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Algebraic Solution to Interval Equilibrium Equations of Truss Structures

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Abstract

This paper considers a recently proposed interval algebraic model of linear equilibrium equations in mechanics. Based on the algebraic completion of classical interval arithmetic (called Kaucher arithmetic), this model provides much smaller ranges for the unknowns than the model based on classical interval arithmetic and fully conforms to the equilibrium principle. The general form of interval equilibrium equations for truss structures is presented. Two numerical approaches for finding the formal (algebraic) solution to the considered class of interval equilibrium equations are proposed. A methodology for adjusting interval parameters so that the equilibrium equations be completely satisfied is also presented. Numerical examples illustrate the theoretical considerations.

Keywords: static, linear equilibrium equations, interval parameters, truss structures, formal (algebraic) solution
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1. Introduction

In the last few decades there is a considerable research interest in modeling, analysis and design of engineering systems involving various types of uncertainty. Interval variables are widely used to represent non-probabilistic uncertainties and methods based on interval analysis are used not only to interval models but also to other generalized uncertainty models, e.g., those based on fuzzy sets and others, cf. [1], [2]. Most of the considered interval models are formulated in terms of classical interval arithmetic, [3]–[6].

The analysis of many engineering problems, for example, finite element formulation of equilibrium and steady state problems, requires the solution of linear systems of equations. The corresponding interval models, that require solving linear systems of algebraic equations involving interval parameters, the latter

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solved by methods of classical interval arithmetic will be called *classical interval models*. It is mentioned in [7] that nowadays the most successful approaches for overestimation reduction are those that relate the dependency of interval quantities to the physics of the problem being considered.

Recently, it was pointed out by several authors that the interval models of static equilibrium basing only on classical interval arithmetic do not conform to the problem physics, cf. [8]. This is because classical interval arithmetic does not possess group properties. A new interval model of linear equilibrium equations in mechanics is proposed in [9]. Since it is based on the algebraic completion of classical interval arithmetic (called Kaucher arithmetic), the newly proposed model is called *interval algebraic model*. This approach replaces straightforward a deterministic model by an interval model in terms of generalized (proper and improper) intervals, fully conforms to the equilibrium principle and, via the formal (algebraic) solution of the interval equilibrium equations, provides sharper enclosure of the unknown quantities than the best solution of classical interval model. For a small number of interval equilibrium equations, their formal solution can be found by *equivalent algebraic transformations* on the equations. This approach is used in [9], [10] to illustrate the application and the properties of the new model. However, for finding an algebraic (formal) solution to a big number of interval equilibrium equations, what is the case in static analysis of truss structures, we need an efficient numerical method. Furthermore, due to the presence of dependencies between the interval model parameters involved in the equilibrium equations, the latter possesses a specific structure which is not studied so far. In other words, the interval linear algebraic systems, for which there are methods for finding an algebraic solution, are not the same as the systems of interval equilibrium equations. This article is motivated also by the work [11], where the authors search an algebraic solution to the classical interval model, however they do not provide interpretation of the obtained improper intervals. The focus of the present work is on the interval algebraic model of linear equilibrium equations, its application to models of truss structures, development of numerical methods finding algebraic solutions and applicable to a large number of interval equilibrium equations.

The paper is organized as follows. Section 2 presents briefly the algebraic completion of classical interval arithmetic with its main properties that are used in the subsequent derivations. References to other articles presenting some topics with more details are provided. Section 3 presents a recently proposed interval algebraic model of linear equilibrium equations. Two interval models of truss structures (the classical one and the newly proposed algebraic one), which are compared in the numerical examples, are also outlined there. At the end of this section we present the general form of interval equilibrium equations for truss structures according to the recent algebraic model. The contribution of this article is contained in Section 4. First two subsections there present two approaches for finding formal (algebraic) solution to the considered interval equilibrium equations. Two numerical methods with necessary and sufficient conditions for existence and uniqueness of a formal solution are derived. A relation to a general class of pseudo-linear interval equations is discussed. Section 4.4 presents a methodology for adjusting interval uncertainty in some model parameters so that the obtained algebraic solution strictly conforms to the equilibrium principle. A hint for applicability to problems of engineering design in presence of interval model parameters is given. Several example problems are considered in Section 5 to illustrate the presented methodology and also to demonstrate the ability of the algebraic interval model to provide sharp bounds for the displacements even in the presence of large uncertainties and large number of interval variables, at the same time conforming fully to the equilibrium principle. The paper ends with some concluding remarks.

2. Algebraic Completion of Interval Arithmetic

2.1. Basic Properties

We assume that the reader is familiar with classical interval arithmetic and its properties, see, e.g., [3]–[6].

The set of classical compact intervals $\mathbb{IR} = \{[a^-, a^+] \mid a^-, a^+ \in \mathbb{R}, a^- \leq a^+\}$, called also *proper* intervals, is extended in [12] by the set $\overline{\mathbb{IR}} := \{[a^-, a^+] \mid a^-, a^+ \in \mathbb{R}, a^- \geq a^+\}$ of *improper* intervals obtaining thus the set $\mathbb{KR} = \mathbb{IR} \bigcup \overline{\mathbb{IR}} = \{[a^-, a^+] \mid a^-, a^+ \in \mathbb{R}\}$ of all ordered couples of real numbers called *generalized* (extended or Kaucher) intervals. For a better understanding we denote the classical intervals by bold face letters (e.g., $\mathbf{a}$) and the intervals from $\mathbb{KR}$ by brackets (e.g., $[a]$). The inclusion order relation ($\subseteq$) between classical intervals is generalized straightforward for $[a], [b] \in \mathbb{KR}$. For $[a] = [a^-, a^+] \in \mathbb{KR}$, define a binary variable *direction* (τ) by

$$\tau([a]) := \begin{cases} + & \text{if } a^- \leq a^+, \\ - & \text{if } a^- \geq a^+. \end{cases}$$

The elements of $\mathbb{KR}$ with positive direction are called proper intervals and the elements with negative direction are called improper intervals. An element-to-element symmetry between proper and improper intervals is expressed by the "dual" operator. For $[a] = [a^-, a^+] \in \mathbb{KR}$, $\mathrm{dual}([a]) := [a^+, a^-]$. For $[a], [b] \in \mathbb{KR}$,

$$\begin{aligned} \mathrm{dual}(\mathrm{dual}([a])) &= [a], && (1) \\ \mathrm{dual}([a] \circ [b]) &= \mathrm{dual}([a]) \circ \mathrm{dual}([b]), \ \circ \in \{+, -, \times, /\}. && (2) \end{aligned}$$

Define proper projection of a generalized interval $[a]$ onto $\mathbb{IR}$ by

$$\mathrm{pro}([a]) := \begin{cases} [a] & \text{if } \tau([a]) = +, \\ \mathrm{dual}([a]) & \text{if } \tau([a]) = -. \end{cases}$$

Denote $\mathcal{T} := \{[a] \in \mathbb{KR} \mid a^-a^+ < 0\}$ and $\mathcal{Z} := \{\mathbf{a} \in \mathbb{IR} \mid a^-a^+ < 0\}$. For $[a] = [a^-, a^+] \in \mathbb{KR} \setminus \mathcal{T}$ define "sign" (σ) by

$$\sigma([a]) := \begin{cases} + & \text{if } \mathrm{pro}([a])^- \geq 0, \\ - & \text{otherwise.} \end{cases}$$

Multiplication of two binary variables $\lambda, \mu \in \{+, -\}$ is defined by

$$\lambda\mu = ++ = -- = +, \qquad \lambda\mu = +- = -+ = -. \tag{3}$$

The conventional interval arithmetic and lattice operations, as well as other interval functions are isomorphically extended onto the whole set $\mathbb{KR}$, cf. [12]. The generalized interval arithmetic structure possesses group properties with respect to the operations addition and multiplication. For $[a] \in \mathbb{KR}$, $[b] \in \mathbb{KR} \setminus \mathcal{T}$,

$$[a] - \mathrm{dual}([a]) = 0, \qquad [b]/\mathrm{dual}([b]) = 1. \tag{4}$$

The complete set of conditionally distributive relations for multiplication and addition of generalized intervals can be found in [13]. Here we present only one that is usually used. For $[a], [b], [s] \in \mathbb{KR} \setminus \mathcal{T}$, $[s] = [a] + [b]$, $[c] \in \mathbb{KR}$

$$([a] + [b])[c]_{\sigma([s])} = [a][c]_{\sigma([a])} + [b][c]_{\sigma([b])}, \tag{5}$$

wherein $[a]_+ = [a]$, $[a]_- = \mathrm{dual}([a])$. In what follows we will often use the subscript notation of the dual operator.

2.2. Arithmetic Operations in Center-Radius Form

For $[a] = [a^-, a^+] \in \mathbb{KR}$, define *center* $\mathrm{mid}([a]) = \check{a}$ and *radius* $\mathrm{rad}([a]) = \hat{a}$, $\check{a}, \hat{a} \in \mathbb{R}$, by

$$\mathrm{mid}([a]) = \check{a} := (a^- + a^+)/2, \qquad \mathrm{rad}([a]) = \hat{a} := (a^+ - a^-)/2.$$

Thus, $[a] = [a^-, a^+] = [\check{a} - \hat{a},\ \check{a} + \hat{a}] = \check{a} + [-\hat{a},\ \hat{a}]$. For the sake of shortening the notation, the symmetric interval $[-\hat{a}, \hat{a}]$ will be also denoted as $[\mp\hat{a}]$. The arithmetic operations between intervals in $\mathbb{KR}$ are presented in center-radius form in [14].

$$[a] + [b] = (\check{a} + \check{b}) + [\mp(\hat{a} + \hat{b})], \qquad [a], [b] \in \mathbb{KR}.$$

Define after [15] a functional $\chi : \mathbb{KR} \to [-1, 1]$ by $\chi([0, 0]) = -1$ and for $[a] \neq [0, 0]$

$$\chi([a]) := a^{-\nu([a])}/a^{\nu([a])}, \qquad \nu([a]) := \begin{cases} + & \text{if } |a^+| = |a^-|, \\ \sigma(|a^+| - |a^-|) & \text{otherwise.} \end{cases}$$

Functional χ admits the geometric interpretation, given in [16], that $[a]$ is more symmetric than $[b]$ if $\chi([a]) \leq \chi([b])$.

