

EXPLICIT DESCRIPTION OF *AE* SOLUTION SETS FOR PARAMETRIC LINEAR SYSTEMS*

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Abstract. Consider linear systems whose input data are linear functions of uncertain parameters varying within given intervals. We are interested in an explicit description of the so-called *AE* parametric solution sets (where all universally quantified parameters precede all existentially quantified ones) by a set of inequalities not involving the parameters. This work presents how to obtain explicit description of *AE* parametric solution sets by combining a modified Fourier–Motzkin type elimination of existentially quantified parameters with the elimination of the universally quantified parameters. Some necessary (and sufficient) conditions for existence of nonempty *AE* parametric solution sets are discussed, as well as some properties of the parametric *AE* solution sets, e.g., shape of the solution set and some inclusion relations. Explicit descriptions of particular classes of *AE* parametric solution sets (tolerable, controllable, any two-dimensional) are given. Numerical examples illustrate the solution sets and their properties.

Key words. linear systems, dependent data, *AE* solution set, tolerable solution set, controllable solution set

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1. Introduction. Consider a linear algebraic system

$$(1.1a) \quad A(p)x = b(p)$$

having linear uncertainty structure

$$(1.1b) \quad a_{ij}(p) := a_{ij,0} + \sum_{\mu=1}^m a_{ij,\mu}p_{\mu}, \quad b_i(p) := b_{i,0} + \sum_{\mu=1}^m b_{i,\mu}p_{\mu},$$

where $a_{ij,\mu}, b_{i,\mu} \in \mathbb{R}$, $\mu = 0, \dots, m$, $i, j = 1, \dots, n$, and the parameters $p = (p_1, \dots, p_m)^\top$ are considered to be uncertain, varying within given intervals

$$(1.1c) \quad p \in [p] = ([p_1], \dots, [p_m])^\top.$$

In a more general case, the dependencies between the parameters in (1.1b) can be nonlinear. Such systems are common in many engineering analysis or design problems, control engineering, robust Monte Carlo simulations, etc., where there are complicated dependencies between the model parameters which are uncertain. The set of solutions to (1.1), called the united parametric solution set, is

$$(1.2) \quad \Sigma_{uni}^p = \Sigma(A(p), b(p), [p]) := \{x \in \mathbb{R}^n \mid \exists p \in [p], A(p)x = b(p)\}.$$

The (*united*) *parametric* solution sets generalize the (*united*) nonparametric solution sets to interval linear systems; the elements of the matrix and of the right-hand side

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in the latter are independent intervals. However, the solutions of many practical problems involving uncertain (interval) data have quantified formulation involving the universal logical quantifier (\forall) besides the existential quantifier (\exists). Examples of several mathematical problems formulated in terms of quantified solution sets can be found in [14] and in the vast literature on quantified constraints satisfaction problems; see, e.g., [5] for references to applications in control engineering, electrical engineering, mechanical engineering, biology, and others.

In this work we focus on linear systems involving linear dependencies between interval parameters and on quantified parametric solution sets where all universally quantified parameters precede all existentially quantified ones. Such solution sets are called AE parametric solution sets, after Shary [14]. AE parametric solution sets generalize both the united parametric solution set and the corresponding nonparametric AE solution sets. Our goal is to describe the parametric AE solution sets by inequalities not involving the interval parameters. This is a fundamental problem with considerable practical importance. The explicit description of a parametric solution set is useful for visualizing the solution set, for exploring the solution set properties, which helps in designing better (sharp and fast) numerical methods, and for finding exact bounds for the solution, which helps in testing new numerical methods.

The description of the parametric solution sets is related to quantifier elimination, which has stimulated a tremendous amount of research. Since Tarski's general theory [15] is EXPSPACE-hard [2], a lot of research is devoted to special cases with polynomial-time decidability. Apart from quantifier elimination, the only known general way of describing the united parametric solution set is a Fourier–Motzkin type parameter elimination process proposed in [1] and modified in [9]. The nonparametric AE solution sets were studied by many authors; see [3], [4], [14] and the references given therein. With the exception of [12], [13], which consider some special cases of tolerable solution sets, and [10], also considering a special case, to our knowledge there are no other studies of the parametric AE solution sets.

In this paper (section 4) we discuss how to obtain an explicit description of parametric AE solution sets by a Fourier–Motzkin type elimination of the existentially quantified parameters (called E -parameters). The methodology for elimination of E -parameters is presented in section 3. Explicit descriptions of particular classes of parametric AE solution sets (tolerable, controllable, any two-dimensional) are given in section 5. Based on the explicit description or the properties of the parameter elimination process, in this section we prove several properties of the parametric AE solution sets. Some necessary or necessary and sufficient conditions for a parametric AE solution set to be nonempty are presented. Also discussed are the shape of the parametric AE solution sets and some inclusion relations. For simplicity of notation we consider square systems. However, all the assertions in the paper are valid for rectangular systems. Numerical examples illustrate the parametric AE solution sets and their properties.

2. Notation. Denote by $\mathbb{R}^n, \mathbb{R}^{n \times m}$ the set of real vectors with n components and the set of real $n \times m$ matrices, respectively. A real compact interval is $[a] = [a^-, a^+] := \{a \in \mathbb{R} \mid a^- \leq a \leq a^+\}$. By $\mathbb{I}\mathbb{R}^n, \mathbb{I}\mathbb{R}^{n \times m}$ we denote the sets of interval n -vectors and interval $n \times m$ matrices, respectively. For $[a] = [a^-, a^+]$, define mid-point $\hat{a} := (a^- + a^+)/2$ and radius $\hat{a} := (a^+ - a^-)/2$. These functions are applied to interval vectors and matrices componentwise. For a given index set $\Pi = \{\pi_1, \dots, \pi_k\}$, denote $p_\Pi = (p_{\pi_1}, \dots, p_{\pi_k})$. \bigwedge and \bigwedge denote the logical “And.”

