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Novel Interval Model Applied to Derived Variables in Static and Structural Problems

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Abstract In this work we further develop a newly proposed interval algebraic approach for analysis or design of structures involving uncertain interval-valued parameters. The methodology is based on an algebraic extension of classical interval arithmetic, namely Kaucher arithmetic, and within it the interval equilibrium equations can be completely satisfied by the primary unknown variables (displacements). Here this method is expanded to derived (secondary) variables – forces, strains and stresses which are of particular practical interest in design. Numerical examples are presented to illustrate the proposed methodology and to compare the algebraic interval approach to that based on classical interval arithmetics.

Keywords uncertainty · interval arithmetic · finite element models · forces · strains · stresses

1 Introduction

Considering uncertainty is inevitable in a realistic analysis or design of engineering structures. Uncertainty analysis is conducted by one of three alternative approaches: probability theory, fuzzy sets theory, and analysis based on convex sets (e.g., interval analysis or ellipsoidal analysis), depending on the nature of the uncertainty considered. While the first two approaches introduce

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some measure of belonging to the set of variation, the third approach does not. A somewhat more general treatment of uncertainty, for generic convex sets, is given in [2] and [4] considers a connection between the two non-probabilistic treatments. Interval analysis influences fuzzy sets theory since the computation of fuzzy quantities by α-cuts requires interval computations.

The uncertain model parameters are represented by interval-valued parameters and interval analysis is applied when the available data do not allow applying either the probabilistic or fuzzy approach. Interval analysis is exposed in various monographs [1], [12], [13], [24], for example. Interval methods are applied for studying variations of structural responses due to fluctuations in structural parameters and loads for more than twenty years now. Most of the interval models considered so far are based on classical interval analysis [1], [12], [13]. Within this setting a lot of effort is put and a variety of special methods are proposed aiming at eliminating the interval dependency problem and obtaining sharp bounds for the unknowns. For an overview of some interval techniques applied to interval finite element models in applied mechanics see, e.g., [11] and the references in [23]. However, even the exact ranges of the unknowns within the models based on classical interval arithmetic may violate some physical laws of the deterministic model. Therefore, some recent works [5], [22], [25], for example, emphasize the need for interval analysis in engineering context.

In [5], [18] the focus is on a mathematical model which is more precise than any model based on classical interval arithmetic. It relates the dependency of interval quantities to the physics of the problem being considered, e.g., linear equilibrium equations. The new model (called algebraic) represents any vector (in geometric sense) model parameter (possessing magnitude and direction) by a directed interval (range + direction) and requires that all kinds of linear equilibrium equations be completely satisfied. Thus, the new interval model is embedded in an isomorphic algebraic extension of classical interval arithmetic which is known under a variety of names: Kaucher arithmetic [9], modal arithmetic [24], directed arithmetic [10], generalized (proper and improper) intervals, reflecting some of its aspects. The generalized interval arithmetic structure possesses group properties with respect to the operation addition and the operation multiplication of intervals not involving zero. This allows the newly proposed interval algebraic model [18], [19] to resemble the corresponding deterministic model, to be applied straightforward to the latter, as well as to yield exact bounds for the unknowns without any overestimation.

While sharp interval enclosure of the primary variables (as displacements in finite element models of structures) is achievable within the classical interval setting with more or less computational effort, the enclosures of secondary (derived) quantities as stresses and strains, considered as functions of the displacements, presents a big challenge for the classical interval model. Due to the dependency, the derived variables are obtained with considerable overestimation, whose reduction requires special approaches [23]. On the other hand, internal forces, strains and stresses are basic characteristics in strength of materials with numerous applications in engineering practice. Their sharp estimation is of particular interest and importance for the design procedures. All these motivated us for the current work.

In this work we extend the interval algebraic approach by expanding it to strength of material problems, in particular for bounding the uncertainties in axial forces, strains and stresses of truss elements considered as functions of the primary obtained variations of the displacements. Our present work continues [20] and provides a general methodology for obtaining exact bounds of axial forces, strains and stresses without extensive computational effort. As in the previous works, here the results obtained by the interval algebraic approach are compared theoretically and by numerical results to various interval models of truss structures based on classical interval arithmetic.

The paper has the following structure. Section 2 contains some basic properties of Kaucher interval arithmetic which are indispensable for the theoretical considerations in this section. Section 2.2 contains a brief overview of the two interval models of mechanical structures — classical and algebraic, and presents the main theoretical contribution of the paper — methodology for obtaining element axial forces, strains and stresses within the algebraic interval model. In Section 3 we consider a simple example of a typical strength of materials problem and compare the two interval models on it. On a small and a larger benchmark examples of truss structures in Section 4 we illustrate the application of the newly proposed methodology. The work ends by some concluding remarks.

2 Theoretical Background

2.1 Algebraic completion of classical interval arithmetic

We assume that the reader is familiar with classical interval arithmetic, [1], [12], [13], and its properties.