Interval multiplication is presented in center-radius form by the following formulae, [14].

$$\begin{aligned} [a][b] &= \check{a}\check{b} + \sigma([a])\sigma([b])\hat{a}\hat{b} \ + \ [\mp(|\check{b}|\hat{a} + |\check{a}|\hat{b})], \quad \text{if } [a], [b] \in \mathbb{KR} \setminus \mathcal{T}, && (6) \\ & \quad \check{a}\check{b} + \tau([a])\sigma([b])\check{a}\hat{b} + [\mp(|\check{b}|\hat{a} + \hat{a}\hat{b})], && (7) \\ & \qquad\qquad \text{if } \begin{cases} [a] \in \mathcal{T}, [b] \in \mathbb{KR} \setminus \mathcal{T}, \text{ or} \\ [a], [b] \in \mathcal{T}, \tau([a]) = \tau([b]), \chi([a]) \leq \chi([b]), \end{cases} \\ & \quad 0, \ \ \text{if } [a], [b] \in \mathcal{T},\ \tau([a]) \neq \tau([b]). \end{aligned}$$

Applying multiplication formula (7) one should be aware that multiplication operation in $\mathbb{KR}$ is commutative, $[a][b] = [b][a]$. The interval multiplication by a scalar can be obtained as a special case of (6).

In the next sections we will use the following equivalent representations, which follow from the above formulae. For $\alpha \in \mathbb{R}$, $[a] \in \mathbb{KR}$ and $\mu \in \{+, -\}$,

$$
\begin{aligned}
[\check{a} - \hat{a}, \check{a} + \hat{a}]_\mu &= \check{a} + [-\mu\hat{a}, \mu\hat{a}], && (8)\\
\alpha[a]_\mu &= \alpha\check{a} + |\alpha|[-\mu\hat{a}, \mu\hat{a}] && (9)\\
&= \alpha\check{a} + [-\mu\sigma(\alpha)\alpha\hat{a}, \mu\sigma(\alpha)\alpha\hat{a}].
\end{aligned}
$$

2.3. Matrix Notation

In this article we follow the notation used in [14]. Denote $\Lambda = \{+, -\}$. All functionals defined in the preceding two subsections are applied to interval matrices (and vectors considered as one column matrices) componentwise. Denote by Λ^n the set of all n-dimensional vectors of binary variables (signs) $\lambda = (\lambda_1, \ldots, \lambda_n)^\top$ with $\lambda_i \in \Lambda$, $i = 1, \ldots, n$; the set of all $n \times n$-dimensional matrices of binary variables is denoted by $\Lambda^{n\times n}$. Matrices of binary variables of same dimension are "multiplied" *componentwise* using the rules (3) for multiplication of binary variables. This is the so-called Hadamard product but, for the sake of a simplified notation, we will not use a special sign to denote it. For example, for $\lambda = (+, -, +, -)^\top$, $\mu = (-, +, +, -)^\top$, we obtain $\lambda\mu = (-, -, +, +)^\top$. Similarly, the product of a matrix of binary values by a an interval matrix of the same dimension is the corresponding componentwise Hadamard product. Also, for $\Gamma, \Delta \in \Lambda^{n\times n}$ and $[A] \in \mathbb{KR}^{n\times n}$, the products $\Gamma\Delta$ and $\Gamma[A]$ are well defined.

In what follows we will denote by dot (.) the interval/real scalar products, respectively the matrix-vector multiplications.

The rules for multiplication by binary-valued matrices are further extended for dualization. For $\lambda = (\lambda_1, \ldots, \lambda_n)^\top \in \Lambda^n$ and $[x] = ([x_1], \ldots, [x_n])^\top \in \mathbb{KR}^n$,

$$
[x]_\lambda := ([x_1]_{\lambda_1}, \ldots, [x_n]_{\lambda_n})^\top. \tag{10}
$$

The rules for dualization are generalized for matrices and to a product of the form $[A].[x]_{\sigma([B])}$. For $[A], [B] \in \mathbb{KR}^{m\times n}$, $[x] \in \mathbb{KR}^n$,

$$
[A].[x]_{\sigma([B])} := \begin{pmatrix} [A_{1\bullet}].[x]_{\sigma([B_{1\bullet}])} \\ \vdots \\ [A_{m\bullet}].[x]_{\sigma([B_{m\bullet}])} \end{pmatrix}. \tag{11}
$$

The conditionally distributive relation (5) is generalized for matrices in [14].

3. Interval Algebraic Model of Linear Equilibrium Equations

The interval model of linear equilibrium equations in mechanics models summation of vectors with different directions and interval magnitude. The one dimensional interval algebraic model for computing a resultant force (and its reaction), developed in [9], can be applied transparently to two- and three-dimensional problems involving vector physical quantities. The interval algebraic model presented here follows [10].

Assume that there is a deterministic model described by some equilibrium equation(s) that involve uncertain vector parameters varying within given proper intervals. Clearly, the unknowns in this model will be also uncertain and we search for proper intervals that are the sharpest interval enclosures of these unknowns and that conform to the physics of the problem (statics or dynamic equilibrium). Conformance to static (dynamic) equilibrium means that the intervals found for the unknowns when replaced in the equation(s) and all operations are performed results in true equality(ies).

Definition 1 ([17]). Interval *algebraic solution* to a (system of) interval equation(s) is an interval (interval vector) which substituted in the equation(s) and performing all interval operations in exact arithmetic (without round-off errors) results in valid equality(ies).

Interval algebraic solutions do not exist in general in classical interval arithmetic [17]. Interval arithmetic on proper and improper intervals $(\mathbb{KR}, +, \times, \subseteq)$ is the natural arithmetic for finding algebraic solutions to interval equations since it is obtained from the arithmetic for classical intervals $(\mathbb{IR}, +, -, \times, /, \subseteq)$ via an algebraic completion. Therefore, we embed the initial problem formulation in the interval space $(\mathbb{KR}, +, \times, \subseteq)$, find an algebraic solution (if exists) and interpret the obtained generalized intervals back in the initial interval space $\mathbb{IR}$. This is a three steps procedure summarized below.

1. The **representation convention** for a model involving interval forces (and/or other physical quantities considered as vectors and possessing magnitude and direction) is:
   - a scalar force component F_x (F_y, F_z) involving any kind of uncertainty is represented by proper interval $\mathbf{F}_x$ ($\mathbf{F}_y$, $\mathbf{F}_z$) if the force component $\underline{F}_x$ ($\underline{F}_y$, $\underline{F}_z$) has the same direction as the positive x (y, z) coordinate axis;
   - a scalar force component F_x (F_y, F_z) involving any kind of uncertainty is represented by the improper interval dual($\mathbf{F}_x$) (dual($\mathbf{F}_y$), dual($\mathbf{F}_z$)) if the force component $\underline{F}_x$ ($\underline{F}_y$, $\underline{F}_z$) has opposite direction to the corresponding positive x (y, z) coordinate axis.
2. **Computing**. Find the *algebraic solution*[1] for the unknown(s) in $(\mathbb{KR}, +, \times, \subseteq)$. For small systems, the approach based on *equivalent algebraic transformations* is transparent and it is used in [9], [10]. In this paper we present two new methods for finding formal solution to the algebraic interval model of equilibrium equations.

[1]The algebraic solution (Definition 1) to an interval equation is often called "*formal*" solution, see, e.g., [18], [11].

3. **Interpretation** of the obtained generalized intervals in the initial space $\mathbb{IR}$ is done according to the physics of the unknowns. If it is a force component, then [9, Theorem 1 ii)] is applied. In general the interpretation projects the generalized interval solution on $\mathbb{IR}$.

If the deterministic model involves more unknowns than the number of equilibrium equations, a *hybrid approach*, proposed and applied in [10], ensures that the unknown uncertain quantities are estimated in a way that the equilibrium equations are satisfied to a highest extent.

In cases when the compatibility equations can be considered as equilibrium equations, e.g., equations of statically indeterminate truss structures, a formal solution to the algebraic interval model of these equations can be sought. Then the interval algebraic solution is transformed by the interpretation convention to proper intervals for the unknowns. This approach is demonstrated in Sections 4 and 5.

3.1. Two interval models of truss structures

We consider statically indeterminate truss structures, where the applied loads (q), some of the structural parameters (modulus of elasticity E_i, cross sectional area A_i, or/and length L_i) of the bars (i) are uncertain and vary within given intervals. The traditional finite element method (FEM) for truss structures relates the force equilibrium equations (namely magnitudes of axial forces in the bars) to the displacements of the nodes and the stiffness of the bars E_iA_i/L_i. Thus, the deterministic model of a statically indeterminate truss structure is represented by a parametric linear system

$$K(E_i, A_i, L_i)u = f(q), \tag{12}$$

where $K(E_i, A_i, L_i)$ is the stiffness matrix depending on the structural parameters for each element, $f(q)$ is the load vector and u is the displacement vector. The parametric stiffness matrix $K(E_i, A_i, L_i)$ can be represented as

$$K(E_i, A_i, L_i) = \sum_{i=1}^{s} \frac{E_iA_i}{L_i} K^i,$$

where $K^i \in \mathbb{R}^{n\times n}$, $i = 1, \dots, s$ are numerical matrices and E_i, A_i, L_i are the structural parameters – some of them may be crisp, other may be uncertain varying within given intervals. Note, that we use superscripts to distinguish the numerical (coefficient) matrices because subscripts are used to denote dualization. In general, the truss may be subjected to several independent external loads $q = (q_1, \dots, q_m)^\top \in \mathbb{R}^m$. Therefore, the load vector $f(q)$ can be represented as $f(q) = F.q$, where $F = (f_i, \dots, f_m) \in \mathbb{R}^{n\times m}$, $f_i = \partial f(q)/\partial q_i$, $i = 1, \dots, m$.

Now, we rename the uncertain (interval) structural parameters of the truss and form a vector $p = (p_1, \dots, p_K)$ of these parameters. The crisp values of all remaining structural parameters E_i, A_i, L_i enter into the corresponding numerical matrix $K^i \in \mathbb{R}^{n\times n}$, which is also renamed to A^i. Then the renamed deterministic system (12) becomes $A(p)u = f(q)$ with parameters p and q.