With the notation $A_{\bullet\bullet\mu} := (a_{ij,\mu}) \in \mathbb{R}^{n \times n}$, $b_{\bullet\mu} := (b_{i,\mu}) \in \mathbb{R}^n$, $\mu = 0, \dots, m$, the system (1.1a) can be rewritten equivalently as

$$\left(A_{\bullet\bullet 0} + \sum_{\mu=1}^m p_{\mu} A_{\bullet\bullet\mu} \right) x = b_{\bullet 0} + \sum_{\mu=1}^m p_{\mu} b_{\bullet\mu}.$$

For a matrix $A \in \mathbb{R}^{n \times n}$, $A_{m\bullet}$ denotes the m th row of A .

For a parametric matrix $A(p)$ (resp., vector $b(p)$), depending on a number of parameters (1.1c), $A([p]), b([p])$ denote the *corresponding* nonparametric matrix (resp., vector)

$$a_{ij}([p]) := a_{ij,0} + \sum_{\mu=1}^m a_{ij,\mu}[p_{\mu}], \quad b_i([p]) := b_{i,0} + \sum_{\nu=1}^m b_{i,\nu}[p_{\nu}].$$

Exactly one nonparametric system $A([p])x = b([p])$ corresponds to a parametric system $A(p)x = b(p)$. However, for a nonparametric system $[A]x = [b]$, there are infinitely many ways by which one can choose the number m of the parameters, the type of the parameter dependencies $A_{\bullet\bullet\mu}$, $\mu = 0, \dots, m$, and the parameter domain $[p] \in \mathbb{I}\mathbb{R}^m$, so that $A([p]) = [A]$ and $b([p]) = [b]$. All parametric systems $A(p)x = b(p)$, such that $A([p]) = [A]$, $b([p]) = [b]$, correspond to the nonparametric system $[A]x = [b]$. Lemma 5.5 below defines a way by which one can obtain a variety of parametric systems that correspond to a nonparametric system.

DEFINITION 2.1. *A parameter p_{μ} , $1 \leq \mu \leq m$, is of 1st class if it occurs in only one equation of the system (1.1a).*

It does not matter how many times a 1st class parameter appears within an equation. A parameter p_{μ} is of 1st class iff the vector $b_{\bullet\mu} - A_{\bullet\bullet\mu}x$ has only one nonzero component (that is, $b_{i\mu} - A_{i\bullet\mu}x \neq 0$ for exactly one i , $1 \leq i \leq n$).

DEFINITION 2.2. *A parameter p_{μ} , $1 \leq \mu \leq m$, is of 2nd class if it is involved in more than one equation of the system (1.1a).*

A parameter p_{μ} is of 2nd class iff the vector $b_{\bullet\mu} - A_{\bullet\bullet\mu}x$ has more than one nonzero component.

DEFINITION 2.3. *A parametric matrix is called row-dependent¹ if for some $\mu \in \{1, \dots, m\}$ and some $i \in \{1, \dots, n\}$, $\text{Card}(\mathcal{J}) \geq 2$, where $\mathcal{J} := \{j \mid 1 \leq j \leq n, a_{ij,\mu} \neq 0\}$. A parametric matrix is called row-independent if for all $\mu \in \{1, \dots, m\}$ and all $i \in \{1, \dots, n\}$, $\text{Card}(\mathcal{J}) < 2$.*

A row-dependent parametric matrix is denoted by $A_{rd}(p)$ and a row-independent one by $A_{ri}(p)$. Examples of row-independent parametric matrices are the symmetric, skew-symmetric, Hankel, Toeplitz, and Hurwitz matrices, as well as the nonparametric matrices.

3. Fourier–Motzkin type elimination of E -parameters. The united parametric solution set (1.2) is characterized as follows by a trivial set of inequalities:

$$\Sigma_{uni}^p = \{x \in \mathbb{R}^n \mid \exists p_{\mu} \in \mathbb{R}, \mu = 1, \dots, m : (3.1)–(3.2) \text{ hold}\},$$

where

$$(3.1) \quad A_{\bullet\bullet 0}x - b_{\bullet 0} + \sum_{\mu=1}^m (A_{\bullet\bullet\mu}x - b_{\bullet\mu}) p_{\mu} \leq 0 \leq A_{\bullet\bullet 0}x - b_{\bullet 0} + \sum_{\mu=1}^m (A_{\bullet\bullet\mu}x - b_{\bullet\mu}) p_{\mu},$$

$$(3.2) \quad p_{\mu}^{-} \leq p_{\mu} \leq p_{\mu}^{+}, \quad \mu = 1, \dots, m.$$

¹By analogy with the column-dependent parametric matrices defined in [7].

Starting from a trivial description of Σ_{uni}^p , the following theorem shows how the existentially quantified parameters in this set of inequalities can be eliminated successively in order to obtain a new description not involving $p_\mu, \mu = 1, \dots, m$.

THEOREM 3.1 (see [9]). *Let $g_\lambda(x), f_{\lambda\nu,1}(x), f_{\lambda\nu,2}(x), f_{\lambda\mu}(x), \lambda = 1, \dots, k (\geq n), \nu = 1, \dots, m_1 - 1, m_1 \geq 1$, be real-valued functions of $x = (x_1, \dots, x_n)^\top$ on some subset $D \subseteq \mathbb{R}^n$. Assume that there exists a nonempty set $\mathcal{T} \subseteq \{1, \dots, k\}$ such that $f_{\lambda m_1}(x) \neq 0$ for all $\lambda \in \mathcal{T}$. For the parameters $p_\mu, \mu = m_1, \dots, m$, varying in \mathbb{R} and for x varying in D define the sets S_1, S_2 by*

$$S_1 := \{x \in D \mid \exists p_\mu \in \mathbb{R}, \mu = m_1, \dots, m : (3.3), (3.4) \text{ hold}\},$$