The set of classical compact intervals $\mathbb{IR} = \{[a^-, a^+] \mid a^-, a^+ \in \mathbb{R}, a^- \leq a^+\}$, called also *proper* intervals, is extended in [9] by the set $\overline{\mathbb{IR}} := \{[a^-, a^+] \mid a^-, a^+ \in \mathbb{R}, a^- \geq a^+\}$ of *improper* intervals obtaining thus the set $\mathbb{KR} = \mathbb{IR} \bigcup \overline{\mathbb{IR}} = \{[a^-, a^+] \mid a^-, a^+ \in \mathbb{R}\}$ of all ordered couples of real numbers called *generalized* (extended or Kaucher) intervals. For a better understanding we denote the classical intervals by bold face letters (e.g., $\mathbf{a}$) and the intervals from $\mathbb{KR}$ by brackets (e.g., $[a]$). For $\mathbf{a} \in \mathbb{IR} \subset \mathbb{KR}$ the assignment $[b] = \mathbf{a} \in \mathbb{KR}$ is a correct one. The inclusion order relation ($\subseteq$) between classical intervals is generalized for $[a], [b] \in \mathbb{KR}$. For $[a] = [a^-, a^+] \in \mathbb{KR}$, define a binary variable *direction* (τ) by

$$\tau([a]) := \begin{cases} + \text{ if } a^- \leq a^+, \\ - \text{ if } a^- > a^+. \end{cases}$$

All elements of $\mathbb{KR}$ with positive direction are called proper intervals and the elements with negative direction are called improper intervals. An element-to-element symmetry between proper and improper intervals is expressed by the "dual" operator. For $[a] = [a^-, a^+] \in \mathbb{KR}$, $\mathrm{dual}([a]) := [a^+, a^-]$. For $[a], [b] \in \mathbb{KR}$,

$$\mathrm{dual}(\mathrm{dual}([a])) = [a], \tag{1}$$

$$\mathrm{dual}([a] \circ [b]) = \mathrm{dual}([a]) \circ \mathrm{dual}([b]), \ \circ \in \{+, -, \times, /\}. \tag{2}$$

Define proper projection of a generalized interval $[a]$ onto $\mathbb{IR}$ by

$$\mathrm{pro}([a]) := \begin{cases} [a] & \text{if } \tau([a]) = +, \\ \mathrm{dual}([a]) & \text{if } \tau([a]) = -. \end{cases}$$

Denote $\mathcal{T} := \{[a] \in \mathbb{KR} \mid a^- a^+ < 0\}$ and $\mathcal{Z} := \{\mathbf{a} \in \mathbb{IR} \mid a^- a^+ < 0\}$. Define the sign functional, $\lambda := \mathrm{sign} : \mathbb{KR} \setminus \mathcal{Z} \to \{+, -\}$ by

$$\lambda([a]) := \begin{cases} + \text{ if } \mathrm{pro}([a])^- \geq 0, \\ - \text{ otherwise.} \end{cases}$$

Note that usually the sign functional is denoted by σ. Here we change the notation so that there is no confusion with the notation for the stress. The conventional interval arithmetic and lattice operations, as well as other interval functions are isomorphically extended onto the whole set $\mathbb{KR}$, cf. [9]. The generalized interval arithmetic structure possesses group properties with respect to the operations of addition and multiplication. For $[a] \in \mathbb{KR}$, $[b] \in \mathbb{KR} \setminus \mathcal{T}$,

$$[a] - \mathrm{dual}([a]) = 0, \qquad [b]/\mathrm{dual}([b]) = 1. \tag{3}$$

The complete set of conditionally distributive relations for multiplication and addition of generalized intervals can be found in [16]. Here we present only one that is usually used. For $[a], [b], [s] = ([a] + [b]) \in \mathbb{KR} \setminus \mathcal{T}$, $[c] \in \mathbb{KR}$

$$([a] + [b])[c]_{\lambda([s])} = [a] \times [c]_{\lambda([a])} + [b] \times [c]_{\lambda([b])}, \tag{4}$$

wherein $[a]_+ = [a]$, $[a]_- = \mathrm{dual}([a])$. In what follows we will often use the subscript notation of the dual operator. Multiplication of two binary variables $\mu, \nu \in \{+, -\}$ is defined by $\mu\nu = ++ = -- = +$, $\mu\nu = +- = -+ = -$. Matrices of binary variables of same dimension are "multiplied" *componentwise* using the rules for multiplication of binary variables. The rules for multiplication by binary-valued matrices are further extended for dualization, cf. [10], [20, Section 2.3]. Some other properties interpretations and applications of generalized interval arithmetic can be found in [9], [10], [24], or the references given on the web site [15].

2.2 Two interval models of truss structures

2.2.1 Classical interval model

We consider statically indeterminate truss structures, where the applied loads (P), some of the structural parameters (modulus of elasticity E_i, cross sectional area A_i, or/and length L_i) of the bars (i) are uncertain and vary within given intervals. The traditional finite element method (FEM) for truss structures relates the force equilibrium equations to the displacements of the nodes and the stiffness of the bars E_iA_i/L_i, leading to a parametric linear system

$$K(E_i, A_i, L_i)u = f(q), \tag{5}$$

where $K(E_i, A_i, L_i)$ is the stiffness matrix depending on the structural parameters for each element, $f(q)$ is the load vector and u is the displacement vector.

The *classical interval model* of an uncertain truss structure with interval valued parameters considers the corresponding parametric interval linear system. The smallest interval vector enclosing the so-called united solution set, defined by

$$\Sigma^p := \{u \in \mathbb{R}^n \mid (\exists E_i \in \mathbf{E}_i, \exists A_i \in \mathbf{A}_i, \exists L_i \in \mathbf{L}_i, \exists q \in \mathbf{q}) : K(E_i, A_i, L_i)u = f(q)\}$$

is identified as the best solution. The smallest interval vector enclosing Σ^p is called interval hull of the solution set and is defined for bounded sets by

$$\Box\Sigma^p := \{\bigcap \mathbf{u} \in \mathbb{IR}^n \mid \Sigma^p \subseteq \mathbf{u}\}.$$

The engineering and interval literature is abundant with numerical methods dedicated to finding a sharp (and possibly guaranteed) interval enclosure of the interval hull of a parametric united solution set to parametric interval linear system; the latter may be resulted from FEM applied to structures with interval parameters, cf., for example, [11], [6], [17], [3], [8] and the references given therein.

