbf{p}_1$, the property (3) gives $[N_2] = 0$.

If $\omega(\mathbf{p}_1) > \omega(\mathbf{p}_2)$,then by T2 property $\mathbf{p}_1 = \mathbf{p}_2 + \mathbf{y}$, where $\mathbf{y} = [p_1^- - p_2^-, p_1^+ - p_2^+]$. Substituting in (8) we obtain

$$\begin{aligned}[N_2] &= [p_1] - \mathrm{Dual}(\mathbf{p}_2) \\ &= \mathbf{p}_2 + \mathbf{y} - \mathrm{Dual}(\mathbf{p}_2) \\ &\overset{\text{comm},(3)}{=} \mathbf{y} = [p_1^- - p_2^-, p_1^+ - p_2^+] = (7). \end{aligned}$$

If $\omega(\mathbf{p}_1) < \omega(\mathbf{p}_2)$, then by T1 property $\mathbf{p}_2 = \mathbf{p}_1 + \mathbf{x}$, where $\mathbf{x} = [p_2^- - p_1^-, p_2^+ - p_1^+]$. Substituting in (8) we obtain

$$\begin{aligned}
[N_2] &= [p_1] - \mathrm{Dual}(\mathbf{p}_2) \\
&= \mathbf{p}_1 - \mathrm{Dual}(\mathbf{p}_1 + \mathbf{x}) \\
&\overset{(2)}{=} \mathbf{p}_1 - \mathrm{Dual}(\mathbf{p}_1) - \mathrm{Dual}(\mathbf{x}) \\
&\overset{(3)}{=} -\mathrm{Dual}(\mathbf{x}) = [p_1^- - p_2^-, p_1^+ - p_2^+] = (7).
\end{aligned}$$

If $\omega(\mathbf{p}_1) = \omega(\mathbf{p}_2)$, $\mathbf{p}_1 \neq \mathbf{p}_2$, then by T3 property $\mathbf{p}_1 = \mathbf{p}_2 + \mathbf{y}$, where $\mathbf{y} = [p_1^- - p_2^-, p_1^+ - p_2^+]$ and $p_1^- - p_2^- = p_1^+ - p_2^+$. Substituting in (8) we obtain

$$\begin{aligned}
[N_2] &= [p_1] - \mathrm{Dual}(\mathbf{p}_2) \\
&= \mathbf{p}_2 + \mathbf{y} - \mathrm{Dual}(\mathbf{p}_2) \\
&\overset{\mathrm{comm},(3)}{=} \mathbf{y} = [p_1^- - p_2^-, p_1^+ - p_2^+] = (7).
\end{aligned}$$

Having the resultant $[N_2]$ and its interpretation, from the interval equilibrium equation $[N_2] + [r] = 0$, we can compute the corresponding reaction. Thus, $[r] = -\mathrm{Dual}([N_2])$. The interpretation of $[r]$ in the initial space is similar to the one-load-model. Due to the equilibrium equation $\overrightarrow{r} + \overrightarrow{N}_k = 0$, $\overrightarrow{r}$ depends on $\overrightarrow{N}_k$, so that $\overrightarrow{r}$ has the same magnitude as $\overrightarrow{N}_k$ and opposite direction to that of $\overrightarrow{N}_k$ if the latter is determined ($0 \notin \mathbf{N}_k$).

Applying iteratively the last interval model of two loads and the computational procedure in $(\mathbb{KR}, +)$, basing on the algebraic properties of the interval structure $(\mathbb{KR}, +)$, we prove the following theorem.

Theorem 1 *Consider a bar subject to a finite number of loads that may be applied in opposite directions and have uncertain magnitude $p_1 \in \mathbf{p_1}, \ldots, p_k \in \mathbf{p_k}$, such that $\mathbf{p}_i \geq 0$, $i = 1, \ldots, k$. Assume that we have chosen a basic direction which (without loss of generality) coincides with the direction in which the first force is applied. Then,*

(i) for every j, $1 \leq j \leq k$, we have $[N_j] = \sum_{i=1}^{j} [p_i]$, wherein

$$[p_i] = \begin{cases} \mathbf{p}_i & \text{if the direction of } \overrightarrow{p}_i \text{ is the direction of } \overrightarrow{p}_1 \\ -\mathrm{Dual}(\mathbf{p}_i) & \text{if the direction of } \overrightarrow{p}_i \text{ is opposite to the direction of } \overrightarrow{p}_1, \end{cases}$$

and $[r] = -\mathrm{Dual}([N_k]) = -\mathrm{Dual}(\sum_{i=1}^{k} [p_i])$.