The *classical interval model* of an uncertain truss structure with interval valued parameters considers the parametric interval linear system

$$A(p)u = f(q), \quad p \in \mathbf{p}, q \in \mathbf{q}, \tag{13}$$

where $A(p) = \sum_{i=1}^{K} p_i A^i$ is the renamed stiffness matrix depending on the renamed interval parameters p. For the sake of unifying the notations, sometimes (see Section 5) the first parameter p_1 should be a crisp one $p_1 \in [1, 1]$.

The classical interval model of an uncertain truss structure searches for the smallest interval vector enclosing the parametric united solution set $\Sigma(A(p), f(q), \mathbf{p}, \mathbf{q})$ to (13), defined by

$$\Sigma(A(p), f(q), \mathbf{p}, \mathbf{q}) := \{u \in \mathbb{R}^n \mid (\exists p \in \mathbf{p}, \exists q \in \mathbf{q}) : A(p)u = f(q)\}.$$

The smallest interval vector enclosing $\Sigma(A(p), f(q), \mathbf{p}, \mathbf{q})$ is called interval hull of the solution set and is defined for bounded sets by

$$\square\Sigma(A(p), f(q), \mathbf{p}, \mathbf{q}) := \{\bigcap \mathbf{u} \in \mathbb{IR}^n \mid \Sigma(A(p), f(q), \mathbf{p}, \mathbf{q}) \subseteq \mathbf{u}\}.$$

The engineering and interval literature is abundant with numerical methods dedicated to finding a sharp and guaranteed interval enclosure of the interval hull of a parametric united solution set to parametric interval linear system; the latter may be resulted from FEM applied to structures with interval parameters, cf., for example, [1], [2], [7], [19] and the references given therein. A computationally feasible method for obtaining the interval hull, which is applicable to parametric interval linear systems involving a large number of interval parameters is recently proposed in [19, Section 5] and demonstrated to the interval model of a large truss structure. When the parametric interval system (13) is a small one, any of the known methods can be applied.

The *algebraic interval model* of a statically indeterminate truss structure starts from the same parametric linear system $A(p)u = f(q)$ with parameters p and q and applies the representation convention in Section 3. We assume that all interval parameters p, q vary within non-degenerate[2] intervals $\mathbf{p} \geq 0$, $\mathbf{q} \geq 0$. Applying the representation convention in Section 3 and the notation of Section 2.3, the load vector in the right-hand side of the algebraic interval model gets the form $F.[q]_{\sigma(F)}$. For each $i = 1, \ldots, n$, the left-hand side of the algebraic interval model becomes

$$\sum_{k=1}^{K} A^k_{i\bullet} . \left([p_k][u]\right)_{\sigma([u])\sigma(A^k_{i\bullet})} .$$

[2] $[a^-, a^-] \in \mathbb{KR}$ is degenerate if $a^- = a^+$.

Note that the dualization above depends on the signs of the components of the unknown displacement vector $[u]$ because the latter can be positive or negative. Since the intervals for the displacements should not involve zero, the sign $\sigma([u])$ can be determined before finding $[u]$ as $\sigma([u]) = \sigma\left(A^{-1}(\check{p}).F.\check{q}\right)$, if $A(\check{p})$ is nonsingular. In matrix form the algebraic interval model of a statically indeterminate truss structure has the form

$$\sum_{k=1}^{K} A^k.\left([p_k][u]\right)_{\sigma([u])\sigma(A^k)} = F.[q]_{\sigma(F)},$$

where $A^k.\left([p_k][u]\right)_{\sigma([u])\sigma(A^k)} = \left(\left([p_k]A^k\right).[u]\right)_{\sigma([u])\sigma(A^k)}$.

4. Formal Solution to Interval Equilibrium Equations

According to the representation convention in Section 3, the algebraic interval model of truss equilibrium equations is

$$\sum_{k=1}^{K} \left(\left([p_k]A^k\right).[x]\right)_{\sigma([x])\sigma(A^k)} = F.[q]_{\sigma(F)}, \tag{14}$$

where $A^k \in \mathbb{R}^{n\times n}$, $[p_k] \in \mathbb{IR}$, $[p_k] \geq 0$, $k = 1, \ldots, K$ and $F \in \mathbb{R}^{n\times m}$, $[q] \in \mathbb{IR}^m$, $[q] \geq 0$. According to the computing convention, we are looking for a formal (algebraic) solution $[x] \in (\mathbb{KR} \setminus \mathcal{T})^n$ to the equilibrium equations (14). If the unique algebraic solution $[x]$ of (14) is such that $[x] \in (\mathbb{IR} \setminus \mathcal{Z})^n$, then it completely satisfies (in exact arithmetic) the equilibrium equations. Otherwise, according to the interpretation convention, for the range of the unknown displacements we take $\mathrm{pro}([x]) \in (\mathbb{IR} \setminus \mathcal{Z})^n$.

For the left-hand side of (14) we have equivalently

$$\sum_{k=1}^{K} \left(\left([p_k]A^k\right).[x]\right)_{\sigma([x])\sigma(A^k)} = \sum_{k=1}^{K} \left([A^k].[x]\right)_{\sigma([x])\sigma(A^k)},$$

where $[A^k] = [p_k]A^k \in (\mathbb{IR} \setminus \mathcal{Z})^{n\times n}$ and, since $[p_k] \geq 0$, $\sigma([A^k]) = \sigma(A^k)$. Thus,

$$\begin{aligned}
\sum_{k=1}^{K} \left([A^k].[x]\right)_{\sigma([x])\sigma(A^k)} &\overset{(2)}{=} \sum_{k=1}^{K} \left([A^k]_{\sigma([x])\sigma(A^k)}.\left([x]_{\sigma([x])}\right)_{\sigma(A^k)}\right) \\
&= \sum_{k=1}^{K} \left([A^k]_{\sigma([x])\sigma(A^k)}.[y]_{\sigma(A^k)}\right),
\end{aligned}$$

where $[y] = [x]_{\sigma([x])}$. Thus, the equilibrium equations (14) are equivalent to the equations

$$\sum_{k=1}^{K} \left([A^k]_{\sigma([x])\sigma(A^k)}.[y]_{\sigma(A^k)}\right) = F.[q]_{\sigma(F)}. \tag{15}$$

In what follows we present two approaches to determine a formal (algebraic) solution to the equilibrium equations (14), respectively (15).

4.1. Formal Solution via Hyperbolic Product

In [20] the so-called *hyperbolic* product is introduced by

$$[a] \times_h [b] = [a^- b^-, a^+ b^+], \qquad [a], [b] \in \mathbb{KR}. \tag{16}$$

The inverse elements $-\mathrm{dual}([a])$ and $1/\mathrm{dual}([a])$ generate operations

$$\begin{aligned} [a] -_h [b] &= [a] - \mathrm{dual}([b]) = [a^- - b^-, a^+ - b^+], \qquad [a], [b] \in \mathbb{KR}, \\ [a]/_h[b] &= [a]/\mathrm{dual}([b]) = [a^-/b^-, a^+/b^+], \qquad [a] \in \mathbb{KR},\ [b] \in \mathbb{KR} \setminus \mathcal{T}, \end{aligned}$$

called *hyperbolic* subtraction and *hyperbolic* division. The interval arithmetic addition together with the hyperbolic product form a field $\{\mathbb{KR}, +, \times_h\}$, [20], where a distributive law

$$[a] \times_h [c] + [b] \times_h [c] \;=\; ([a] + [b]) \times_h [c] \tag{17}$$

holds true for arbitrary $[a], [b], [c] \in \mathbb{KR}$. In [15] a transition formula expressing interval multiplication by the hyperbolic multiplication is used to find formal (algebraic) solutions to classes interval equations. In this subsection we will use the same approach basing on the transition formula

$$[a][b] = [a]_{\sigma([b])} \times_h [b]_{\sigma([a])}, \qquad [a], [b] \in \mathbb{KR} \setminus \mathcal{T}. \tag{18}$$

Similarly to the generalization done in Section 2.3, the hyperbolic arithmetic operations and the distributive law (17) are generalized to interval vectors and matrices.

Theorem 1. *The formal (algebraic) solution $[x] \in (\mathbb{KR} \setminus \mathcal{T})^n$ to the equilibrium equations (14) is determined by*

$$[x] = [y]_{\sigma([x])} = [y^-, y^+]_{\sigma([y])}$$

and the solution of the real-valued equation

$$\begin{pmatrix} A^-, & 0 \\ 0, & A^+ \end{pmatrix} \cdot \begin{pmatrix} y^- \\ y^+ \end{pmatrix} = \begin{pmatrix} F.q^- \\ F.q^+ \end{pmatrix}, \tag{19}$$

wherein

$$[A] = [A^-, A^+] \;\; := \;\; \sum_{k=1}^{K} \left([p_k] A^k\right)_{\sigma(A^k)} \in (\mathbb{KR} \setminus \mathcal{T})^{n \times n}. \tag{20}$$

Proof. Applying (18), we transform the interval scalar products in (15) into hyperbolic scalar products $(\cdot_h)$ and obtain equivalently to (14)

$$\sum_{k=1}^{K} \left([A^k]_{\sigma([x])\sigma(A^k)\sigma([y])} \cdot_h [y]_{\sigma(A^k)\sigma(A^k)}\right) \;\; = \;\; F \cdot_h [q]_{\sigma(F)\sigma(F)},$$

which, due to $\sigma([x]) = \sigma([y])$, is $\sum_{k=1}^{K} \left([A^k]_{\sigma(A^k)} \cdot_h [y]\right) = F \cdot_h [q]$. Due to the complete distributivity (17) in the hyperbolic interval space, (14) is equivalent to

$$\left(\sum_{k=1}^{K} [A^k]_{\sigma(A^k)}\right) \cdot_h [y] \quad = \quad F \cdot_h [q].$$

With the notation (20) of $[A]$, the equation (14) is equivalent to $[A] \cdot_h [y] = F \cdot_h [q]$ and applying the formulae of the hyperbolic arithmetic operations we get equivalently

$$\left[A^-.y^-, A^+.y^+\right] = \left[F.q^-, F.q^+\right], \tag{21}$$

which gives (19). □

Corollary 1. *The equilibrium equations (14) have a unique formal (algebraic) solution $[x] \in (\mathbb{KR} \setminus \mathcal{T})^n$ if and only if the endpoint matrices*

$$A^- = \sum_{k=1}^{K} \left(p_k^- A^k\right), \qquad A^+ = \sum_{k=1}^{K} \left(p_k^+ A^k\right)$$

of the interval matrix $[A] = \sum_{k=1}^{K} \left([p_k] A^k\right)_{\sigma(A^k)} \in (\mathbb{KR} \setminus \mathcal{T})^{n \times n}$ are nonsingular matrices.