While sharp interval enclosure of the primary variables (displacements in FE models of structures) is achievable within the classical interval setting with more or less computational effort, the enclosures of secondary (derived) quantities as stresses and strains, which are functions of the displacements, presents a big challenge for the classical interval model [23]. Due to the dependency phenomenon, the derived variables are obtained with considerable overestimation, whose reduction requires special approaches [23]. We show below that all the drawbacks of the models based on classical interval arithmetic vanish when considering an interval model embedded in the generalized interval space $\{\mathbb{KR}, +, \times, \subseteq\}$.

2.2.2 Algebraic interval model

The interval FEM models based on classical interval arithmetic do not conform to the physics of equilibrium equations because replacing the obtained intervals in the equilibrium equations does not result in true equality, cf. [5]. A new approach is proposed in [5] and further elaborated in [18], [19], [20] to interval model of linear equilibrium equations. Since the new model is based on the algebraic completion of classical interval arithmetic, it is called *algebraic interval model*.

Assume that there is a deterministic model described by some equilibrium equation(s) that involve uncertain vector (in geometric sense) parameters varying within given proper intervals. Clearly, the unknowns in this model will be also uncertain and we search for proper intervals that are the sharpest interval enclosures of these unknowns and that conform to the physics of the problem (statics or dynamic equilibrium). Conformance to the equilibrium means that the intervals found for the unknowns when replaced in the equation(s) and all operations are performed results in true equality(ies). Such a solution is called formal (algebraic) solution.

Interval algebraic solutions do not exist in general in classical interval arithmetic, cf. [10]. Generalized interval arithmetic $(\mathbb{KR}, +, \times, \subseteq)$ is the natural one for finding algebraic solutions to interval equations since it is obtained from the arithmetic for classical intervals via an algebraic completion. Therefore, the newly proposed algebraic interval model embeds the initial problem formulation in the interval space $(\mathbb{KR}, +, \times, \subseteq)$, finds an algebraic solution and interprets the obtained generalized intervals back in the initial interval space $\mathbb{IR}$. This is a three step procedure summarized below.

1. The **representation convention** for vector physical quantities (in geometric sense, having magnitude and direction, e.g., forces, momenta, etc.):
   – if a scalar force component F_x (F_y, F_z) has the same direction as the positive x (y, z) coordinate axis, it is represented by proper interval $\mathbf{F}_x$ ($\mathbf{F}_y$, $\mathbf{F}_z$);
   – if a scalar force component F_x (F_y, F_z) has opposite direction to the positive x (y, z) coordinate axis, it is represented by improper interval $\mathrm{dual}(\mathbf{F}_x)$ $(\mathrm{dual}(\mathbf{F}_y), \mathrm{dual}(\mathbf{F}_z))$.
2. **Computing**. Find the formal (algebraic) solution for the unknown(s) in $(\mathbb{KR}, +, \times, \subseteq)$. For small systems, equivalent algebraic transformations can be applied as in [18], [19]. Efficient numerical method for large-scale problems is proposed in [20].
3. **Interpretation** of the obtained generalized intervals is in the initial space $\mathbb{IR}$ according to the physics of the unknowns, projecting the generalized intervals on $\mathbb{IR}$.

Following the above general methodology, below we further develop it for the problems of strength of materials. More precisely, we present in detail the methodology for obtaining secondary (derived) variables such as forces, stresses and strains of the conventional displacement FEM from the primary variables (displacements) in the algebraic interval model. For simplicity, the presentation here will be based on the truss model. When a structure is modeled as being composed of unimodal components, the equations are identical in the form of those of a truss, [7]. This implies that many techniques, including the present general methodology, and properties applicable to trusses can migrate to frame analysis.

The standard displacement method of truss analysis leads to a set of linear equations (5). The construction of this system starts from two sets of equations.

I. The force equilibrium equation at every node i

$$\overline{Q}_i + \sum_j \overline{F}_{ij} \;=\; 0, \tag{6}$$

where $\overline{Q}_i$ is the external load at node i, $\overline{F}_{ij}$ is the reaction force exerted by the bar (ij) at node i. In order to simplify the notations, overlines of the vector quantities (in geometrical sense) will be omitted.

II. The equations relating magnitudes of the element axial forces to the displacements at the nodes

$$F_k = F_{ij} = g_k\left((u_j^x - u_i^x)c_{ij} + (u_j^y - u_i^y)s_{ij}\right) \;=\; g_k t_k^\top . u_k, \tag{7}$$

where $g_k = A_k E_k / L_k$ is the stiffness of the k-th element, $u_k = (u_i^x, u_i^y, u_j^x, u_j^y)^\top$ is the vector of the displacements in the nodes i, j and $t_k = (-1, 1).T_k$ is a numerical vector, $T_k = \begin{pmatrix} c_k & s_k & 0 & 0 \\ 0 & 0 & c_k & s_k \end{pmatrix}$, $c_k = c_{ij} = \cos(\measuredangle_k)$, $s_k = s_{ij} = \sin(\measuredangle_k)$.