(ii) The interpretation of $[N_j] \in \mathbb{KR}$, $1 \leq j \leq k$, and similarly of $[r]$, is as follows.

- *If $0 \in int(pro([N_j]))$, then $\overrightarrow{N_j}$ may have positive or negative direction and its magnitude varies in $pro([N_j])$.*
- *If $0 \notin int(pro([N_j]))$, the magnitude of $\overrightarrow{N_j}$ varies in $Abs(pro([N_j]))$, while the direction of $\overrightarrow{N_j}$ coincides with the sign of $[N_j]$ (if $[N_j] \geq 0$ the direction of $\overrightarrow{N_j}$ is the chosen basic direction, otherwise it is opposite to the basic direction).*

The interval model of force equilibrium equation is essentially based on a chosen basic direction, similarly to the chosen coordinate system for the deterministic model. Changing the basic direction changes the representation of each force magnitude as proper/improper interval. However, the direction obtained for an unknown force is the same as the direction of this force in the deterministic model, unless the interval magnitude involves zero in the interior.

3.2 Numerical issues and examples

The model presented in Theorem 1 can be implemented in an environment supporting Kaucher interval arithmetic, e.g., the *Mathematica*® package `directed.m` [18], JInterval library [13], or any other software environment that allows implementing proper and improper intervals, operations and functions on them, for example, this may be the environment of C-XSC [5].

Example 1 We consider the example presented in Fig. 5 which illustrates in [3, section 5] the non-uniqueness of the canceling methodology proposed therein.

Fig. 5 Bar loaded with four forces after [3, Fig. 16]

Depending on which intervals are canceled, the methodology from [3] yields

$$\mathbf{N}_4^{'} = (\mathbf{p}_1 - \mathbf{p}_3) + \mathbf{p}_2 - \mathbf{p}_4 = 0 + [5, 10] - [15, 30] = [-25, 5]$$
$$\mathbf{N}_4^{''} = ((\mathbf{p}_1 + \mathbf{p}_2) - \mathbf{p}_3) - [10, 20] = 0 - [10, 20] = -[10, 20].$$

The interval model from Theorem 1 yields in a unique way

$$\begin{aligned}
[N_4] &= \mathbf{p}_1 + \mathbf{p}_2 + \mathrm{Dual}(-\mathbf{p}_3) + \mathrm{Dual}(-\mathbf{p}_4)\\
&= [N_2] + [-10, -20] + \mathrm{Dual}(-\mathbf{p}_4) \qquad (\text{where } [N_2] = [15, 30])\\
&= [N_3] + \mathrm{Dual}(-\mathbf{p}_4) = [5, 10] + [-15, -30] = -[20, 10],
\end{aligned}$$

showing that the resultant of the four loads has direction opposite to the direction of the first load and uncertain magnitude which varies in the interval $[10, 20]$.
Opposite to the discussion in [3, section 6], which illustrates what can happen if there is not a concept defining how to use the Dual operator, the present interval model and Theorem 1 are simple and without any ambiguity.

The following example demonstrates that choosing a basic direction (or considering the coordinate system) is essential for the interval model and that the model from section 3.1 is consistent with any choice of the basic direction (coordinate system).

Example 2 Consider the same problem as in Example 1. We choose a basic direction which is opposite to the direction of the first load shown in Fig. 5. Then, the application of Theorem 1 yields

$$\begin{aligned}[N_4] &= \mathrm{Dual}(-\mathbf{p}_1) + \mathrm{Dual}(-\mathbf{p}_2) + \mathbf{p}_3 + \mathbf{p}_4 \\ &= [-15, -30] + [10, 20] + \mathbf{p}_4 \\ &= [N_3] + \mathbf{p}_4 = [-5, -10] + [15, 30] = [10, 20],\end{aligned}$$

the interpretation of which gives the same result as in Example 1.
Similarly to the deterministic model, if we do not have a basic direction (coordinate system) we would not know where to apply the Dual operator.