$$S_2 := \{x \in D \mid \exists p_\mu \in \mathbb{R}, \mu = m_1 + 1, \dots, m : (3.5), (3.6), (3.7) \text{ hold}\},$$

where inequalities (3.3), (3.4) and (3.5), (3.6), (3.7), respectively, are given by

$$(3.3) \quad g_\lambda(x) + \sum_{\nu=1}^{m_1-1} f_{\lambda\nu,1}(x)\dot{p}_\nu \mp \sum_{\nu=1}^{m_1-1} f_{\lambda\nu,2}(x)\hat{p}_\nu$$

$$(3.4) \quad + \sum_{\mu=m_1+1}^m f_{\lambda\mu}(x)p_\mu \leq -f_{\lambda m_1}(x)p_{m_1} \leq \dots, \quad \lambda = 1, \dots, k,$$

$$(3.5) \quad \dot{p}_\mu - \hat{p}_\mu \leq p_\mu \leq \dot{p}_\mu + \hat{p}_\mu, \quad \mu = m_1, \dots, m,$$

$$(3.5) \quad g_\lambda(x) + \sum_{\nu=1}^{m_1-1} f_{\lambda\nu,1}(x)\dot{p}_\nu \mp \sum_{\nu=1}^{m_1-1} f_{\lambda\nu,2}(x)\hat{p}_\nu + f_{\lambda m_1}(x)\dot{p}_{m_1} \mp |f_{\lambda m_1}(x)|\hat{p}_{m_1}$$

$$(3.5) \quad + \sum_{\mu=m_1+1}^m f_{\lambda\mu}(x)p_\mu \leq 0 \leq \dots, \quad \lambda = 1, \dots, k,$$

and for $\alpha, \beta \in \mathcal{T}, \alpha < \beta$,

$$(3.6) \quad g_\alpha(x)f_{\beta m_1}(x) - g_\beta(x)f_{\alpha m_1}(x) + \sum_{\nu=1}^{m_1-1} (f_{\beta m_1}(x)f_{\alpha\nu,1}(x) - f_{\alpha m_1}(x)f_{\beta\nu,1}(x))\dot{p}_\nu$$

$$(3.6) \quad \mp \sum_{\nu=1}^{m_1-1} (|f_{\beta m_1}(x)|f_{\alpha\nu,2}(x) + |f_{\alpha m_1}(x)|f_{\beta\nu,2}(x))\hat{p}_\nu$$

$$(3.6) \quad + \sum_{\mu=m_1+1}^m (f_{\alpha\mu}(x)f_{\beta m_1}(x) - f_{\beta\mu}(x)f_{\alpha m_1}(x))p_\mu \leq 0 \leq \dots,$$

$$(3.7) \quad \dot{p}_\mu - \hat{p}_\mu \leq p_\mu \leq \dot{p}_\mu + \hat{p}_\mu, \quad \mu = m_1 + 1, \dots, m.$$

The “ \dots ” in the right-hand side inequalities denotes the left-side expression in the left inequality with the bottom sign (+) in front of the terms involving a parameter radius, while “ \mp ” in the left inequality should be read “-.”² (Trivial inequalities which are true for any $x \in \mathbb{R}^n$ can be omitted.) Then $S_1 = S_2$.

²For example, the expanded (3.3) is

$$g_\lambda(x) + \sum_{\nu=1}^{m_1-1} f_{\lambda\nu,1}(x)\dot{p}_\nu - \sum_{\nu=1}^{m_1-1} f_{\lambda\nu,2}(x)\hat{p}_\nu + \sum_{\mu=m_1+1}^m f_{\lambda\mu}(x)p_\mu \leq -f_{\lambda m_1}(x)p_{m_1}$$

$$\leq g_\lambda(x) + \sum_{\nu=1}^{m_1-1} f_{\lambda\nu,1}(x)\dot{p}_\nu + \sum_{\nu=1}^{m_1-1} f_{\lambda\nu,2}(x)\hat{p}_\nu + \sum_{\mu=m_1+1}^m f_{\lambda\mu}(x)p_\mu.$$

The inequalities (3.5) are called *end-point inequalities* because they are obtained by combining (3.3) with (3.4). The inequalities (3.6) are called *cross inequality pairs* because they are obtained by combining two inequality pairs (3.3). Note that the resulting inequalities (3.5) and (3.6) have the form (3.3), which allows the elimination process to continue with the next parameters.

The parameter elimination process resembles the so-called Fourier–Motzkin elimination of variables; see, e.g., [11]. It was first proposed in [1] in a form based on the parameter inequalities (3.2) which leads to a tremendous number of solution set characterizing inequalities. In order to reduce the number of characterizing inequalities, the modified parameter elimination in Theorem 3.1 is based on the equivalent parameter inequalities (3.4) in midpoint/radius representation. Thus, in the parameter elimination process we apply the relation

$$(3.8) \quad \lambda \hat{p}_\mu - |\lambda| \hat{p}_\mu \leq \lambda p_\mu \leq \lambda \hat{p}_\mu + |\lambda| \hat{p}_\mu \quad \text{for } \lambda \in \mathbb{R}$$

without the necessity to consider the particular sign of λ . Therefore, the modified parameter elimination does not depend on a particular orthant. Furthermore, Theorem 3.1 gives a compact representation of the characterizing inequalities which will be illustrated below.