Following the representation convention, the deterministic force equilibrium equations in x, y coordinates (below, left) are transformed to interval algebraic equations (below, right) in $\mathbb{KR}$

$$\begin{matrix} Q_i^x + \sum_k c_k F_k \;=\; 0 \\ Q_i^y + \sum_k s_k F_k \;=\; 0 \end{matrix} \qquad \Longrightarrow \qquad \begin{matrix} (\mathbf{Q}_i^x)_{\lambda(Q_i^x)} + \sum_j c_k [F_k]_{\lambda(c_k)} = 0 \\ (\mathbf{Q}_i^y)_{\lambda(Q_i^y)} + \sum_j s_k [F_k]_{\lambda(s_k)} = 0 \end{matrix} \tag{8}$$

Since (7) involves summation of differently directed vector parameters (displacements), according to the representation convention the algebraic interval model of the element axial forces is

$$[F_k] = (\mathbf{g}_k)_{\lambda([\delta_k])}(t_k^\top .(\mathbf{u}_k)_{\lambda(u_k)\lambda(t_k)}), \qquad [\delta_k] := t_k^\top .(\mathbf{u}_k)_{\lambda(u_k)\lambda(t_k)}. \tag{9}$$

It was proven in [20] that if the interval equilibrium equations for the displacements are completely satisfied by $\mathbf{u} = [u]_{\lambda(u)}$, then the interval expression for $[F_k]$ in (9) is distributive. The latter means

$$(\mathbf{g}_k)_{\lambda([\delta_k])}(t_k^\top .(\mathbf{u}_k)_{\lambda(u)\lambda(t_k)}) = t_k^\top .(\mathbf{g}_k \mathbf{u}_k)_{\lambda(u)\lambda(t_k)}.$$

Replacing the expression at the right-hand side above into interval equilibrium equations (8), we obtain the interval algebraic equations for the displacements considered in [20].

Since $\mathbf{g}_k > 0$, we have $\lambda([F_k]) = \lambda([\delta_k])$ and

$$\mathbf{F}_k = \mathrm{pro}([F_k]) = \left((\mathbf{g}_k)_{\lambda([\delta_k]}[\delta_k]\right)_{\lambda([\delta_k])} \;=\; \mathbf{g}_k[\delta_k]_{\lambda([\delta_k])}.$$

Thus, due to the equivalence in the above derivation, we proved the following theorem.

Theorem 1 *If the formal (algebraic) solution* $[u]$ *to interval algebraic equilibrium equations (8) satisfies* $[u]_{\lambda(u)} \in \mathbb{IR}$, *then* $\mathbf{F}_k = [F_k]_{\lambda(F_k)} \in \mathbb{IR}$ *and* $[F_k] = (\mathbf{F}_k)_{\lambda(F_k)}$ *completely satisfies (8).*

It should be noted that, in general, $\mathrm{pro}([\delta_k]) \neq [\delta_k]_{\lambda([\delta_k])}$. With the interval element deformation $[\delta_k]$, defined in (9), the corresponding interval element stress can be obtained in two equivalent ways:

$$[\sigma_k] = \left(\frac{\mathbf{E}_k}{\mathbf{L}_k}\right)_{\lambda(\delta_k)} [\delta_k] \tag{10}$$

$$[\sigma_k] = \frac{[F_k]}{(\mathbf{A}_k)_{-\lambda(\delta_k)}} = \frac{\left(\frac{\mathbf{E}_k \mathbf{A}_k}{\mathbf{L}_k}\right)_{\lambda(\delta_k)}}{(\mathbf{A}_k)_{-\lambda(\delta_k)}}[\delta_k] \overset{(3)}{=} \left(\frac{\mathbf{E}_k}{\mathbf{L}_k}\right)_{\lambda(\delta_k)} [\delta_k]. \tag{11}$$

Since intervals $\mathbf{E}_k, \mathbf{A}_k, \mathbf{L}_k$ do not contain zero, relation (11) applies the inverse multiplication property (3). Thus, instead of using the left-hand side of (11), we can use (10) for obtaining $[\sigma_k]$. As for the deformations $[\delta_k]$, in general, we have $\mathrm{pro}([\sigma_k]) \neq [\sigma_k]_{\lambda([\sigma_k])}$. Therefore,

$$\mathrm{pro}([\sigma_k]) = \frac{\mathbf{E}_k}{\mathbf{L}_k} \mathrm{pro}\left([\delta_k]\right).$$

Similarly we find intervals for the element normal strain by

$$\mathrm{pro}([\varepsilon_k]) = \frac{\mathrm{pro}([\delta_k])}{\mathbf{L}_k}.$$

From implementation point of view, neither obtaining the interval displacements $\mathbf{u} = [u]_{\lambda(u)}$ discussed in [20], nor obtaining $\mathrm{pro}([\delta_k])$ requires an environment supporting Kaucher interval arithmetic. Computations can be done separately for the endpoints of $[\delta_k^-, \delta_k^+]$ applying the following proposition.