Due to various sources of uncertainty two forces may have almost the same magnitude but be represented by intervals that are not equal. For example, $\mathbf{p}_1 = [7.0359, 18.5678]$, $\mathbf{p}_2 = [7.0324, 18.5631]$. If $\overrightarrow{p}_1$ and $\overrightarrow{p}_2$ have opposite directions, the new interval operation and the computational procedure proposed in [3] will fail to deliver a tight interval for the sum of these forces, while the model of Theorem 1 gives the tightest interval enclosure. Beside representing uncertainty in the model parameters, interval arithmetic (classical and generalized) accounts for both the input/output and computational round-off errors via appropriate directed rounding of the interval end-points. For implementation of Kaucher arithmetic in floating point environment see [19]. Thus, the model of Theorem 1 applied in a software environment supporting correctly implemented computer interval arithmetic will account for all the uncertainties that arise.

3.3 Interval forces in IEEE 1788™-2015 environment

As we presented in Section 2.3, the set $\mathbb{KR}$ of proper and improper intervals can be considered as a set of "directed" intervals with the representation $(\mathbf{p}, \alpha_p)$. Since the two interval structures $(\mathbb{IR}, +, +^-, \times, \times^-)$ and $(\mathbb{KR}, +, \times)$ are closely related, we obtain the following theorem.

Theorem 2 *Consider a bar subject to k loads that may be applied in opposite directions and have uncertain magnitude $p_1 \in \mathbf{p_1}, \ldots, p_k \in \mathbf{p_k}$, $\mathbf{p}_i \geq 0$, $i = 1, \ldots, k$. Assume that we have chosen a basic direction $\alpha_x \in \{+, -\}$. Then, for every j, $1 \leq j \leq k$, we have $(\mathbf{N}_j, \alpha_{N_j}) = \sum_{i=1}^{j} (\mathbf{a}_i, \alpha_i)$, wherein*

$$(\mathbf{a}_i, \alpha_i) = \begin{cases} (\mathbf{p}_i, +) & \text{if } \alpha_x \text{ is the direction of } \overrightarrow{p}_i \\ (-\mathbf{p}_i, -) & \text{if } \alpha_x \text{ is opposite to the direction of } \overrightarrow{p}_i, \end{cases}$$

and the summation is defined by (4)–(6). Also, $(\mathbf{r}, \alpha_r) = (-\mathbf{N}_k, -\alpha_{N_k})$*. The magnitude of* $\overrightarrow{N}_j$ *is* $\mathbf{N}_j$*, the direction of* $\overrightarrow{N}_j$ *is determined by the sign of* $\mathbf{N}_j$*. The magnitude of* $\overrightarrow{r}$ *is equal to* $\mathbf{N}_k$*, while the direction of* $\overrightarrow{r}$ *is opposite to that of* $\overrightarrow{N}_k$*.*

Proof The proof follows from Theorem 1 and the transition formulae between $(\mathbb{KR}, +)$ and $(\mathbb{IR}, +, +^-)$, see [2].

It follows from Theorem 1 and the properties of the corresponding interval arithmetic that computing a sum of k forces, having magnitude $\mathbf{p}_i \in \mathbb{IR}$, $1 \le i \le k$, such that some of them ($i \in K_1$) have positive direction and other ($i \in K_2$, $\mathrm{Card}(K_1)+\mathrm{Card}(K_2) = k$) have negative direction can be equivalently computed by

$$\left(\sum_{i\in K_1} \mathbf{p}_i\right) + (-1)\mathrm{Dual}(\sum_{i\in K_2} \mathbf{p}_i),$$

or (basing on Theorem 2) by

$$\left(\sum_{i\in K_1} \mathbf{p}_i\right) +^- (-1)\left(\sum_{i\in K_2} \mathbf{p}_i\right).$$

In both formulae above, the summation in the brackets is summation between classical proper intervals. $+^-$ denotes the so-called inner addition between proper intervals defined by (5).