Proof. The proof follows from Theorem 1. We will prove the endpoint representation.

$$\begin{aligned}
\sum_{k=1}^{K} \left([p_k] A^k\right)_{\sigma(A^k)} &\overset{(9)}{=} \sum_{k=1}^{K} \left(\check{p}_k A^k + \left[\mp \left(\hat{p}_k |A^k|\right)\right]\right)_{\sigma(A^k)} \\
&\overset{(8)}{=} \sum_{k=1}^{K} \left(\check{p}_k A^k + \left[\mp \left(\hat{p}_k \sigma(A^k) |A^k|\right)\right]\right) \\
&= \sum_{k=1}^{K} \left(\check{p}_k A^k + \left[\mp \left(\hat{p}_k A^k\right)\right]\right) \\
&= \sum_{k=1}^{K} \left(\check{p}_k A^k\right) + \left[\mp \sum_{k=1}^{K} \left(\hat{p}_k A^k\right)\right].
\end{aligned} \tag{22}$$

Thus, similarly for A^- and A^+, and with the notation $A(\check{p}) := \sum_{k=1}^{K} \left(\check{p}_k A^k\right)$,

$$\begin{aligned}
A^- &= \sum_{k=1}^{K} \left(\check{p}_k A^k\right) - \sum_{k=1}^{K} \left(\hat{p}_k A^k\right) = A(\check{p}) - \sum_{k=1}^{K} \left(\hat{p}_k A^k\right) \\
&= \sum_{k=1}^{K} \left(\left(\check{p}_k - \hat{p}_k\right) A^k\right) = \sum_{k=1}^{K} \left(p_k^- A^k\right).
\end{aligned} \tag{23}$$

□

Proposition 1. *The conditionally distributive law (5) holds true for the left-hand side of the equilibrium equations (14). That is*

$$\sum_{k=1}^{K}\left(\left([p_k]A^k\right).[x]\right)_{\sigma([x])\sigma(A^k)} = \left(\sum_{k=1}^{K}[A^k]_{\sigma([x])\sigma(A^k)}\right).[x]_{\sigma([x])\sigma(\sum_{k=1}^{K}[A^k]_{\sigma([x])\sigma(A^k)})}. \quad (24)$$

With the notation $[A^k] = [p_k]A^k$, $[B^k] = [A^k]_{\sigma([x])\sigma(A^k)}$, $k = 1, \ldots, K$, $[y] = [x]_{\sigma([x])}$, the distributive relation (24) gets the following form

$$\sum_{k=1}^{K}\left([B^k].[y]_{\sigma([B^k])}\right) = \left(\sum_{k=1}^{K}[B^k]\right).[y]_{\sigma(\sum_{k=1}^{K}[B^k])},$$

where $\sigma([B^k]) = \sigma(A^k)$, $k = 1, \ldots, K$, and $\sigma(\sum_{k=1}^{K}[B^k]) = \sigma(A(\check{p}))$.

PROOF. Consider the right-hand side of (24).

$$\begin{aligned}
\sum_{k=1}^{K}[A^k]_{\sigma([x])\sigma(A^k)} &= \left(\sum_{k=1}^{K}[A^k]_{\sigma(A^k)}\right)_{\sigma([x])} \\
&\overset{(22)}{=} \left(\sum_{k=1}^{K}\left(\check{p}_k A^k\right) + \left[\mp\sum_{k=1}^{K}\left(\hat{p}_k A^k\right)\right]\right)_{\sigma([x])} \\
&= A(\check{p}) + \left[\mp\sum_{k=1}^{K}\left(\hat{p}_k A^k\right)\right]_{\sigma([x])}.
\end{aligned}$$

The latter implies that $\sigma\left(\sum_{k=1}^{K}[A^k]_{\sigma([x])\sigma(A^k)}\right) = \sigma\left(A(\check{p})\right)$. Then, by the transition formula (18), the right-hand side of (24) is equivalent to

$$\left(A(\check{p}) + \left[\mp\sum_{k=1}^{K}\left(\hat{p}_k A^k\right)\right]_{\sigma([x])}\right).[y]_{\sigma(A(\check{p}))} = \left(A(\check{p}) + \left[\mp\sum_{k=1}^{K}\left(\hat{p}_k A^k\right)\right]_{\sigma([x])}\right)_{\sigma([y])} \cdot_h [y]_{\sigma(A(\check{p}))\sigma(A(\check{p}))},$$

which is equivalent to

$$\left(A(\check{p}) + \left[\mp\sum_{k=1}^{K}\left(\hat{p}_k A^k\right)\right]\right) \cdot_h [y] = \left[\left(A(\check{p}) - \sum_{k=1}^{K}\left(\hat{p}_k A^k\right)\right).y^-,\ \left(A(\check{p}) + \sum_{k=1}^{K}\left(\hat{p}_k A^k\right)\right).y^+\right].$$

The latter interval matrix is exactly the matrix in (21) to which the left-hand side of (24) evaluates in the proof of Theorem 1. □

With Proposition 1 and the notation $[y] = [x]_{\sigma([x])}$, the interval system of linear equilibrium equations (14) is equivalent to the system

$$[A]_{\sigma([y])}.[y]_{\sigma([A])} = F.[q]_{\sigma(F)}, \tag{25}$$

wherein $[A] \in (\mathbb{KR} \setminus \mathcal{T})^{n\times n}$ is defined in (20), $\sigma([y]) = \sigma([x])$, $\sigma([A]) = \sigma(A(\check{p}))$. With the additional notation $[B] := [A]_{\sigma([y])}$, where $\sigma([B]) = \sigma([A])$, and $[b] = F.[q]_{\sigma(F)}$, equation (25) is equivalent to

$$[B].[y]_{\sigma([B])} = [b]. \tag{26}$$

The formal (algebraic) solution to such a system (25) or system (26) in $\mathbb{KR}$ has not been considered so far. The system (25) is different from the systems considered by S. Shary in [18] and elsewhere because the formal solution $[y]$ needs a different quantification for each equation. Denote by $[B^{(+)}]$ and $[B^{(-)}]$ the nonnegative, respectively nonpositive, sub-matrices of $[B]$. Then, the equation (26) gets the form

$$[B^{(+)}].[y] + [B^{(-)}].[y]_- = [b]. \tag{27}$$

The latter system presents a special case of the general normal form of a system of m pseudo-linear interval equations in n variables presented in [15, end of Section 3]. Thus, the algebraic interval model of linear equilibrium equations of truss structures involving interval parameters presents a real-life application of this general class of pseudo-linear interval equations in $\mathbb{KR}$. Although [15] considers in details the formal solution to all basic types pseudo-linear interval equations in only one variable, above we have used the general approach used therein and confirmed its efficiency. Next, we will find the formal solution to (14) and its equivalent forms by the center-radius approach introduced in [14].

4.2. Formal Solution in Center-Radius Form

Theorem 2. *The formal (algebraic) solution $[x] \in (\mathbb{KR} \setminus \mathcal{T})^n$ to the equilibrium equations (14) is determined by*

$$[x] = [y]_{\sigma([x])} = \check{y} + [-\sigma(\check{y})\hat{y}, \sigma(\check{y})\hat{y}]$$

and the solution of the real-valued equation

$$\begin{pmatrix} A(\check{p}), & \sum_{k=1}^{K} \hat{p}_k A^k \\ \sum_{k=1}^{K} \hat{p}_k A^k, & A(\check{p}) \end{pmatrix} . \begin{pmatrix} \check{y} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} F.\check{q} \\ F.\hat{q} \end{pmatrix}. \tag{28}$$

Proof. Applying formulae (9) and (6) to the equation (15) we get the following subsequent equivalent equations

$$\begin{aligned}
\sum_{k=1}^{K}\left(\left(\check{p}_k A^k + [\mp(\hat{p}_k\sigma([x])|A^k|)]\right).[y]\right)_{\sigma(A^k)} &= F.\check{q} + [\mp(|F|.\hat{q})]_{\sigma(F)}, \\
\sum_{k=1}^{K}\left(\check{p}_k A^k.\check{y} + \sigma(A^k)\sigma([y])\left(\hat{p}_k\sigma([x])|A^k|\right).\hat{y} + \right. & \\
\left.\left[\mp\left(\hat{p}_k\sigma([x])|A^k|.|\check{y}| + |\check{p}_k A^k|.\hat{y}\right)\right]\right)_{\sigma(A^k)} &= F.\check{q} + [\mp(\sigma(F)|F|.\hat{q})], \\
\sum_{k=1}^{K}\left(\check{p}_k A^k.\check{y} + \hat{p}_k A^k.\hat{y} + \left[\mp\left(\hat{p}_k|A^k|.\check{y} + \check{p}_k|A^k|.\hat{y}\right)\right]\right)_{\sigma(A^k)} &= F.\check{q} + [\mp(F.\hat{q})], \\
\sum_{k=1}^{K}\left(\check{p}_k A^k.\check{y} + \hat{p}_k A^k.\hat{y} + \left[\mp\left(\hat{p}_k A^k.\check{y} + \check{p}_k A^k.\hat{y}\right)\right]\right) &= F.\check{q} + [\mp(F.\hat{q})], \\
\left(\sum_{k=1}^{K}\check{p}_k A^k\right).\check{y} + \left(\sum_{k=1}^{K}\hat{p}_k A^k\right).\hat{y} + & \\
\left[\mp\left(\left(\sum_{k=1}^{K}\hat{p}_k A^k\right).\check{y} + \left(\sum_{k=1}^{K}\check{p}_k A^k\right).\hat{y}\right)\right] &= F.\check{q} + [\mp(F.\hat{q})].
\end{aligned}$$

Since

$$(\check{a} + [\mp\hat{a}]) = (\check{b} + [\mp\hat{b}]) \Leftrightarrow \check{a} = \check{b} \text{ and } \hat{a} = \hat{b}, \tag{29}$$

and $\sum_{k=1}^{K}\check{p}_k A^k = A(\check{p})$, we obtain (28). $\square$

Corollary 2. *The system of interval equilibrium equations (14) has a unique formal (algebraic) solution $[x] \in (\mathbb{KR} \setminus \mathcal{T})^n$ if and only if the left-hand side matrix in (28) is nonsingular.*

One can prove Proposition 1 also by equivalent transformation based on the midpoint-radius representations (9) and (6).