Consider the parametric system (1.1), the united parametric solution set of which is described by the trivial set of characterizing inequalities (3.1) and for $\mu = 1, \dots, m$, (3.4). Let for $M_1 \cup M_2 = \{1, \dots, m\}$, $M_1 \cap M_2 = \emptyset$, p_μ , $\mu \in M_1$, be 1st class E -parameters and p_μ , $\mu \in M_2$, be 2nd class E -parameters. By Theorem 3.1, the elimination of all p_μ , $\mu \in M_1$, updates the inequality pairs (3.1) so that they become

$$(3.9) \quad A_{\bullet\bullet 0}x - b_{\bullet 0} + \sum_{\mu \in M_1} (A_{\bullet\bullet \mu}x - b_{\bullet \mu}) \hat{p}_\mu \mp \sum_{\mu \in M_1} |A_{\bullet\bullet \mu}x - b_{\bullet \mu}| \hat{p}_\mu + \sum_{\mu \in M_2} (A_{\bullet\bullet \mu}x - b_{\bullet \mu}) p_\mu \leq 0 \leq \dots$$

The end-point inequality pairs (3.9) are equivalent to single absolute-value inequalities (3.10) and vice versa:

$$(3.10) \quad \left| A_{\bullet\bullet 0}x - b_{\bullet 0} + \sum_{\mu \in M_1} (A_{\bullet\bullet \mu}x - b_{\bullet \mu}) \hat{p}_\mu + \sum_{\mu \in M_2} (A_{\bullet\bullet \mu}x - b_{\bullet \mu}) p_\mu \right| \leq \sum_{\mu \in M_1} |A_{\bullet\bullet \mu}x - b_{\bullet \mu}| \hat{p}_\mu.$$

Let for p_{ν_1} , $\nu_1 \in M_2$, $\mathcal{T}_{\nu_1} \subseteq \{1, \dots, n\}$, $\text{Card}(\mathcal{T}_{\nu_1}) = k$, be the index set of the inequalities (3.9) (resp., (3.10)) involving p_{ν_1} . By Theorem 3.1, the elimination of p_{ν_1} updates the end-point inequalities (3.9) (resp., (3.10)), which become

$$(3.11) \quad \left| A_{\bullet\bullet 0}x - b_{\bullet 0} + \sum_{\mu \in M_1 \cup \{\nu_1\}} (A_{\bullet\bullet \mu}x - b_{\bullet \mu}) \hat{p}_\mu + \sum_{\mu \in M_2 \setminus \{\nu_1\}} (A_{\bullet\bullet \mu}x - b_{\bullet \mu}) p_\mu \right| \leq \sum_{\mu \in M_1 \cup \{\nu_1\}} |A_{\bullet\bullet \mu}x - b_{\bullet \mu}| \hat{p}_\mu$$

and for $\alpha, \beta \in \mathcal{T}_{\nu_1}$ generate $k(k - 1)/2$ cross inequality pairs

$$\begin{aligned}
 (3.12) \quad & \Delta_{0,\nu_1}(\alpha, \beta, x) + \sum_{\mu \in M_1} \Delta_{\mu,\nu_1}(\alpha, \beta, x) \dot{p}_\mu \\
 & \mp \sum_{\mu \in M_1} (|f_{\nu_1}(\beta, x)| |f_\mu(\alpha, x)| + |f_{\nu_1}(\alpha, x)| |f_\mu(\beta, x)|) \hat{p}_\mu \\
 & + \sum_{\mu \in M_2 \setminus \{\nu_1\}} \Delta_{\mu,\nu_1}(\alpha, \beta, x) p_\mu \leq 0 \leq \dots,
 \end{aligned}$$

wherein $f_\mu(\alpha, x) := (A_{\alpha \bullet \mu} x - b_{\alpha \mu})$, and similarly for $f_{\nu_1}(\alpha, x)$, $f_\mu(\beta, x)$, $f_{\nu_1}(\beta, x)$, and $\Delta_{\mu,\nu_1}(\alpha, \beta, x) := f_{\nu_1}(\beta, x) f_\mu(\alpha, x) - f_{\nu_1}(\alpha, x) f_\mu(\beta, x)$ for $\mu \in \{0\} \cup M_1$ or $\mu \in M_2 \setminus \{\nu_1\}$. The cross inequality pairs (3.12) also can be written as equivalent single absolute-value inequalities.

The elimination of the next 2nd class E -parameters updates similarly the end-point inequalities (3.11) and introduces more cross inequalities. The cross inequalities can be more complicated than the inequalities (3.12). However, the solution set characterizing inequalities (both end-point and cross inequalities), obtained by the Fourier–Motzkin type elimination of E -parameters, have the same general form, which can be presented as follows.

For $\lambda \in \mathcal{T} := \{1, \dots, n\} \cup \mathcal{T}_c$, where $\{1, \dots, n\}$ is the index set of the end-point characterizing inequalities and \mathcal{T}_c is the index set of the characterizing cross inequalities, the set of all solution set characterizing inequalities obtained by the Fourier–Motzkin type elimination of E -parameters is

$$\begin{aligned}
 (3.13) \quad & \bigwedge_{\lambda \in \mathcal{T}} u_{\lambda,0}(x) + \sum_{\mu \in L_1} u_{\lambda,\mu}(x) \dot{p}_\mu - \sum_{\mu \in L_1} v_{\lambda,\mu}(x) \hat{p}_\mu \leq \sum_{\mu \in L_2} w_{\lambda,\mu}(x) p_\mu \\
 & \leq u_{\lambda,0}(x) + \sum_{\mu \in L_1} u_{\lambda,\mu}(x) \dot{p}_\mu + \sum_{\mu \in L_1} v_{\lambda,\mu}(x) \hat{p}_\mu,
 \end{aligned}$$

wherein L_1 is the set of indexes of all eliminated E -parameters, $u_{\lambda,0}(x)$, $u_{\lambda,\mu}(x)$, $v_{\lambda,\mu}(x)$, $w_{\lambda,\mu}(x)$ are corresponding real-valued functions of $x = (x_1, \dots, x_n)^\top$, and L_2 is the set of indexes of the noneliminated parameters. A more general representation of the inequality pairs (3.13) is

$$(3.14) \quad \bigwedge_{\lambda \in \mathcal{T}} u_\lambda(x, L_1) \leq \sum_{\mu \in L_2} w_{\lambda,\mu}(x) p_\mu \leq v_\lambda(x, L_1),$$

where $u_\lambda(x, L_1)$ is the expression in the left side of the left inequality of (3.13), and $v_\lambda(x, L_1)$ is the expression in the right side of the right inequality of (3.13); $u_\lambda(x, L_1)$ and $v_\lambda(x, L_1)$ differ only in the signs of the terms involving the radius of a parameter.