While there are not so many implementations of Kaucher interval arithmetic, the recently approved IEEE standard 1788$^{\mathrm{TM}}$-2015 for interval arithmetic [6] aims at improving the availability of reliable computing in modern hardware and software environments. The standard requires two interval functions called `cancelPlus` and `cancelMinus` intended to facilitate the implementation of Kaucher interval arithmetic. For two intervals $\mathbf{x} = [x^-, x^+]$, $\mathbf{y} = [y^-, y^+] \in \mathbb{IR}$, $\texttt{cancelMinus}(\mathbf{x}, \mathbf{y}) = \mathbf{z}$ is defined in [6] if and only if $\omega(x) \ge \omega(y)$ by $\mathbf{z} = [x^- - y^-, x^+ - y^+]$. $\texttt{cancelPlus}(\mathbf{x}, \mathbf{y}) = \texttt{cancelMinus}(\mathbf{x}, -\mathbf{y})$. Comparing these definitions to (5) it is obvious that IEEE 1788$^{\mathrm{TM}}$-2015 functions `cancelPlus` and `cancelMinus` provide only half of the operations inner addition/subtraction. In order to be able to simulate the operations addition/subtraction between directed (or Kaucher) intervals, and respectively, the operations inner addition/subtraction between proper intervals, in a software environment supporting IEEE 1788$^{\mathrm{TM}}$-2015 standard, we present the following proposition.

Proposition 1 *Let* $\texttt{cancelPlus}(\mathbf{x}, \mathbf{y})$ *be the IEEE 1788*$^{\mathrm{TM}}$*-2015 function. Then for* $(\mathbf{x}, \alpha_x)$*,* $(\mathbf{y}, \alpha_y)$*,* $\alpha_x, \alpha_y \in \{+, -\}$ *we have*

$$(\mathbf{x}, \alpha_x) + (\mathbf{y}, \alpha_y) = \begin{cases} (\mathbf{x} + \mathbf{y}, \alpha_x) & \text{if } \alpha_x = \alpha_y, \\ (\texttt{cancelPlus}(\mathbf{x}, \mathbf{y}), \alpha_x) & \text{if } \alpha_x \ne \alpha_y, \omega(\mathbf{x}) \ge \omega(\mathbf{y}), \\ (\texttt{cancelPlus}(\mathbf{y}, \mathbf{x}), \alpha_y) & \text{if } \alpha_x \ne \alpha_y, \omega(\mathbf{x}) < \omega(\mathbf{y}). \end{cases}$$

Similarly for `cancelMinus`$(\mathbf{x}, \mathbf{y})$ *if in the above formula* $+$ *is replaced by* $-$ *and* `Plus` *is replaced by* `Minus`*.*

Operation inner addition corresponds to the two cases $\alpha_x \neq \alpha_y$ in the formula of Proposition 1. Namely, for $\mathbf{x}, \mathbf{y} \in \mathbb{IR}$

$$\mathbf{x} +^{-} \mathbf{y} = \begin{cases} \texttt{cancelPlus}(\mathbf{x}, \mathbf{y}) & \text{if } \omega(\mathbf{x}) \geq \omega(\mathbf{y}), \\ \texttt{cancelPlus}(\mathbf{y}, \mathbf{x}) & \text{if } \omega(\mathbf{x}) < \omega(\mathbf{y}). \end{cases}$$

So far, IEEE Std. 1788$^{\mathrm{TM}}$-2015 has two prototype implementations: a C++ class library [14] and an interval package for GNU Octave (GNU Octave is a free Matlab replacement) [4]. One can use Proposition 1 to simulate addition/subtraction between Kaucher intervals in any of its two forms (normal or directed), as well as the operations inner addition/subtraction, in an environment supporting IEEE Std. 1788$^{\mathrm{TM}}$-2015. Therefore, the interval model of force equilibrium can be implemented in the above two software environments, too.

Comparing the formula for addition of two generalized intervals $[a], [b] \in \mathbb{KR}$, $[a] + [b] = [a^- + b^-, a^+ + b^+]$, to the formula in Proposition 1, it becomes obvious that simulating the former operation in an environment supporting IEEE Std. 1788$^{\mathrm{TM}}$-2015 is possible but will be highly inefficient. The same is true for the simulation of the inner operations addition and subtraction. How to provide a credible implementation of inner interval operations in floating-point is discussed in [17].

In [15] Neumaier writes about the algebraic extensions of classical interval arithmetic "Therefore, the improved algebraic properties are difficult to exploit in real applications. The practically useful properties of Kaucher arithmetic all arise from their modal semantics, not from the fact that one has these two groups around. Thus the group properties are nice but seem to be nearly irrelevant in applications." The proposed model of interval equilibrium equations refutes this claim. Thus, the set of real-life success stories for the algebraic extensions of classical interval arithmetic (which includes inward rounding of interval operations and inner estimation of solution sets, models with quantified interval parameters, exact range of monotone rational functions with multi-incident interval variables, to name a few, see also [15]) is expanded by the proposed model of interval force equilibrium equation and its many applications in various models with uncertain (interval) data in mechanics, robotics, etc.