Proposition 2. *The necessary and sufficient conditions for uniqueness of the algebraic solution to system (14), defined by Corollaries 1 and 2, are equivalent.*

Proof. With the notation $\check{A} = \sum_{k=1}^{K}\check{p}_k A^k$, $A(\hat{p}) = \sum_{k=1}^{K}\hat{p}_k A^k$, relation (20) implies

$$A^- = \check{A} - A(\hat{p}), \qquad A^+ = \check{A} + A(\hat{p}).$$

Consider the matrix in (28), which is a block $2n \times 2n$ matrix. Add the first block row of this matrix to its second block row and obtain equivalently

$$\mathrm{mat}((28)) = \begin{pmatrix} \check{A} & A(\hat{p}) \\ A(\hat{p}) & \check{A} \end{pmatrix} = \begin{pmatrix} \check{A} & A(\hat{p}) \\ A(\hat{p}) + \check{A} & \check{A} + A(\hat{p}) \end{pmatrix}.$$

Next, we subtract the second block column of the last matrix from its first block column and obtain equivalently

$$\begin{pmatrix} \check{A} & A(\hat{p}) \\ A(\hat{p}) & \check{A} \end{pmatrix} = \begin{pmatrix} \check{A} - A(\hat{p}) & A(\hat{p}) \\ 0 & \check{A} + A(\hat{p}) \end{pmatrix}.$$

It is well-known from linear algebra that the above transformations do not change the regularity/singularity property of the matrix. Since $\det(\mathrm{mat}((28))) = \det(A^-)\det(A^+)$, the proof is completed. $\square$

Definition 2. A parametric matrix $A(p)$ is regular (nonsingular) on the interval box $\mathbf{p}$ if $A(\tilde{p})$ is regular (nonsingular) for each $\tilde{p} \in \mathbf{p}$.

Proposition 3. *If $A(p)$ is regular on $\mathbf{p}$, then equilibrium equations (14) have a unique algebraic (formal) solution for any $\mathbf{p}' \subseteq \mathbf{p}$, such that $\check{p}' = \check{p}$ and $0 \leq \hat{p}' \leq \hat{p}$, and for any $F.[q]_{\sigma(F)}$.*

Proof. Regularity (nonsingularity) of $A(p)$ on $\mathbf{p}$ implies regularity (nonsingularity) of $\check{A} - \sum_{k=1}^{K}(\hat{p}_k - \tilde{p})A^k$ and $\check{A} + \sum_{k=1}^{K}(\hat{p}_k - \tilde{p})A^k$ for every $\tilde{p}$, $0 \leq \tilde{p} \leq \hat{p}$. Then, the proof follows from the representation $\begin{pmatrix} \check{A} & A(\hat{p} - \tilde{p}) \\ A(\hat{p} - \tilde{p}) & \check{A} \end{pmatrix}$ and Theorem 1 or 2. $\square$

4.3. Implementation

Let $A(p)u = f(q)$ be the deterministic parametric model of a truss structure, where $A(p) = \sum_{k=0}^{K} p_k A^k$, $f(q) = \sum_{i=0}^{m} q_k f_i$, and $\mathbf{p} \in \mathbb{IR}^{K+1}$, $\mathbf{q} \in \mathbb{IR}^{m+1}$ be the intervals for the uncertain parameters in the model, cf. Section 3.1.

Algorithm 1. *Intervals for the displacements according to the interval algebraic model of truss structures and Theorem 1.*
Input*: matrices $A^k \in \mathbb{R}^{n\times n}$, $k = 0, \ldots, K$,*
vectors $f_i \in \mathbb{R}^{n\times m}$, $i = 0, \ldots, m$,
intervals $\mathbf{p}_0 = [1, 1]$, $[p_k^-, p_k^+]$, $k = 1, \ldots, K$,
$\mathbf{q}_0 = [1, 1]$, $[q_i^-, q_i^+]$, $i = 1, \ldots, m$.
Output*: interval vector $\mathbf{u} = [u^-, u^+] \in \mathbb{IR}^n$ of the unknown displacements, or the message "The parametric interval matrix may not be regular.".*

1. **Generate** *the crisp matrices $A^- = \sum_{k=0}^{K} p_k^- A^k$ and $A^+ = \sum_{k=0}^{K} p_k^+ A^k$, as well as the crisp vectors $a^- = \sum_{i=0}^{m} q_i^- f_i$ and $a^+ = \sum_{i=0}^{m} q_i^+ f_i$;*
2. **Compute** *y^- and y^+ by solving $A^-.y^- = a^-$ and $A^+.y^+ = a^+$;*
3. **If** *solving some of the systems in step 2. fails,* **then** *Message "The parametric interval matrix may not be regular.";* **Exit***;*
4. **else**
   4.1 **for** $i = 1$ *to* n **do**
   If $y_i^- > y_i^+$, **then** $\mathbf{u}_i = [y_i^+, y_i^-]$
   else $\mathbf{u}_i = [y_i^-, y_i^+]$;
   4.2 **Return***: $\mathbf{u}$.*

Algorithm 2. *Intervals for the displacements according to the interval algebraic model of truss structures and Theorem 2.*
Input: *matrices* $A^k \in \mathbb{R}^{n\times n}$, $k = 0, \ldots, K$,
vectors $f_i \in \mathbb{R}^{n\times m}$, $i = 0, \ldots, m$,
intervals $\mathbf{p}_0 = [1, 1]$, $[p_k^-, p_k^+]$, $k = 1, \ldots, K$,
$\mathbf{q}_0 = [1, 1]$, $[q_i^-, q_i^+]$, $i = 1, \ldots, m$.
Output: *interval vector* $\mathbf{u} = [u^-, u^+] \in \mathbb{IR}^n$ *of the unknown displacements, or the message "The parametric interval matrix may not be regular.".*

1. $\check{A} = \sum_{k=0}^{K} \frac{p_k^- + p_k^+}{2} A^k$; $\hat{A} = \sum_{k=0}^{K} \frac{p_k^+ - p_k^-}{2} A^k$;
   $\check{q} = \sum_{i=0}^{m} \frac{q_i^- + q_i^+}{2} f_i$; $\hat{q} = \sum_{i=0}^{m} \frac{q_i^+ - q_i^-}{2} f_i$;
2. **Compute** $\check{y}$, $\hat{y}$ *by solving the linear system* $\begin{pmatrix} \check{A}, & \hat{A} \\ \hat{A}, & \check{A} \end{pmatrix} \cdot \begin{pmatrix} \check{y} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} \check{q} \\ \hat{q} \end{pmatrix}$;
3. **If** *solving the system of step 2. fails,* **then** *Message "The parametric interval matrix may not be regular.";* **Exit**;
4. **else**
   4.1 **for** $i = 1$ *to* n **do**
   If $\mathrm{sign}(\check{y}_i) = \mathrm{sign}(\hat{y}_i)$, **then** $\mathbf{u}_i = \check{y}_i + [-\mathrm{sign}(\check{y}_i)\hat{y}_i, \mathrm{sign}(\check{y}_i)\hat{y}_i]$
   else $\mathbf{u}_i = [\check{y}_i + \mathrm{sign}(\check{y}_i)\hat{y}_i, \check{y}_i - \mathrm{sign}(\check{y}_i)\hat{y}_i]$;
   4.2 **Return**: $\mathbf{u}$.

The implementation of Algorithms 1 and 2 do not require Kaucher interval arithmetic. It may not require classical interval arithmetic, too, if the input parameter intervals and the output interval vector are represented by their lower $(\cdot)^-$ and upper $(\cdot)^+$ endpoints (endpoint vectors).

In the models of truss structures the parameter coefficient matrices A^k, $1 \le k \le K$, have sparse structures and rank one. A memory efficient implementation of Algorithms 1 and 2 should use these properties. The computational complexity of Algorithms 1 and 2 is determined by their step two. It is well known from linear algebra that the complexity of solving a linear algebraic system in floating point arithmetic depends on the size n of the system. The complexity is at most $O(n^3)$ and at least $O(n^2)$ depending on the method and the implementation. Since $O(2n^3) \ll O((2n)^3)$, Algorithm 1 is much more efficient (computationally and in memory demand) than Algorithm 2, especially for large number of equilibrium equations. We recommend the implementation of Algorithms 1 and 2 in software environments which support rigorously implemented floating point interval arithmetic and methods with automatic result verification, see e.g., [5, Sect. 3.3–3.4] and `http://cs.utep.edu/interval-comp/intsoft.html`. In such environments, step two of the algorithm should be implemented by calling a corresponding function that proves nonsingularity of the system matrix and provides guaranteed bounds for the solution vector. With much less computational $O(n^3)$ and memory demand compared to a linear system solving in exact arithmetic, linear system solvers with automatic result verification will provide guaranteed results.

4.4. Keeping a Complete Equilibrium

Let the equilibrium equations (14) have a unique formal (algebraic) solution $[x] \in (\mathbb{KR} \setminus \mathcal{T})^n$

$$[x] = \check{y} + [-\sigma(\check{y})\hat{y}, \sigma(\check{y})\hat{y}]\,. \tag{30}$$

According to the interpretation convention, we have to take its proper projection pro$([x])$ for the range of the unknown displacement components. If $[x]$ is a proper interval vector, $[x] \in (\mathbb{IR} \setminus \mathcal{Z})^n$, then the distributive relation of Proposition 1 holds true and the equilibrium equations (14) are completely satisfied in exact arithmetic. In (30), a component $[x]_i$ is a proper interval iff

$$\sigma\left(\hat{y}_i\right) = \sigma\left(\check{y}_i\right). \tag{31}$$

A component $[x]_i$ in (30) is an improper interval iff[3]

$$\sigma\left(\hat{y}_i\right) <^s 0 \leq^s \sigma\left(\check{y}_i\right), \quad s = \sigma(\check{y}_i). \tag{32}$$

If for some $i = 1, \ldots, n$ (31) does not hold true, we can change the uncertainty (radius) of some interval parameters $[p_1], \ldots, [p_K]$, $[q_1], \ldots, [q_m]$ in the algebraic model of equilibrium equations, so that (31) holds true for each $i = 1, \ldots, n$. In this section we present some techniques for obtaining a proper interval vector as algebraic solution to interval equilibrium equations. For simplicity, F in (14) will be considered as one column matrix.