4. Description of parametric AE solution sets. We start this section with a general definition.

DEFINITION 4.1. *Quantified solution sets to a parametric linear system $A(p)x = b(p)$, involving either linear or nonlinear dependencies between the parameters $p = (p_1, \dots, p_m)^\top$, are sets of the form*

$$\{x \in \mathbb{R}^n \mid (Q_1 p_1 \in [p_1]) \dots (Q_m p_m \in [p_m]) (A(p)x = b(p))\},$$

where $Q_i \in \{\forall, \exists\}$, $i = 1, \dots, m$.

The total number of quantified parametric solution sets exceeds 2^m since the existential and the universal quantifiers do not commute. In this work we consider only linear systems involving linear dependencies between the uncertain parameters and quantified solutions sets of such systems where all occurrences of the universal quantifier precede all occurrences of the existential quantifier. After the terminology used in [14], we call these solution sets *AE* parametric solution sets. Thus, a parametric *AE* solution set of the system (1.1a)–(1.1c) is defined by

$$(4.1) \quad \Sigma_{AE}^p := \{x \in \mathbb{R}^n \mid (\forall p_{\mathcal{A}} \in [p_{\mathcal{A}}])(\exists p_{\mathcal{E}} \in [p_{\mathcal{E}}])(A(p)x = b(p))\},$$

where \mathcal{A} and \mathcal{E} are sets of indexes such that $\mathcal{A} \cup \mathcal{E} = \{1, \dots, m\}$, $\mathcal{A} \cap \mathcal{E} = \emptyset$. There are exactly 2^m parametric *AE* solution sets.

THEOREM 4.2. *For given index sets \mathcal{A} and \mathcal{E} , the parametric *AE* solution set (4.1) of the system (1.1a)–(1.1c) is described by the set of inequality pairs*

$$(4.2) \quad \bigwedge_{\lambda \in \mathcal{T}} \left(u_{\lambda}(x, \mathcal{E}) - \sum_{\mu \in \mathcal{A}} (w_{\lambda, \mu}(x)\dot{p}_{\mu} - |w_{\lambda, \mu}(x)|\hat{p}_{\mu}) \leq 0 \right. \\ \left. \leq v_{\lambda}(x, \mathcal{E}) - \sum_{\mu \in \mathcal{A}} (w_{\lambda, \mu}(x)\dot{p}_{\mu} + |w_{\lambda, \mu}(x)|\hat{p}_{\mu}) \right),$$

where

$$S_{\mathcal{E}} := \bigwedge_{\lambda \in \mathcal{T}} u_{\lambda}(x, \mathcal{E}) \leq \sum_{\mu \in \mathcal{A}} w_{\lambda, \mu}(x)p_{\mu} \leq v_{\lambda}(x, \mathcal{E})$$

is the set of inequality pairs obtained by Fourier–Motzkin type elimination of all *E*-parameters, $\mathcal{T} = \{1, \dots, t\}$, $t \geq n$.

Proof.

$$\begin{aligned} \Sigma_{AE}^p &:= \{x \in \mathbb{R}^n \mid (\forall p_{\mathcal{A}} \in [p_{\mathcal{A}}])(\exists p_{\mathcal{E}} \in [p_{\mathcal{E}}])(A(p)x = b(p))\} \\ &= \left\{ x \in \mathbb{R}^n \mid (\forall p_{\mathcal{A}} \in [p_{\mathcal{A}}]) \left(\bigwedge_{\lambda \in \mathcal{T}} u_{\lambda}(x, \mathcal{E}) \leq \sum_{\mu \in \mathcal{A}} w_{\lambda, \mu}(x)p_{\mu} \leq v_{\lambda}(x, \mathcal{E}) \right) \right\} \\ &= \{x \in \mathbb{R}^n \mid (4.2)\}. \end{aligned}$$

The first equality above follows from the Fourier–Motzkin type elimination of all *E*-parameters. The second equality follows from the distributivity of the universal quantifiers over conjunction, the parameter inequality pairs for the *A*-parameters, and the relation

$$(4.3) \quad (\forall p \in [p] : b_1 \leq f(p) \leq b_2) \Leftrightarrow \left(b_1 \leq \min_{p \in [p]} f(p) \right) \wedge \left(\max_{p \in [p]} f(p) \leq b_2 \right). \quad \square$$

COROLLARY 4.3. *The elimination of the universally quantified parameters does not introduce new characterizing inequalities to the description of Σ_{AE}^p obtained by elimination of all existentially quantified parameters.*

It was proved in [9] that the elimination of 1st class \mathcal{E} -parameters does not introduce the so-called cross inequalities. These inequalities are generated only by the elimination 2nd class \mathcal{E} -parameters and the degree of the polynomials involved in the

cross inequalities may increase with each eliminated 2nd class \mathcal{E} -parameter. Thus, we can estimate the shape of a parametric AE solution set, i.e., the maximal degree of the polynomial equations describing the solution set boundary.

COROLLARY 4.4. *The nonlinear shape of Σ_{AE}^p is determined by the 2nd class \mathcal{E} -parameters.*

The next important corollary follows from the elimination theorems for the 1st class \mathcal{E} -parameters and the \mathcal{A} -parameters.