It follows from the above consideration that a straightforward implementation of Kaucher arithmetic is much more efficient than its simulation in an environment supporting the new IEEE 1788$^{\mathrm{TM}}$-2015 standard. On the other hand, classical interval arithmetic is a special case (projection) of Kaucher arithmetic. Therefore, a future revision of IEEE Std. 1788$^{\mathrm{TM}}$-2015 should take this into account.

4 Further generalization and applications

In two- and three-dimensional problems involving several forces, the determination of their resultant $\overrightarrow{N}$ is best carried out by first resolving each force into rectangular components, [1]. Then, the one-dimensional interval model of Theorem 1 can be applied for obtaining each component of the resultant force.

Example 3 Four forces act on a particle as shown in Fig. 6. Determine the resultant of the forces on the particle if $F_1 \in [129, 131]N$, $F_2 \in [59, 61]N$, $F_3 \in [109, 111]N$, $F_4 \in [99, 101]N$ and the angles in Fig. 6 are $\theta_1 = 30°$, $\theta_2 = 20°$, $\theta_3 = 15°$.

Fig. 6 Four forces acting on a particle

With the coordinate system shown in Fig. 6, the x and y components of each force are determined by trigonometry as follows:

$$
\begin{aligned}
F_{1,x} &\in \mathbf{F}_{1,x} = [129, 131]\cos(30°) = [129\sqrt{3}/2, 131\sqrt{3}/2]N, \\
F_{1,y} &\in \mathbf{F}_{1,y} = [129, 131]\sin(30°) = [129/2, 131/2]N, \\
-F_{2,x} &\in -\mathbf{F}_{2,x} = -[59, 61]\sin(20°) = -[20.1791, 20.8633]N, \\
F_{2,y} &\in \mathbf{F}_{2,y} = [59, 61]\cos(20°) = [55.4418, 57.3213]N, \\
F_{3,x} &\in \mathbf{F}_{3,x} = 0N, \qquad -F_{3,y} \in -\mathbf{F}_{3,y} = -[109, 111]N, \\
F_{4,x} &\in \mathbf{F}_{4,x} = [99, 101]\cos(15°) = [\frac{99(1+\sqrt{3})}{2\sqrt{2}}, \frac{101(1+\sqrt{3})}{2\sqrt{2}}]N, \\
-F_{4,y} &\in -\mathbf{F}_{4,y} = -[999, 131]\sin(15°) = -[\frac{99(-1+\sqrt{3})}{2\sqrt{2}}, \frac{101(-1+\sqrt{3})}{2\sqrt{2}}]N.
\end{aligned}
$$

Since the angles are assumed to be accurate, we represented the intervals $\mathbf{F}_{i,x}$, $\mathbf{F}_{i,y}$, $i = 1, 3, 4$, with rational endpoints. However, since $\sin(20°)$, $\cos(20°)$

do not have explicit rational representation, floating point intervals are introduced to account for the round-off errors in the representation of $\sin(20°)$ and $\cos(20°)$.

We apply Theorem 1 in computing the x, y components of the resultant force

$$[N_x] = \mathbf{F}_{1,x} + \mathrm{Dual}(-\mathbf{F}_{2,x}) + 0 + \mathbf{F}_{4,x} = [187.1647, 190.1447]N,$$
$$[N_y] = \mathbf{F}_{1,y} + \mathbf{F}_{2,y} + \mathrm{Dual}(-\mathbf{F}_{3,y}) + \mathrm{Dual}(-\mathbf{F}_{4,y}) = -[14.3194, 14.6813]N.$$

Interpreting the generalized intervals $[N_x]$, $[N_y]$, we obtain that the resultant of the four forces is

$$\overrightarrow{N} = N_x \overrightarrow{i} - N_y \overrightarrow{j},$$

wherein $N_x \in [187.1647, 190.1447]N$, $N_y \in [14.3194, 14.6813]N$. The magnitude and direction of $\overrightarrow{N}$ can be determined from the triangle shown in Fig. 7.