Theorem 3. *Let for given parameter intervals* $\mathbf{p}$, $\mathbf{q}$, *the equilibrium equations (14) have a unique formal (algebraic) solution* $[x] = [y]_{\sigma(\check{y})} \in (\mathbb{KR} \setminus \mathcal{T})^n$ *and* $[x]_i$ *be the only improper interval component. The equation*

$$\tilde{q}\left(((A^+)^{-1})_{i\bullet}.F + ((A^-)^{-1})_{i\bullet}.F\right) = y_i^- - y_i^+$$

has a unique solution $\tilde{q}$, *which is the minimal positive* $\tilde{q}$, *providing that the algebraic solution to (14), with* $\mathbf{p}$, $\mathbf{q'} = \mathbf{q} + [-\tilde{q}, \tilde{q}]$, *is proper interval vector.*

Proof. Consider system (19). It is equivalent to

$$\begin{aligned} y^- &= (\check{q} - \hat{q})(A^-)^{-1}.F, \\ y^+ &= (\check{q} + \hat{q})(A^+)^{-1}.F. \end{aligned} \tag{33}$$

Let $[y_i]_{\sigma([y_i])}$ be improper interval. By (31) or (32) this is equivalent to

$$y_i^+ - y_i^- <^s 0 <^s y_i^+ + y_i^-, \qquad s = \sigma([y_i]). \tag{34}$$

For the parameter q we consider a new interval $q \in \mathbf{q'} = \mathbf{q} + [-\tilde{q}, \tilde{q}]$. Since the matrix is unchanged, there exists a unique algebraic solution $[u]_{\sigma([u])}$ to (14) with the new interval for q

$$\begin{aligned} u^- &= (\check{q} - \hat{q})(A^-)^{-1}.F - \tilde{q}(A^-)^{-1}.F = y^- - \tilde{q}(A^-)^{-1}.F, \\ u^+ &= (\check{q} + \hat{q})(A^+)^{-1}.F + \tilde{q}(A^+)^{-1}.F = y^+ + \tilde{q}(A^+)^{-1}.F. \end{aligned}$$

[3]For $s \in \{-, +\}$, $<^s$ reads $\begin{cases} < & \text{if } s = +, \\ > & \text{otherwise.} \end{cases}$ Similarly for $\leq, >, \geq$.

We search for a $\tilde{q} > 0$, which provides $u_i^+ - u_i^- = 0$ in (32). Replacing u_i^+, u_i^- by their equivalent from above, we obtain the equation

$$\tilde{q}\left(((A^+)^{-1})_{i\bullet}.F + ((A^-)^{-1})_{i\bullet}.F\right) = y_i^- - y_i^+.$$

From (33) and (34) it follows that $y_i^- - y_i^+ >^s 0$ and $0 <^s ((A^+)^{-1})_{i\bullet}.F + ((A^-)^{-1})_{i\bullet}.F$. This implies that the above equation has a unique solution

$$\tilde{q} = \frac{y_i^- - y_i^+}{((A^+)^{-1})_{i\bullet}.F + ((A^-)^{-1})_{i\bullet}.F}.$$

As required, we have $[u_i] = y_i^- - \tilde{q}((A^-)^{-1})_{i\bullet}.F = y_i^- + \tilde{q}((A^+)^{-1})_{i\bullet}.F$, $[u_i] \in \mathbb{R}$. Since the vectors $(A^-)^{-1}.F$ and $(A^+)^{-1}.F$ are the same for both y^-, u^- and y^+, u^+, respectively, we have the inclusion relation

$$[y_j]_{s_j} \subseteq^{s_j} [u_j]_{s_j}, \qquad s_j = \sigma([y_j]) \tag{35}$$

for each component $1 \le j \le n$. The latter ensures that $[u]_{\sigma([u])}$ is a proper interval vector. It is obvious that any $\tilde{q}' > \tilde{q}$ will increase the width of $[u_i]$ and will retain the inclusion (35). $\square$

Theorem 4. *Let $A(p)$ be regular on $\mathbf{p}$. Let for the parameter intervals $\mathbf{p}$, $\mathbf{q}$, the unique formal (algebraic) solution $[x]$ to equilibrium equations (14) have improper intervals at the components $[x]_i$, $i \in \mathcal{I} \subseteq \{1, \ldots, n\}$. For* $\mathrm{Card}(\mathcal{I}) \le K$ *prescribed interval parameters p_k, $1 \le k \le K$, define $\tilde{p} = (\tilde{p}_1, \ldots, \tilde{p}_K)^\top$, where $0 \le \tilde{p}_k \le \hat{p}_k$ and $\tilde{p}_k = 0$ if p_k is not among the prescribed parameters.*

The following system of polynomial equations and inequalities

$$\begin{pmatrix} A(\check{p}), & \sum_{k=1}^K (\hat{p}_k - \tilde{p}_k)A^k \\ \sum_{k=1}^K (\hat{p}_k - \tilde{p}_k)A^k, & A(\check{p}) \end{pmatrix} \cdot \begin{pmatrix} \check{y} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} F.\check{q} \\ F.\hat{q} \end{pmatrix},$$
$$0 \le \tilde{p}_k \le \hat{p}_k, \qquad 1 \le k \le K,$$

where $\hat{y}_i = 0$ for $i \in \mathcal{I}$, has a unique solution with respect to $2n$ unknowns $\check{y}$, $\hat{y}_j$, $j \notin \mathcal{I}$, $\tilde{p}_k \neq 0$. This solution provides the maximal positive $\tilde{p}$, such that $[x] = (\check{y} + [-\hat{y}, \hat{y}])_{\sigma(\check{y})}$ is proper interval vector that completely satisfies the equilibrium equations (14) for the new intervals $\mathbf{p'} = \check{p} + [-(\hat{p} - \tilde{p}), \hat{p} - \tilde{p}]$, $\mathbf{q}$.

Proof. The existence and uniqueness follow from Proposition 3. Relation (31) and maximality of $\tilde{p}$ are provided by the requirement $\hat{y}_i = 0$ for $i \in \mathcal{I}$. $\square$

For the sake of simplified notation, Theorem 4 is specified only for the parameters p. Theorems 3 and 4 can be joined if $\mathrm{Card}(\mathcal{I}) \ge 2$. Furthermore, applying Theorem 4 one can change more than $\mathrm{Card}(\mathcal{I})$ parameters p assuming that $\tilde{p}$ is the same for more than one parameter p_k.

The joint of Theorems 3 and 4 could be used for solving some design problems. Following the Taguchi's ideas [21] for robust engineering design, we may consider the interval parameters involved in the algebraic interval model of equilibrium equations as belonging to two classes:

- *noise* interval parameters whose interval uncertainty we do not want to change (or cannot) in the model, and
- *control* interval parameters whose interval uncertainty can be controlled during the manufacturing process.

Then, we can combine the algebraic interval model of linear equilibrium equations with Theorems 3, 4 in solving the following design problem of truss structures involving interval uncertainties.

> *Find what is the maximal allowed uncertainty of specified control model parameters p, so that, for the given noise interval parameters, we can find the range of the unknown displacements and the interval equilibrium equations of the model with these intervals be completely satisfied.*

Some optimization and anti-optimization techniques of structures under interval uncertainty are presented in [2]. While the application of the above theory to particular design problems of truss structures requires consideration in a separate paper, the interval algebraic model of equilibrium equations can be used at least for checking how realistic are the parameter interval uncertainties chosen for a given model of structure; in other words for checking compatibility of the parameter uncertainties. This will be illustrated in the next section.

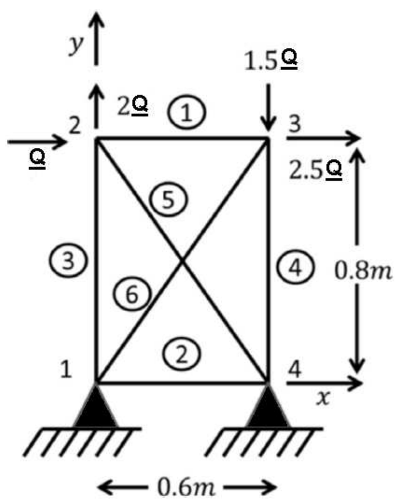

Figure 1: A 6-bar truss structure after [22].

5. Numerical Examples

The first two examples of this section are based on a truss construction (Fig. 1) of small size with three interval parameters chosen so that the reader can easily understand the interval algebraic model, to reproduce the computations and to comprehend the new ideas presented in the article. A cantilever truss, as shown in Figure 2, is chosen as a third example involving 121 interval parameters. This truss structure is proposed in [23] as a benchmark problem for testing the applicability, computational efficiency and scalability of the numerical approaches applied to structures with complex configuration and a large number of interval parameters.

Most of the computations are done in exact arithmetic in order to avoid the influence of round-off errors. The exact results are rounded appropriately and presented by floating-point numbers with a big number of digits so that the reader can follow the computations and verify that the equilibrium equations are satisfied, that is

$$\sum_{k=1}^{K} \left(\left([p_k]A^k\right).[x]\right)_{\sigma([x])\sigma(A^k)} - F.[q]_{-\sigma(F)} = [0,0].$$

Consider a 6-bar truss structure as presented in Fig. 1 after [22]. The structure consists of 6 elements. The crisp values of the parameters of the truss are presented in Table 1.

Parameter	Value
Modulus of elasticity for all elements E_i, $i = 1, \ldots, 6$ (kN/m^2)	2.1×10^8
Cross sectional area A_1, A_2, A_3, A_4 (m^2)	1.0×10^{-3}
Cross sectional area A_5, A_6 (m^2)	1.05×10^{-3}
Load Q (kN)	20.5
Length of the first and second element L_1, L_2 (m)	0.6
Length of the third and fourth element L_3, L_4 (m)	0.8
Length of the fifth and sixth element L_6, L_6 (m)	1

Table 1: Crisp values of the parameters for the 6-bar truss structure.