COROLLARY 4.5. *The infimum/supremum of a parametric AE solution set is attained at particular end-points of the intervals for the 1st class \mathcal{E} -parameters and the \mathcal{A} -parameters.*

Due to the above, we sometimes say that the boundary of a parametric AE solution set is linear with respect to the 1st class \mathcal{E} -parameters and the \mathcal{A} -parameters. Despite this property, the parametric AE solution set may not depend linearly on these parameters.

The application of Theorem 4.2 will be illustrated in the next section, where we consider some classes of parametric AE solution sets, give their explicit description, and derive some of their properties.

5. Properties of the parametric AE solution sets. The first two general theorems below are proved in [10] and are not based on the description of the parametric AE solution sets. The following theorem gives a set-theoretical description of AE parametric solution sets (4.1) and generalizes a corresponding theorem [14, Theorem 3.1] for nonparametric AE solution sets.

THEOREM 5.1 (see [10]).

$$\Sigma_{AE}^p = \bigcap_{p_A \in [p_A]} \bigcup_{p_E \in [p_E]} \{x \in \mathbb{R}^n \mid A(p_A, p_E) \cdot x = b(p_A, p_E)\}.$$

The next theorem gives some analytic necessary conditions for a general AE parametric solution set to be nonempty.

THEOREM 5.2 (see [10]). *If a parametric AE solution set (4.1) is nonempty, then for any $x \in \Sigma_{AE}^p$*

$$(5.1) \quad \sum_{\nu \in \mathcal{A}} (A_{\bullet\bullet\nu}x - b_{\bullet\nu})[p_\nu] \subseteq b_{\bullet 0} - A_{\bullet\bullet 0}x + \sum_{\mu \in \mathcal{E}} (b_{\bullet\mu} - A_{\bullet\bullet\mu}x)[p_\mu].$$

The interval inclusion (5.1) is equivalent to the inequality

$$(5.2) \quad |A(\hat{p})x - b(\hat{p})| \leq \sum_{\mu=1}^m \delta_\mu |A_{\bullet\bullet\mu}x - b_{\bullet\mu}| \hat{p}_\mu,$$

where $\delta_\mu := \{1 \text{ if } \mu \in \mathcal{E}, -1 \text{ if } \mu \in \mathcal{A}\}$.

The inequality (5.2) presents the end-point inequalities in the explicit characterization of a parametric AE solution set. The following theorem and corollary follow from Theorem 4.2 and a property proved in [9] that the elimination of 1st class E-parameters does not generate any cross inequalities.

THEOREM 5.3. *A parametric AE solution set of the linear system (1.1a)–(1.1c) is nonempty iff the solution set describing inequalities (4.2), defined in Theorem 4.2, holds true.*

COROLLARY 5.4. *Let the definition of a parametric AE solution set to the linear system (1.1a)–(1.1c) involve only 1st class existentially quantified parameters. Such parametric AE solution set is nonempty iff the inequality (5.2) holds true.*

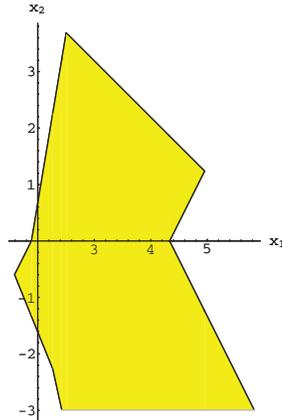


FIG. 5.1. The parametric AE solution set for the system from Example 5.1.

The nonempty parametric AE solution sets from Corollary 5.4 have linear shape but they are not convex in the general case.

Example 5.1. Consider the parametric linear system $A(p)x = b(q)$, where

$$A(p) = \begin{pmatrix} 2p_1 & p_{12} - p_1 \\ 2.5p_{21} + p_2 & p_2 \end{pmatrix}, \quad b(q) = \begin{pmatrix} 2q \\ 2q \end{pmatrix},$$

$$p_1 \in \left[\frac{1}{2}, \frac{3}{2} \right], \quad p_2 \in \left[\frac{7}{10}, \frac{17}{10} \right], \quad p_{12}, p_{21} \in [0, 1], \quad q \in \left[\frac{13}{6}, \frac{17}{6} \right].$$

The solution set $\Sigma_{\forall q \exists p_1, p_2, p_{12}, p_{21}}$ is presented on Figure 5.1. Its boundary is linear but neither the whole solution set nor its intersection with the fourth orthant is convex. Furthermore, the solution set is unbounded in the fourth orthant.

It is well known that a parametric united solution set is a subset of its corresponding nonparametric solution set, but we have never seen a formal proof of this fact. Below, for completeness, we give the proof of a more general inclusion relation.

LEMMA 5.5. Let $f(p)$ and $g(p)$ be linear functions of the interval parameters $p \in [p] \in \mathbb{IR}^m$ such that they involve at least two different parameters p_{i_1}, p_{i_2} :

$$f(p) = \alpha_0 + \alpha p_{i_1} + f_0(p \setminus \{p_{i_1}, p_{i_2}\}), \quad g(p) = \beta_0 + \beta p_{i_2} + g_0(p \setminus \{p_{i_1}, p_{i_2}\}),$$

where $\alpha_0, \beta_0, \alpha, \beta \in \mathbb{R}$, and f_0, g_0 are linear functions of p_{i_3}, \dots, p_{i_m} . Then for $\tilde{f}(q), \tilde{g}(q)$, defined by

$$\tilde{f}(q) := q_1 + q_2 + f_0(p \setminus \{p_{i_1}, p_{i_2}\}),$$

$$\tilde{g}(q) := q_1 + q_3 + g_0(p \setminus \{p_{i_1}, p_{i_2}\}),$$

where $q = (q_1, q_2, q_3, p_{i_3}, \dots, p_{i_m})$, $q_1 \in [q_1]$ is arbitrary, $q_2 \in [q_2]$, such that $\hat{q}_2 = \alpha_0 + \alpha \hat{p}_{i_1} - \hat{q}_1$, $\hat{q}_2 = |\alpha| \hat{p}_{i_1} - \hat{q}_1$, $q_3 \in [q_3]$, such that $\hat{q}_3 = \beta_0 + \beta \hat{p}_{i_2} - \hat{q}_1$, $\hat{q}_3 = |\beta| \hat{p}_{i_2} - \hat{q}_1$, the following relations hold:

$$f([p]) = \tilde{f}([q]), \quad g([p]) = \tilde{g}([q]).$$

Proof. The proof is trivial and follows from the relation $a = \hat{a} + \hat{a}e$, where $a \in [a]$, $e \in [-1, 1]$, and the relation (3.8). \square