Fig. 7 The resultant force in Example 3 and its x, y components

$$\tan(\theta) \in \frac{\mathbf{N}_y}{\mathbf{N}_x}, \implies \theta \in [4.3067, 4.4852]°$$

and $N = \frac{N_y}{\sin(\theta)} \in [183.1137, 195.5002]N$.

The above rule for deterministic addition of several forces also applies to the addition of other vector quantities in mechanics, such as velocities, accelerations, or momenta. The next example is similar to the examples considered in [11] and illustrates that the proposed interval model accounts for the change in the direction of the resultant force.

Example 4 Consider a box, with mass m, resting on a horizontal plane. Assume that there is no friction force. Let three robotic arms apply uncertain forces on the box as shown in Fig. 8. Determine: the resultant force applied on the box; does the box move and in what direction; if the box moves, how much is the acceleration when a) $F_3 = 300N$, b) $F_3 = 550N$, c) $F_3 = 1200N$ and all forces are measured with 5% uncertainties.

Applying Theorem 1, with basic direction which is the same as the direction of $\overrightarrow{F}_1$, we obtain the following resultant force.

a) $[N_a] = [142.5, 157.5] - \mathrm{Dual}([665, 735]) + [285, 315] = [-237.5, -262.5]N$. The interpretation of this generalized interval gives a resultant force with direction opposite to the basic direction and magnitude $[237.5, 262.5]N$, which implies that the box moves in the direction of $\overrightarrow{F}_3$.

Fig. 8 Three robotic arms acting on a box which rests on a horizontal plane

b) $[N_b] = [142.5, 157.5] - \text{Dual}([665, 735]) + [522.5, 577.5] = [0, 0]N$, which means that the system is in equilibrium and the box does not move.
c) $[N_c] = [142.5, 157.5] - \text{Dual}([665, 735]) + [1140, 1260] = [617.5, 682.5]N$. The interpretation of the last generalized interval gives a resultant force with the same direction as that of $\overrightarrow{F}_1$ and magnitude $[617.5, 682.5]N$, which implies that the box moves in the direction of $\overrightarrow{F}_1$.

We apply Newton's second law, $\sum \overrightarrow{F} = m\overrightarrow{a}$, to determine an object's acceleration $\overrightarrow{a}$ if we know its mass m and the sum of the forces that other objects exert on it. We use the inverse with respect to multiplication operation (3) and obtain for the magnitude of the corresponding acceleration $\mathbf{a} = \text{pro}(\mathbf{N}/\text{Dual}(\mathbf{m}))\ m/s^2$.

More complicated practical examples involving linear equilibrium equations are presented in [16] where the one- dimensional model proposed here is generalized and the applications are discussed with many details; in particular, [16, Section 5] presents an example of statically indeterminate structure. Note, however, that the more complicated examples require the four arithmetic operations on generalized intervals[3], properties (3) and the conditionally distributive relations in $\mathbb{KR}$.

Interval models of equilibrium equations defined in Kaucher interval arithmetic or in directed interval arithmetic can be computed in any environment that supports or allows emulation of any of the two interval arithmetic structures. In this, the results obtained by the two approaches will be identical up to the round-off errors.

5 Conclusion

In this paper we showed that interval arithmetic, via its algebraic extensions, is completely capable to provide straightforward and efficient interval models of physical phenomena and processes whose deterministic setting (model) is based on algebraic properties of the arithmetic operations between real numbers. In several papers, cf. [9] and others, S. Markov makes analogy between the completion of the set of nonnegative real numbers by the negative ones

[3] The IEEE interval Std. 1788 does not prescribe functionality for simulating multiplication/division of generalized (Kaucher) intervals.

and the algebraic completions of classical interval arithmetic. Markov's articles show that the "invention" of new interval operations, if successful, would lead to some of the already proposed interval algebraic extensions. Therefore the new interval operation proposed in [3] is only partially successful.

The proposed interval model of force equilibrium equations opens up an array of new possibilities for efficient handling of interval uncertainties in models involving linear equilibrium equations. Static equilibrium is fundamental for many disciplines and appears as part of a diversity of complicated models. The proposed interval approach allows accounting for the dependencies between interval parameters from the very beginning of modeling process and efficient reduction of the interval overestimation that may arise.

Although intervals allow the simplest way of representing uncertainties, their correct and efficient application in models requires deep understanding of classical interval arithmetic and its algebraic extensions — properties of the algebraic structure and various possibilities for their interpretation and application with respect to the particular physical problem under consideration.