The traditional finite element method (FEM) for this structure leads to a linear system

$$K(E, A, L)u = f(Q),$$

where $K(E, A, L)$ is the reduced stiffness matrix depending on the structural parameters (modulus of elasticity, cross sectional area, length) for each element,

$f(Q)$ is the load vector and u is the displacement vector. Namely,

$$K(E,A,L) = \begin{pmatrix} \frac{E_1A_1}{L_1}+0.36\frac{E_5A_5}{L_5} & -0.48\frac{E_5A_5}{L_5} & -\frac{E_1A_1}{L_1} & 0 \\ -0.48\frac{E_5A_5}{L_5} & \frac{E_3A_3}{L_3}+0.64\frac{E_5A_5}{L_5} & 0 & 0 \\ -\frac{E_1A_1}{L_1} & 0 & \frac{E_1A_1}{L_1}+0.36\frac{E_6A_6}{L_6} & 0.48\frac{E_6A_6}{L_6} \\ 0 & 0 & 0.48\frac{E_6A_6}{L_6} & \frac{E_4A_4}{L_4}+0.64\frac{E_6A_6}{L_6} \end{pmatrix},$$

$$f(Q) = (Q, 2Q, 2.5Q, -1.5Q)^\top, \qquad u = (ux_2, uy_2, ux_3, uy_3)^\top. \quad (36)$$

With crisp values for all the parameters, the horizontal and vertical displacements at nodes 2 and 3 are presented in Table 2. In [22] the corresponding values are presented with 2 digits after the decimal point.

Displacement (m)	value $\times 10^4$
ux_2	8.584575731452372
uy_2	3.266913174314749
ux_3	8.957928677718704
uy_3	−3.110864603463029

Table 2: Horizontal and vertical displacements of the 6-bar truss structure with crisp values for all the parameters.

Example 1. Consider the 6-bar truss structure presented in Fig. 1, where the force parameter Q is unknown-but-bounded in the interval $\mathbf{Q} = [20, 21]kN$ and the cross sectional areas A_5, A_6 are also uncertain varying in the intervals $[1.008, 1.092] \times 10^{-3}\ m^2$, $[1, 1.1] \times 10^{-3}\ m^2$, respectively. Find interval bounds for the unknown displacement components by the classical interval model and by the algebraic interval model (14).

Renaming the interval parameters as $p_1 = 1$, $p_2 = A_5$, $p_3 = A_6$, so that $p = (p_1, p_2, p_3)^\top \in \mathbf{p} = ([1, 1], [1.008, 1.092]\times 10^{-3}, [1, 1.1]\times 10^{-3})^\top$, the classical interval model considers the parametric interval linear system

$$A(p)u = f(Q), \qquad p \in \mathbf{p}, Q \in \mathbf{Q}$$

and searches for the interval hull of the united parametric solution set $\Sigma(A(p), f(Q), \mathbf{p}, \mathbf{Q})$, where $f(Q)$ and u are defined in (36), and $A(p) = A^1p_1 +$

$A^2 p_2 + A^3 p_3$ with

$$A^1 = 10^8 \begin{pmatrix} 3.5 & 0 & -3.5 & 0 \\ 0 & 2.625 & 0 & 0 \\ -3.5 & 0 & 3.5 & 0 \\ 0 & 0 & 0 & 2.625 \end{pmatrix}, \; A^2 = 10^8 \begin{pmatrix} 756 & -1008 & 0 & 0 \\ -1008 & 1344 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A^3 = 10^8 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 756 & 1008 \\ 0 & 0 & 1008 & 1344 \end{pmatrix}.$$

Since the considered parametric interval system is a small one, any of the known methods for finding the exact interval hull of its parametric united solution set can be applied. The exact ranges for the unknown displacements that correspond to the classical interval model are presented in Table 3.

Displacement (m)	value $\times 10^4$
$\mathbf{ux}_2$	[8.191069812880123, 9.005105588841962]
$\mathbf{uy}_2$	[3.1401416045511555, 3.3968465038536545]
$\mathbf{ux}_3$	[8.551466199186033, 9.391967390236717]
$\mathbf{uy}_3$	−[2.9871303137901704, 3.23618464855462]

Table 3: According to classical interval model, intervals for the unknown displacements of the 6-bar truss structure with interval parameters defined in Example 1.

According to the representation convention in Section 3, the algebraic interval model (14) of the equilibrium equations for the considered truss structure is

$$\begin{aligned} 3.5\,\mathbf{ux}_2 + 756\,\mathbf{p}_2\mathbf{ux}_2 - 1008\,\mathrm{dual}(\mathbf{p}_2\mathbf{uy}_2) - 3.5\,\mathrm{dual}(\mathbf{ux}_3) &= 10^{-8}\mathbf{Q}, \\ -1008\,\mathrm{dual}(\mathbf{p}_2\mathbf{ux}_2) + 2.625\,\mathbf{uy}_2 + 1344\,\mathbf{p}_2\mathbf{uy}_2 &= 10^{-8}2\mathbf{Q}, \\ -3.5\,\mathrm{dual}(\mathbf{ux}_2) + 3.5\,\mathbf{ux}_3 + 756\,\mathbf{p}_3\mathbf{ux}_3 + 1008\,\mathrm{dual}(\mathbf{p}_3\mathbf{uy}_3) &= 10^{-8}\frac{25}{10}\mathbf{Q}, \\ 1008\,\mathbf{p}_3\mathbf{ux}_3 + 2.625\,\mathrm{dual}(\mathbf{uy}_3) + 1344\,\mathrm{dual}(\mathbf{p}_3\mathbf{uy}_3) &= -10^{-8}\frac{15}{10}\mathrm{dual}(\mathbf{Q}). \end{aligned} \tag{37}$$

By the numerical methods defined in Theorem 1 or Theorem 2 we find the formal (algebraic) solution to the above equations. Since all solution components are proper intervals, they present the range of the unknown displacements presented in Table 4. Comparing the intervals in Table 3 and in Table 4, it is readily seen that the intervals in Table 4 are much tighter than the intervals in Table 3 and completely satisfy the algebraic interval model of the equilibrium equations.

It should be mentioned that both the classical interval model and the algebraic interval model of a truss structure are mathematically correct models

Displacement (m)	value $\times 10^4$
$\mathbf{ux}_2$	[8.576291036992344, 8.60062330352413]
$\mathbf{uy}_2$	[3.194686647864135, 3.3394065877710314]
$\mathbf{ux}_3$	[8.944730847844491, 8.979039509145336]
$\mathbf{uy}_3$	$-$[3.027535574358087, 3.1939267455623015]

Table 4: Intervals for the unknown displacements of the 6-bar truss structure with interval parameters defined in Example 1, which are obtained by formal solution to the algebraic interval model (14), Theorems 1 and 2.

which differ in their logical interpretation, cf. [8], and the scope of application. The classical model is the major interval model (Section 3.1) applied for worst-case analysis of static responses of structures, see, e.g., [1], [2], [7], [22], [24]. The application of the algebraic interval model depends on the problem physics, when satisfaction of interval equilibrium equations is required, e.g., in some engineering design problems, see Example 4 below.

The best solution of classical interval model is (Section 3.1) the interval hull of the solution set (in exact arithmetic) or a sharp enclosure of it (in floating point arithmetic). Classical interval model of the truss structure in Fig. 1 is considered also in [24]. The approach presented there yields the best solution of classical interval model. This solution is presented in Table 3 for Example 1. The interval hull solution for the displacements of truss structures can be obtained by various other methods, see, e.g., [1], [2], [19], however in general these methods are NP-hard (exponential with respect to the number of interval parameters) and not feasible even for modest structures with many interval parameters.

In [11] the authors consider the classical interval model but not its united solution set, presented in Section 3.1, which corresponds to a worst-case analysis. They search for a proper formal solution to an interval linear system by classical interval methods. The solution method proposed in [11, Theorem 1] is proven in [14]. It is discussed in the Introduction and in Section 3 that classical interval arithmetic cannot always provide proper interval formal solution; presence of improper interval components means no solution. That is why, [11] concludes "it is a question how to interpret the improper intervals from the structural mechanics point of view"; also "In both numerical examples presented, there appear improper intervals in the solutions.". The newly proposed interval algebraic model of equilibrium equations [9], [10] answers the question and the present article presents efficient techniques for obtaining algebraic (formal) solution that completely satisfies the interval equilibrium equations. Examples 2 and 3 below compare the results of [11] and the present approach. It is mentioned in [11] that the two examples considered there "are also solved by Chakraverty and Behera [25] and Friedman et al. [26] (with the same results) substituting $\alpha = 0$ in their methods, as for $\alpha = 0$ these (fuzzy-set) methods are converted to the interval system of linear equations." This implies that the interval algebraic model of equilibrium equations has a potential for application to problems formulated in terms of fuzzy-set theory.

Example 2. Consider the truss structure presented in Fig. 1, where after [11] the force parameter Q is unknown-but-bounded in the interval $\mathbf{Q} = [20, 21]kN$ and all other parameters have the crisp values presented in Table 1. Table 5 presents the results for the unknown displacements obtained by the approach of [11] and by the interval algebraic model and the present methods. Note, that $\mathbf{ux}_2$ and $\mathbf{uy}_3$ in [11, Table 3] are improper intervals, which implies that the problem considered there has no solution. The present approach results in proper intervals for the displacements, which completely satisfy (in exact arithmetic) the interval algebraic model of equilibrium equations.

Displacement (m)	[11, Table 3]	present$\times 10^3$
$\mathbf{ux}_2$	$[0.8655, 0.8514]$	$[0.83751, 0.87940]$
$\mathbf{uy}_2$	$[0.3224, 0.3310]$	$[0.31872, 0.33466]$
$\mathbf{ux}_3$	$[0.8871, 0.9046]$	$[0.87394, 0.91765]$
$\mathbf{uy}_3$	$[-0.3106, -0.3115]$	$-[0.30349, 0.31868]$

Table 5: Intervals for the displacements of the 6-bar truss structure with interval parameters defined in Example 2, which are obtained by the present approach and by that of [11].