THEOREM 5.6. *For two parameter vectors $u \in [u] \in \mathbb{IR}^{m_1}$, $v \in [v] \in \mathbb{IR}^{m_2}$, such that $A([u]) = A([v]) = [A]$, $b([u]) = b([v]) = [b]$ and $A(u)$, $b(u)$ are obtained from $A(v)$, $b(v)$ by successive application of Lemma 5.5. Similarly, $A(v)$, $b(v)$ are obtained from $[A]$, $[b]$; then*

$$\Sigma_{uni}(A(u), b(u), [u]) \subseteq \Sigma_{uni}(A(v), b(v), [v]) \subseteq \dots \subseteq \Sigma_{uni}([A], [b]).$$

Proof. The application of Lemma 5.5 to two elements $a_{ij_1}(\cdot)$, $a_{ij_2}(\cdot)$ from the i th row of the matrix A (or to $a_{ij_1}(\cdot)$ and $b_i(\cdot)$) implies the introduction of a parameter (say, u_μ) having row-dependencies. Then the inclusion relation follows from the inequality

$$|a_{ij_1\mu}x_{j_1} + a_{ij_2\mu}x_{j_2}| \leq |a_{ij_1\mu}x_{j_1}| + |a_{ij_2\mu}x_{j_2}|$$

applied to the right side of the i th absolute-value end-point inequality characterizing the solution set.

The application of Lemma 5.5 to two elements of different rows of the matrix A (and the vector b) implies the introduction of column-dependencies. Then the elimination of the parameter having more nonzero components in the coefficient vector will generate additional characterizing cross inequalities which may additionally restrict the solution set. \square

Since by Theorem 4.2 all \mathcal{E} -parameters are eliminated first and the elimination of all \mathcal{A} -parameters does not introduce any cross inequalities, we can also apply Theorem 5.6 to parametric AE solution sets as specified by the next corollary.

COROLLARY 5.7. *Theorem 5.6 is applicable to parametric AE solution sets which have the same structure of the dependencies between the \mathcal{A} -parameters and the same domain $[p_{\mathcal{A}}]$.*

Proof. Let us have two parametric systems (resp., AE solution sets) such that the requirements of the corollary hold, that is, $A_{\bullet\bullet\mu}^{(1)} = A_{\bullet\bullet\mu}^{(2)}$, $b_{\bullet\mu}^{(1)} = b_{\bullet\mu}^{(2)}$ for $\mu \in \mathcal{A}$. If for a $\tilde{p}_{\mathcal{A}} = 0$, by Theorem 5.6 we have

$$\Sigma(A^{(1)}(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}_1}), b^{(1)}(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}_1}), [p_{\mathcal{E}_1}]) \subseteq \Sigma(A^{(2)}(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}_2}), b^{(2)}(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}_2}), [p_{\mathcal{E}_2}]),$$

then we will have the same inclusion for every $\tilde{p}_{\mathcal{A}} \in [p_{\mathcal{A}}]$ and thus for the corresponding parametric AE solution sets. \square

5.1. Parametric tolerable solution sets. For $p = (p_1, \dots, p_{m_1})$ and $q = (q_1, \dots, q_{m_2})$, the general parametric tolerable solution set is defined by

$$\Sigma_{tol}(A(p), b(q), [p], [q]) := \{x \in \mathbb{R}^n \mid \forall p \in [p], \exists q \in [q], A(p)x = b(q)\}.$$

Denote by $\Sigma_{tol}(A_{ri}(p), [p], [b])$ the tolerable solution set of a system involving a row-independent parametric matrix and a right-hand-side vector with independent interval components. Since $A([p])$ is the interval hull of $A_{ri}(p)$ and in view of Definition 2.3, by Theorem 4.2 the two tolerable solution sets $\Sigma_{tol}(A([p]), [b])$ and $\Sigma_{tol}(A_{ri}(p), [p], [b])$ have the same explicit representation

$$\Sigma_{tol}(A([p]), [b]) = \Sigma_{tol}(A_{ri}(p), [p], [b]) = \left\{ x \in \mathbb{R}^n \mid \left| \dot{A}x - \dot{b} \right| \leq \hat{b} - \hat{A}|x| \right\}.$$

PROPOSITION 5.8. *Let $q = (q_1, \dots, q_{m_2})$. If q_1, \dots, q_{m_2} are 1st class parameters, then*

$$\Sigma_{tol}(A(p), b(q), [p], [q]) = \left\{ x \in \mathbb{R}^n \mid \left| \dot{A}x - \dot{b} \right| \leq \sum_{\mu=1}^{m_2} \hat{q}_\mu |b_{\bullet\mu}| - \sum_{\mu=1}^{m_1} \hat{p}_\mu |A_{\bullet\bullet\mu}x| \right\},$$

where $\sum_{\mu=1}^{m_2} \hat{q}_\mu |b_{\bullet\mu}| = \text{rad}(b([q]))$.

If the parametric tolerable solution set involves 2nd class E -parameters, then its description contains cross inequalities with respect to these parameters. However, since all 2nd class E -parameters (if any) are involved in the right-hand side of the system, the cross inequalities with respect to these parameters will be linear, which proves the following theorem.