Theorem 2 *Let* $\mathbf{u}_k \in \mathbb{IR}$ *be obtained from the formal (algebraic) solution to interval algebraic equilibrium equations for the truss. With* $[\delta_k^-, \delta_k^+] := t_k^\top.(\mathbf{u}_k)_{\lambda(u_k)\lambda(t_k)}$, *we have*

$$\delta_k^- = t_k^\top.\check{u}_k - t_k^\top.(\lambda(u_k)\hat{u}_k), \qquad \delta_k^+ = t_k^\top.\check{u}_k + t_k^\top.(\lambda(u_k)\hat{u}_k),$$

where $\check{u}_k$, $\hat{u}_k$ *are the corresponding vectors of midpoints and radiuses of* $\mathbf{u}_k$. *Multiplication of a binary-valued vector* $\lambda(u_k)$ *and a real-valued vector of the same dimension is a Hadamard product.*

Proof The proof follows from [20, Eqn. (9)]. □

3 Clamped Bar with a Gap

Consider a clamped bar subjected to a concentrated loading as presented in Fig. 1. The bar has a cross-sectional area A, modulus of elasticity E and length L. The loading is P and the bar has a gap equal to d at its tip. The goal is to find the distribution of stresses at the cross-section x, which is at a distance L_1 from the upper clamping, assuming that all the parameters of the problem have some kind of uncertainty and their values vary within given

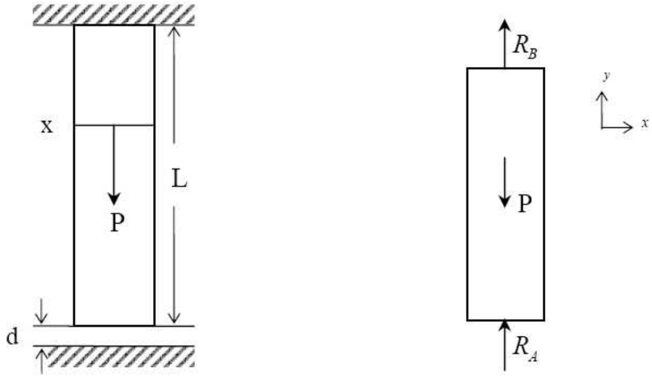

Fig. 1 Clamped bar with a gap, a), and its free body diagram, b).

intervals, $P \in \mathbf{P}$, $A \in \mathbf{A}$, $E \in \mathbf{E}$, $L \in \mathbf{L}$, $d \in \mathbf{d}$. For the sake of simplicity we assume that $L = 2L_1$. We assume also that the interval value (the range) of the elongation of the tip, $\delta_A = \frac{P}{g_A}$, $g_A = \frac{EA}{L_1}$, satisfies the relation

$$\mathbf{d} \leq \frac{\mathbf{PL}_1}{\mathbf{AE}}, \tag{12}$$

so that the interval problem is completely statically indeterminate, with d denoting the gap due to manufacturing error, for example.

In deterministic settings we have one equilibrium equation with two unknowns, once the gap is closed

$$R_A - P + R_B = 0 \tag{13}$$

and the geometric compatibility equation

$$\frac{PL_1}{AE} - \frac{R_B L}{AE} = d. \tag{14}$$

From equations (13), (14) we determine R_A and R_B. Then, the stress in the upper part of the bar is $\sigma^{(I)} = R_A/A$ and the stress in the lower part of the bar is $\sigma^{(II)} = -R_B/A$.

Since $L_1 = L/2$ and in order to avoid multiple interval parameters we substitute $t = \frac{1}{g_B} = \frac{L}{AE}$ in (13), (14) and consider the following linear algebraic system

$$A(t)x = a(P, d, t)$$
$$x = \begin{pmatrix} R_A \\ R_B \end{pmatrix}, \quad A(t) = \begin{pmatrix} 1 & 1 \\ 1 & -t \end{pmatrix}, \quad a(P, d, t) = \begin{pmatrix} P \\ d - \frac{1}{2}Pt \end{pmatrix}.$$

The classical interval model considers the interval system

$$A(t)x = a(P, d, t), \quad t \in \mathbf{t}, P \in \mathbf{P}, d \in \mathbf{d}$$

and searches for $\square\Sigma^p$ of the parametric solution set Σ^p to this system As mentioned in Section 2.2.1, there are various approaches to find $\square\Sigma^p$. Most of them have high computational complexity and are not appropriate for solving large-scale problems. Nevertheless, when comparing the results of classical interval model, the best interval solution enclosure vector $\square\Sigma^p$ will be used.

The algebraic interval model requires that all equilibrium equations involved in the deterministic description of the problem be completely satisfied. The same requirement is for those compatibility equations which involve summation of vector quantities. For example, since the displacement of section B due to the force P and the displacement of the entire bar due to force R_B are vector quantities, we consider the compatibility equation (14) as an equilibrium equation. According to the algebraic model, we consider two interval equations

$$R_A - \mathtt{Dual}(\mathbf{P}) + R_B = 0, \tag{15}$$

$$\frac{\mathbf{PL}_1}{\mathbf{AE}} - \mathrm{Dual}\left(\frac{R_B\mathbf{L}}{\mathbf{AE}}\right) = \mathbf{d} \tag{16}$$

and look for their algebraic solution in Kaucher interval arithmetic. That is, we search for proper intervals $\mathbf{R}_A, \mathbf{R}_B$ which completely satisfy (15) and (16). For this small example we derive the analytic algebraic solution by algebraic transformations. An efficient numerical approach is proposed in [20] for large-scale problems. From (16) we obtain

$$[R_B] = \left(\frac{\mathbf{P}}{2}\frac{\mathbf{L}}{\mathbf{AE}} - \mathrm{dual}(\mathbf{d})\right)\frac{\mathrm{dual}(\mathbf{AE})}{\mathrm{dual}(\mathbf{L})}.$$