References

1. Beer FP, Johnston ER, Mazurek DF, Cornwell PJ, Eisenberg ER (2010) Vector Mechanics for Engineers: Statics and Dynamics, 9th edition, McGraw-Hill
2. Dimitrova N, Markov SM, Popova ED (1992) Extended interval arithmetics: new results and applications. In: Computer Arithmetic and Enclosure Methods (Eds L. Atanassova, J. Herzberger), North-Holland, 225-234.
3. Elishakoff I, Gabriele S, Wang Y (2015) Generalized Galileo Galilei problem in interval setting for functionally related loads. Archive of Applied Mechanics 86(7):1203-1217.
4. Heimlich O (2015) GNU Octave Interval Package, version 1.1.0, release 2015-08-03, http://octave.sourceforge.net/interval/. Accessed 25 January 2016
5. Hofschuster W, Krämer W (2004) C-XSC 2.0: A C++ library for extended scientific computing. In: Lecture Notes in Computer Science 2991, Springer-Verlag, Heidelberg, pp 15-35
6. IEEE Standard Association (2015) IEEE Std 1788$^{\mathrm{TM}}$-2015, IEEE Standard for Interval Arithmetic. IEEE Computer Society
7. Kaucher E (1980) Interval analysis in the extended interval space IR. Computing Suppl. 2:33-49. http://www.math.bas.bg/~epopova/Kaucher-80-CS_33-49.pdf Accessed 25 January 2016
8. Markov SM (1977) Extended interval arithmetic. Compt. rend. Acad. bulg. Sci. 30(9):1239-1242
9. Markov SM (1995) On directed interval arithmetic and its applications. Journal of Universal Computer Science 1(7):514-526
10. Markov SM (1996) On the foundations of interval arithmetic. In: Alefeld G, Frommer A, Lang B (eds) Scientific Computing and Validated Numerics, Academie Verlag, pp 307-313
11. Mazandarani M (2015) IEEE standard 1788-2015 vs. multidimensional RDM interval arithmetic, posting to IEEE P1788 working group. http://grouper.ieee.org/groups/1788/email/msg08439.html Accessed 25 January 2016
12. Moore RE, Kearfott RB, Cloud MJ (2009) Introduction to Interval Analysis, SIAM Press, Philadelphia
13. Nadezhin DY, Zhilin SI (2014) JInterval library: principles, development, and perspectives, Reliable Computing 19:229-247.
14. Nehmeier M (2014) libieeep1788: A C++ Implementation of the IEEE interval standard P1788. In: Norbert Wiener in the 21st Century (21CW), 2014 IEEE Conference on, June 2014, pages 1-6. doi:10.1109/NORBERT.2014.6893854

15. Neumaier A (2009) Computer graphics, linear interpolation, and nonstandard intervals. Manuscript, Univarsität Wien. http://www.mat.univie.ac.at/~neum/ms/nonstandard.pdf. Accessed 25 January 2016
16. Popova ED (2016) Interval Model of Equilibrium Equations in Mechanics, in S. Freitag, R. L. Muhanna, R. L. Mullen (Eds) Proceedings of REC'2016, Ruhr University Bochum, pp. 241-255. http://rec2016.rub.de/downloads/rec2016_proceedings.pdf
17. Popova ED, Markov SM (1997) Towards credible implementation of inner interval operations. In: A. Sydow (Ed.), 15th IMACS World Congress on Scientific Computation, Modelling and Applied Mathematics. Volume 2 Numerical Mathematics, W&T Verlag, pp 371-376. http://www.math.bas.bg/~epopova/papers/imacs97r.pdf Accessed 25 January 2016
18. Popova ED, Ullrich CP (1996) Directed interval arithmetic in Mathematica. Implementation and applications. Technical Report 96-3, Universität Basel, Switzerland. http://www.math.bas.bg/~epopova/papers/tr96-3.pdf. Accessed 25 January 2016
19. Popova ED, Ullrich CP (1997) Generalizing BIAS Specification. Journal of Universal Computer Science 3(1):23-41
20. Sainz MA, Armengol J, Calm R, Herrero P, Jorba L, Vehi J (2014) Modal Interval Analysis: New Tools for Numerical Information, Lecture Notes in Mathematics 2091, Springer