Displacement (m)	[11, Table 5]	present$\times 10^6$
$\mathbf{x}_2$	$[9.538, 9.721]10^{-6}$	$[9.43686, 9.82205]$
$\mathbf{y}_2$	$[1.907, 1.552]10^{-6}$	$[1.69525, 1.76445]$
$\mathbf{x}_3$	$[8.634, 8.780]10^{-6}$	$[8.53273, 8.88101]$
$\mathbf{y}_3$	$[-0.364, 2.652]10^{-6}$	$[0.113016, 0.117630]$

Table 6: Intervals for the displacements of the rectangular sheet considered in [11, Example 2], which are obtained by the present approach and by that of [11].

Example 3. For a uniform rectangular sheet considered in [11, Example 2], Table 6 presents the results for the unknown displacements obtained by the approach of [11] and by the interval algebraic model and the present methods. Note, that $\mathbf{y}_2$ in [11, Table 5] is improper interval, which implies that the problem considered there has no solution. Furthermore, the interval $\mathbf{y}_3$ in [11, Table 5] contains zero. The present approach results in proper intervals for the displacements, which completely satisfy (in exact arithmetic) the interval algebraic model of equilibrium equations.

Example 4. Consider the 6-bar truss structure presented in Fig. 1, where the force parameter Q is unknown-but-bounded in the interval $\mathbf{Q} = [20, 21]kN$ and the cross sectional areas $A_5 = p_2$, $A_6 = p_3$ are also uncertain varying in the intervals $\mathbf{p}_2 = [1, 1.1] \times 10^{-3}\ m^2$, $\mathbf{p}_3 = [1, 1.1] \times 10^{-3}\ m^2$, respectively.

The algebraic interval model of this example has the same explicit form as the corresponding model in Example 1. The difference is in the intervals for the parameters. The intervals $\mathbf{p}_2, \mathbf{p}_3$ have approximately 4.76% relative uncertainty, that is $\mathbf{p}_i = \check{p}_i + 4.76 \times 10^{-2}\check{p}_i[-1,1]$, $i = 2, 3$. Find interval bounds for the unknown displacement components by the algebraic interval model (14).

With the specified interval values, both Theorem 1 and Theorem 2 give the same formal solution to (14), which is presented in Table 7.

displacement component (m)	10^4 center	10^7 radius
$[ux_2]$	8.589591492754394	−5.117501954600647
$[uy_2]$	3.266894188981651	76.30042061010485
$[ux_3]$	8.962933759770859	2.0871989242976163
$[uy_3]$	−3.1108835887961266	79.25513494545071

Table 7: Formal (algebraic) solution to the equilibrium equations (14) with interval parameters defined in Example 4.

Due to the negative radius, $[ux_2]$ is an improper interval. According to the interpretation convention (Section 3), for the range of ux_2 we have to take $\mathrm{pro}([ux_2]) = \mathrm{mid}([ux_2]) + [-r, r]$, where $r = -\mathrm{rad}([ux_2])$ and $[ux_2]$ is in Table 7. With the obtained proper intervals (ranges) for the uncertain displacements, the equilibrium equations (37) are not completely satisfied. This motivates us to study what uncertainty in interval parameters provides a complete satisfaction of the interval algebraic model of equilibrium equations.

For this example we have only one component $[ux_2]$ of $[y]$, which is improper interval. By Theorem 3 we find what is the minimal uncertainty in Q that provides proper algebraic solution to (37). Solving the equation in Theorem 3 we find $\tilde{q} = 894764 \times 10^3/73305047$ and a new exact interval $Q \in \mathbf{Q} + [-\tilde{q}, \tilde{q}]$. Applying Theorem 1 with the new interval parameter we obtain a new algebraic solution, which is a proper interval vector and completely satisfies the equilibrium equations (37).

Note, that in floating-point arithmetic $\tilde{q}$ must be rounded upwardly in order to provide the required properties. Let the above exact value of $\tilde{q}$ be rounded upwardly to $\tilde{q} \approx 12.2061$. For the new parameter interval $Q \in \mathbf{Q'} = [19987.7938, 21012.2061]$, Theorem 1 yields the algebraic solution

$$[y] = 10^{-4}\left([8.58946355786, 8.58946365499], [3.18864651708, 3.34513781758],\right.$$
$$\left.[8.95537766658, 8.9702318131], [-3.02977822069, -3.19199296909]\right)^{\top},$$

and $[y]_{\sigma([y])}$ is proper interval vector. Note that $[y_1]$ is very close but not a degenerate (point) interval due to the rounded $\tilde{q}$ and $\mathbf{Q'}$. Evaluating left-hand side of (37) minus dual of the right-hand side with the intervals $\mathbf{p}_2, \mathbf{p}_3$, the new interval for Q and the obtained $[y]_{\sigma([y])}$ for the unknown displacements, we obtain an interval vector $[-\epsilon, \epsilon] \in \mathbb{KR}^4$, where $|\epsilon| < 3.3 \times 10^{-7}$ is due to the above rounded results.

By Theorem 4 we find what is the maximal uncertainty in p_1, p_2 that provides proper algebraic solution to (37). Solving the polynomial system of Theorem 4 with the constraint $0 \le \tilde{p} \le 1/20000$ we obtain $\tilde{p} \approx 1.18902619344885 \times 10^{-6}$ and approximate values for the new algebraic solution $\check{y}' + [-\hat{y}', \hat{y}']$. In order to verify the equilibrium equations (37) in exact arithmetic, one can represent the obtained $\tilde{p}$ as a rational number. This will give new exact intervals $p_i \in \check{p}_i + [\mp(\hat{p}_i - \tilde{p})]$, $i = 1, 2$. With these exact data Theorem 1 yields exact bounds for the unknown displacements and the complete satisfaction of the equilibrium equations (37) can be verified.

The interval algebraic model of equilibrium equations can be used (as above) for checking how realistic are the parameter interval uncertainties chosen for a given model of structure. In Example 4 there is no compatibility between the parameter uncertainties. The assumed level of uncertainty in the two interval parameters A_5 and A_6 is higher than the uncertainty in the external load Q. The obtained improper interval for the displacement component ux_2 shows that the uncertainty in $\mathbf{Q}$ cannot compensate the uncertainties in $\mathbf{A}_5$, $\mathbf{A}_6$, so that the equilibrium be reached by proper intervals for the displacements.

Example 5. Consider a finite element model of a one-bay 20-floor truss cantilever presented in Fig. 2, after [23]. The structure consists of 42 nodes and 101

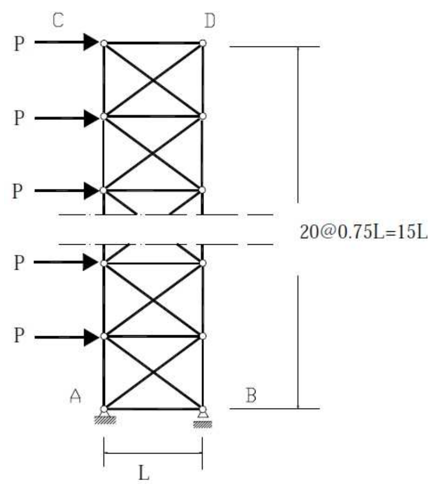

Figure 2: One-bay 20-floor truss cantilever after [23].

elements. The bay is $L = 1$m, every floor is $0.75L$, the element cross-sectional area is $A = 0.01$ m^2, and the crisp value for the element Young modulus is $E = 2 \times 10^{11}$N/m^2. Twenty horizontal loads with nominal value $P = 1000$ N are applied at the left nodes. The boundary conditions are determined by the supports: at A the support is a pin, at B the support is roller. We assume δ_E% uncertainty in the modulus of elasticity E_k of each element, that is $E_k \in E[1 - \delta_E/2, 1 + \delta_E/2]$, $k = 1, \ldots, 101$, and δ_P% uncertainty in the loads, that is $P_m \in P[1-\delta_P/2, 1+\delta_P/2]$, $m = 1, \ldots, 20$. The goal is to find interval estimate for the normalized (nondimensional) displacements U, that is $U\frac{EA}{PL}$, by the interval algebraic model.

Computing the formal solution of the algebraic interval model, it was observed that for any uncertainty $\delta_E = \delta_P$ applied to all interval parameters $k = 1, \ldots, 101$, $m = 1, \ldots, 20$, the formal (algebraic) solution coincides with the crisp midpoint solution. For $\delta_E > \delta_P$ the formal solution is improper interval for all solution components. For $\delta_E < \delta_P$ the formal solution is proper interval vector and the equilibrium equations are completely satisfied. Varying the level of uncertainty at the different E_k, $k = 1, \ldots, 101$, and/or the different P_m, $m = 1, \ldots, 20$, we may obtain improper intervals at some components of the formal solution, which was illustrated in the previous example.

For $\delta_E = 5\%$, $k = 1, \ldots, 101$, and $\delta_P = 10\%$, $k = 1, \ldots, 20$, the normalized horizontal and vertical displacements at the right upper corner (node D) of the truss are

$$U_{D,x} \in [17741.5, 18652.6], \qquad U_{D,y} \in -[785.978, 826.337].$$

In contrast to the classical interval model, the obtained algebraic displacements together with parameter intervals completely satisfy the interval equilibrium equations.

6. Conclusion

Considered is the interval algebraic model of equilibrium equations to statically indeterminate truss structures.

We presented two approaches to finding a formal (algebraic) solution to this class of interval equilibrium equations. The first approach is based on the transition formulae [15, Eqn (4)] between the interval operations in the space of proper and improper intervals $\mathbb{KR}$ and the so-called hyperbolic interval operations [20]. The second approach is based on center-radius representation of interval operations in $\mathbb{KR}$, [14]. Due to the conditionally distributive relations in $\mathbb{KR}$ (cf. [13]), it was proven that the considered parameter-dependent interval equilibrium equations are equivalent to a general class of nonparametric interval linear system of equations in $\mathbb{KR}$, formulated in [15]. The latter means that the interval algebraic model of truss equilibrium equations gives a practical justification of this general class interval linear systems, whose formal solution has not been considered so far.

The numerical method, based on Theorem 1 is a scalable one. It can be used for finding formal solutions to very large systems of equilibrium equations involving many interval parameters, what are the interval models of real-life truss structures.

The two techniques used in Section 4 of this article, one based on the hyperbolic interval operations and the other based on center-radius representation of interval operations in $\mathbb{KR}$, could be also applied to any algebraic interval model of linear equilibrium equations which involves linear dependencies between interval parameters.

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