In order to avoid the multiple occurrence of $\mathbf{t}$ we check the interval distributive relation (4) and apply it to the last equation. From relation (12) we obtain $\frac{\mathbf{P}}{2}\frac{\mathbf{L}}{\mathbf{AE}} - \mathrm{dual}(\mathbf{d}) > 0$, and then

$$\begin{aligned}[R_B] &= \frac{\mathbf{P}}{2}\frac{\mathbf{L}}{\mathbf{AE}}\frac{\mathrm{dual}(\mathbf{AE})}{\mathrm{dual}(\mathbf{L})} - \mathrm{dual}(\mathbf{d})\left(\frac{\mathrm{dual}(\mathbf{AE})}{\mathrm{dual}(\mathbf{L})}\right)_- \\ &= \frac{\mathbf{P}}{2} - \mathrm{dual}(\mathbf{d})\frac{\mathbf{AE}}{\mathbf{L}}.\end{aligned}$$

The unknown reaction $\mathbf{R}_A$ is obtained from the *interval equilibrium equation* (15) by algebraic transformations in Kaucher interval arithmetic

$$\begin{aligned}[R_A] &= \mathbf{P} - \mathrm{dual}(\mathbf{R}_B) \\ &= \mathbf{P} - \mathrm{dual}(\frac{\mathbf{P}}{2} - \mathrm{dual}(\mathbf{d})\frac{\mathbf{AE}}{\mathbf{L}}) = \frac{\mathbf{P}}{2} + \mathbf{d}\times\mathrm{dual}(\frac{\mathbf{AE}}{\mathbf{L}}).\end{aligned}$$

We take $\mathbf{R}_A = \mathrm{pro}([R_A])$, $\mathbf{R}_B = \mathrm{pro}([R_B])$ If $[R_A], [R_B]$ are proper intervals, the interval equations (15), (16) will be completely satisfied within exact arithmetic.

The stresses in both models are computed as $\mathbf{R}_A/\mathbf{A}$, $-\mathbf{R}_B/\mathbf{A}$.

Next we present numerical computations with both models.

Example 1 Let $P = 200 \times 10^3$ N, $d = 3 \times 10^{-4}$ m, $L = 3$ m, $A = 25 \times 10^{-4}\text{m}^2$, $E = 2 \times 10^{11}$. Assume that there is a 5% relative uncertainty in each of the parameters. That is, for each $t \in \{P, d, L, A, E\}$, $t \in \mathbf{t} = [t - t/20, t + t/20]$.

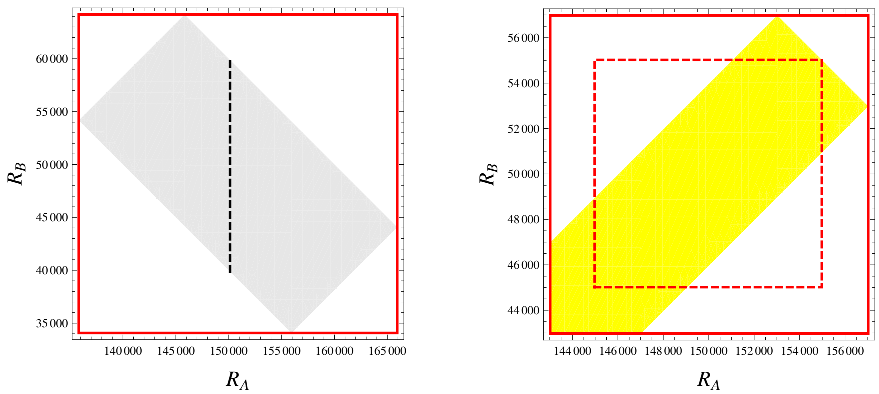

Fig. 2 For the clamped bar with a gap, the solution set Σ^p (in gray/yellow) and its minimal enclosing interval vector (solid line box) of the classical model and the interval algebraic solution according to the algebraic model, dashed line. Left: data of Example 1, right: data of Example 2.

Exact bounds for the reactions in the classical model are

$$\mathbf{R}_A = \frac{\mathbf{P}}{2} + \frac{\mathbf{d}}{\mathbf{t}} \subset [135827.38, 165927.64], \; \mathbf{R}_B = \frac{\mathbf{P}}{2} - \frac{\mathbf{d}}{\mathbf{t}} \subset [34072.36, 64172.62]. \tag{17}$$

The solution set Σ^p and its minimal enclosing interval vector are presented in the left graphics of Figure 2. Interval reactions according to the algebraic model are

$$\mathbf{R}_A = [R_A] = [150125,\ 150125], \qquad \mathbf{R}_B = [R_B] = [39875,\ 59875]$$

and presented by the dashed line segment in the left graphics of Figure 2. While the latter $\mathbf{R}_A, \mathbf{R}_B$ completely satisfy equations (15), (16), interval reaction from (17) replaced in these equations give

$$\mathbf{R}_A + \mathbf{R}_B - \text{dual}(\mathbf{d}) \subset [-20100.3, 20100.3],$$
$$\frac{1}{2}\mathbf{P}\mathbf{t} - \text{dual}(\frac{\mathbf{R}_B}{\mathbf{t}}) - \text{dual}(\mathbf{d}) \subset [-4.88 \times 10^{12},\ -1.25 \times 10^{13}].$$

The respective enclosures of the stresses according to the algebraic interval model are

$$\frac{\mathbf{R}_A}{\mathbf{A}} \in 10^7[5.71904,\ 6.32106], \qquad -\frac{\mathbf{R}_B}{\mathbf{A}} \in -10^7[1.51904,\ 2.52106].$$

Example 2 Consider the same problem with specified levels of uncertainty. Assume that there is a 5% relative uncertainty in P and 2% relative uncertainty in d and A. Find enclosures for the unknown reactions.

The solution set Σ^p and its minimal enclosing interval vector according to the classical model, as well as the interval algebraic solution according to the algebraic model are presented in the right-hand side graphics of Figure 2. As in Example 1, the interval box for R_A, R_B in the algebraic model (dashed line in Fig. 2 right) is smaller than the corresponding interval (solid line) box according to the classical interval model. Furthermore, the former one completely satisfies the equilibrium and compatibility equations contrary to the latter one.

4 Truss Structures

4.1 A 6-bar truss structure

Consider a 6-bar truss structure presented in Fig. 3, where the force parameter Q is unknown-but-bounded in the interval $\mathbf{Q} = [20, 21]kN$ and the cross sectional areas A_5, A_6 are also uncertain varying in the intervals $[1.008, 1.092] \times 10^{-3}\ m^2$, $[1, 1.1] \times 10^{-3}\ m^2$, respectively. An efficient numerical method find-

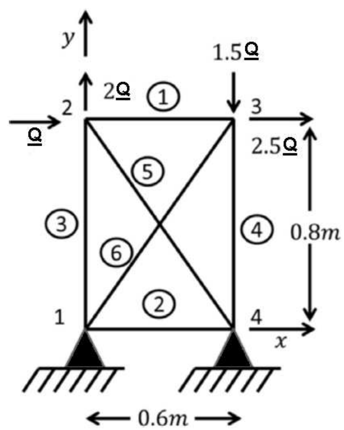

Fig. 3 A 6-bar truss structure after [21].

ing the algebraic (formal) solution to interval equilibrium equations of truss structures is proposed in [20]. This example is considered also therein and the exact bounds for the displacements obtained by the interval algebraic model are compared in [20, Example 1] to the exact bounds of the displacements obtained by the classical interval model. Here we illustrate this comparison on Figure 4. Since the derived variables are functions of the displacements, it is clear that the corresponding ranges obtained by the algebraic model will be narrower than those obtained by the classical model. Therefor, here we find interval bounds for the unknown element axial forces, element stains and stresses by the algebraic interval model.

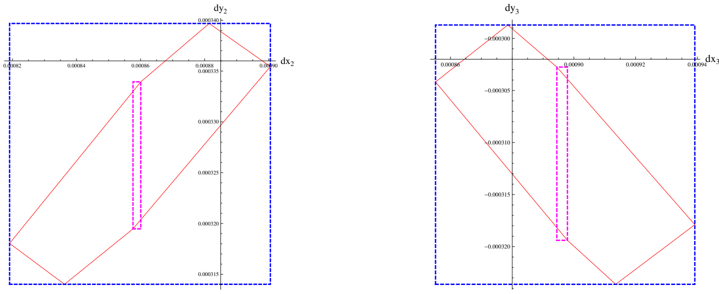

Fig. 4 For the 6-bar example, the solution set Σ^p and its minimal enclosing interval vector (dashed blue box) of the classical model and the interval algebraic solution according to the algebraic model, dashed line.

For the considered example the deterministic force equilibrium equations are

$$\begin{aligned} Q + c_1F_1 + c_5F_5 &= 0, \\ 2Q + s_3F_3 + s_5F_5 &= 0, \\ 2.5Q - c_1F_1 - c_6F_6 &= 0, \\ -1.5Q + s_4F_4 - s_6F_6 &= 0, \end{aligned} \tag{18}$$

where

$$\begin{aligned} &F_1 = F_{23} = t_1^\top(g_1u) = (-1, 0, 1, 0)(g_1u), && \measuredangle_1 = 0^\circ, \\ &F_3 = F_{21} = t_3^\top(g_3u) = (0, 1, 0, 0)(g_3u), && \measuredangle_3 = 270^\circ, \\ &F_4 = -F_{34} = t_4^\top(g_4u) = (0, 0, 0, 1)(g_4u), && \measuredangle_4 = 270^\circ, \\ &F_5 = F_{24} = t_5^\top(g_5u) = (-\frac{6}{10}, \frac{8}{10}, 0, 0)(g_5u), && c_5 = \frac{6}{10},\ s_5 = -\frac{8}{10} \\ &F_6 = F_{13} = t_6^\top(g_6u) = (0, 0, \frac{6}{10}, \frac{8}{10})(g_6u) && c_6 = \frac{6}{10},\ s_6 = \frac{8}{10} \end{aligned}$$

$g_k = A_kE_k/L_k$ and $u = (u_2^x, u_2^y, u_3^x, u_3^y)^\top$.

With the intervals for the displacements from [20, Table 4], we obtain

$$\begin{aligned} [F_1] &= -\mathrm{dual}(\mathbf{g}_1\mathbf{u}_2^x) + \mathbf{g}_1\mathbf{u}_3^x \in [12895.3, 13244.6], \\ [F_3] &= \mathbf{g}_3\mathbf{u}_2^y \in [83860.5, 87659.5], \\ [F_4] &= \mathrm{dual}(\mathbf{g}_4\mathbf{u}_3^y) \in [-79472.9, -83840.5], \\ [F_5] &= -\frac{6}{10}\mathrm{dual}(\mathbf{g}_5\mathbf{u}_2^x) + \frac{8}{10}\mathbf{g}_5\mathbf{u}_2^y \in [-54825.7, -57074.2], \\ [F_6] &= \frac{6}{10}\mathbf{g}_6\mathbf{u}_3^x + \frac{8}{10}\mathrm{dual}(\mathbf{g}_6\mathbf{u}_3^y) \in [61841., 65425.8]. \end{aligned}$$

Verification of the obtained numerical values (and of the algebraic interval model) can be done in two equivalent ways: (a) replacing the obtained $[F_k]$ in the interval equilibrium equations (8), or (b): replacing $[F_k]_{\lambda(c_k)}$ by $(\mathbf{F}_k)_{\lambda(c_k)\lambda(F_k)}$ in (8).

With the algebraic interval deformation $[\delta_k]$, defined in (9), we obtain

$$\mathrm{Diag}(\frac{E}{L}).[\delta] \in \mathrm{Diag}\left(10^{10}\begin{pmatrix}35\\26.25\\26.25\\21\\21\end{pmatrix}\right).10^{-5}\begin{pmatrix}[3.68439, 3.78417]\\[31.9468, 33.3941]\\[-30.2754, -31.9392]\\[-25.9003, -24.8884]\\[29.4481, 28.3229]\end{pmatrix},$$

$$\sigma \in 10^7\,([1.28953, 1.32446], [8.38605, 8.76595],$$
$$[-8.38406, -7.9472], [-5.43906, -5.22658], [5.94779, 6.18411])^\top .$$

Similarly we compute intervals for the element normal strain $\mathrm{pro}([\delta_k])/\mathbf{L}_k$.

4.2 One-bay 20-floor truss cantilever

As a large truss example we consider a one-bay 20-floor truss cantilever, presented in Fig. 5, and considered in [23] as a benchmark problem for the applicability, computational efficiency and scalability of the approach proposed therein for structures with complex configuration and a large number of interval parameters. The structure consists of 42 nodes and 101 elements. The bay

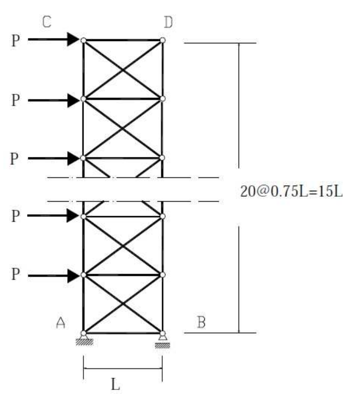

Fig. 5 One-bay 20-floor truss cantilever after [14].

is $L = 1$m, every floor is $0.75L$, the element cross-sectional area is $A = 0.01$ m^2, and the crisp value for the element Young modulus is $E = 2 \times 10^8$kN/m^2. Twenty horizontal loads with nominal value $P = 10$ kN are applied at the left nodes. The boundary conditions are determined by the supports: at A the support is a pin, at B the support is roller. It is assumed 10% uncertainty in the modulus of elasticity E_k of each element ($\mp 5\%$ from the corresponding mean value) and 10% uncertainty in the twenty loads. The goal is to obtain bounds for the axial force (F_{40}) in element 40 by the algebraic interval model and to compare the obtained interval to the bounds yield by the approaches based on classical interval arithmetic presented in [23, Example 2].

According to the algebraic interval model of the considered truss, we first apply the efficient numerical method proposed in [20] and obtain interval values for the unknown node displacements. Then, with these interval node displacements and the corresponding numerical transformation vector for the element 40, following the interval algebraic methodology presented in this paper, we obtain $\mathbf{F}_{40}$ presented in Table 1. This table presents also the interval obtained by a specially designed methodology and the model based on classical interval arithmetic.

uncertainty	$\mathbf{F}_{40}$ by [23]	$\mathbf{F}_{40}$ (algebraic)
0%	79.821	79.821
10%	[60.652, 98.991]	[75.8303, 83.8125]

Table 1 Axial force F_{40} (kN) in element 40 of the cantilever truss obtained by the algebraic interval model and by a numerical approach [23] based on classical interval model.

5 Conclusion

Naive application of interval arithmetic (classical or generalized) is not recommended — each interval space has specific properties that have to be accounted for by the mathematical models and the computations. Therefore, a major contribution of the present work is providing a sound methodology for obtaining exact bounds of secondary quantities (variables), like element axial forces, strains and stresses, in the algebraic interval FEM model of truss structures with displacements as primary variables. These secondary quantities present a background in strength of materials theory, which determines the scope of application of the presented methodology to a diversity of problems involving analysis or design in structural mechanics. The following works:

– [5], [18] formulating the basis of the new *algebraic* interval model, where any vector (in geometric sense) parameter in mechanics (possessing magnitude and direction) is represented by a directed interval (range + direction), thus embedded in an algebraically rich space of proper and improper intervals;
– [20] presenting efficient methodology and numerical procedures computing the formal (algebraic) solution to the equations for the displacements in large-scale problems; and
– the present work expanding the algebraic methodology to secondary variables,

formulate a background of the new *algebraic* interval model, present its application to models of truss structures, and trace the road toward a variety of other applications. In summary, the algebraic interval model:

– fully conforms to the physics of linear equilibrium and compatibility equations,

– applies straightforward and transparently to the corresponding deterministic formulations,
– the computational numerical procedures are fast and applicable to large-scale problems involving high-dimensional interval parameters and large uncertainties,
– while the theory of generalized (proper and improper) intervals is indispensable in the development of the corresponding methodologies and computational procedures, the implementation of the latter does not require software environment supporting Kaucher interval arithmetic,
– the obtained intervals for the unknown quantities are exact (without any overestimation) in exact arithmetic,
– replacing the obtained exact intervals for the unknowns in the particular algebraic interval model results in true equalities, which verifies the corresponding model and the computations.

Most of the properties of the proposed novel methodology are not attributable to the models based on classical interval arithmetic although they are required by practical engineering in analysis and design.

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