THEOREM 5.9. *The parametric tolerable solution sets have linear shape.*

Next we prove some inclusion relations between different tolerable solution sets.

THEOREM 5.10. *Let $A_{ri}(u), A_{rd}(v) \in \mathbb{R}^{n \times n}$, and $[A] \in \mathbb{IR}^{n \times n}$ be such that for given parameter vectors $u \in [u] \in \mathbb{IR}^{m_1}$, $v \in [v] \in \mathbb{IR}^{m_2}$, $A_{ri}([u]) \subseteq [A]$, and $A_{rd}(v)$ is obtained from $A_{ri}(u)$ or from $A_{ri}([u])$ by successive application of Lemma 5.5. If the parameters $q \in [q] \in \mathbb{IR}^{m_3}$ are of 1st class, then*

$$(5.3) \quad \Sigma_{tol}([A], b([q])) \subseteq \Sigma_{tol}(A_{ri}([u]), b([q])) \\ = \Sigma_{tol}(A_{ri}(u), b([q]), [u]) \subseteq \Sigma_{tol}(A_{rd}(v), b([q]), [v]).$$

If $A(v)$ is obtained from $A(u)$ by successive application of Lemma 5.5, then for an arbitrary $q \in [q] \in \mathbb{IR}^{m_3}$ which may involve 2nd class E -parameters

$$(5.4) \quad \Sigma_{tol}(A(u), b(q), [u], [q]) \subseteq \Sigma_{tol}(A(v), b(q), [v], [q]).$$

Proof. The equality relation in (5.3) follows from the equivalent explicit description of the two solution sets.

The proof of $\Sigma_{tol}(A_{ri}(u), b([q]), [u]) \subseteq \Sigma_{tol}(A_{rd}(v), b([q]), [v])$ is similar to the proof of Theorem 5.6 for row-dependencies. However, since the radiuses of the universally quantified parameters appear in the right-hand side of the solution set characterizing inequalities with negative sign, the inclusion relation is reversed.

The inclusion relation (5.4) follows similarly if we consider also the characterizing cross inequalities for the 2nd class E -parameters.

We prove $\Sigma_{tol}([A], b([q])) \subseteq \Sigma_{tol}(A_{ri}([u]), b([q]))$. If $[a_{ij}] \not\supseteq a_{ij}([u])$, there exist at least one interval $[t] \neq [0, 0]$ such that $[a_{ij}] = a_{ij}([u]) + [t]$ and $\hat{a}_{ij} = \hat{a}_{ij}([u]) + \hat{t}$. Then the inclusion follows from $-\hat{a}_{ij} \leq -\hat{a}_{ij}([u])$. \square

Example 5.2. Consider the nonparametric interval linear system $[A]x = [b]$, where

$$[A] = \begin{pmatrix} [0, 1] & [\frac{1}{2}, \frac{3}{2}] \\ [-2, 0] & [1, 2] \end{pmatrix}, [b] = \begin{pmatrix} [-1, 2] \\ [-3, 3] \end{pmatrix}.$$

The nonparametric interval matrix $[A]$ presents an interval hull of the following parametric matrices (and of infinitely many other parametric matrices):

$$A_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 1 + a_{11} \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_{11} & a + \frac{1}{2} \\ -2a & 1 + a_{11} \end{pmatrix}, \quad A_3 = \begin{pmatrix} a & a + \frac{1}{2} \\ -2a & 1 + a \end{pmatrix},$$

where $a_{11}, a \in [0, 1]$, $a_{12} \in [\frac{1}{2}, \frac{3}{2}]$, $a_{21} \in [-2, 0]$. Since both A_1 and A_2 are row-independent matrices with the same interval hull, the parametric tolerable solution sets $\Sigma_{tol}(A_1, [b])$ and $\Sigma_{tol}(A_2, [b])$ have the same explicit description which is equivalent to the description of the corresponding nonparametric tolerable solution set $\Sigma_{tol}([A], [b])$. The parametric matrix A_3 is row-dependent and has the same interval hull as the matrices A_1, A_2 . Therefore, by Theorem 5.10, relation (5.3),

$$\Sigma_{tol}([A], [b]) = \Sigma_{tol}(A_1, [b]) = \Sigma_{tol}(A_2, [b]) \subseteq \Sigma_{tol}(A_3, [b]).$$

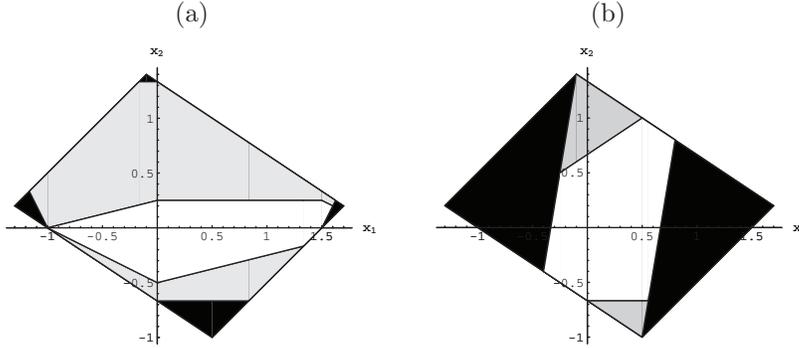


FIG. 5.2. Inclusion relations between the parametric tolerable solution sets from Example 5.2: (a) the inclusions (5.5), (b) the inclusions (5.6).

If we consider a system with matrix $[B] = \begin{pmatrix} [0,1] & [-4,2] \\ [-2,0] & [1,2] \end{pmatrix}$ which encloses the matrix $[A]$, we obtain the inclusions

$$(5.5) \quad \Sigma_{tol}([B], [b]) \subseteq \Sigma_{tol}([A], [b]) = \Sigma_{tol}(A_1, [b]) = \Sigma_{tol}(A_2, [b]) \subseteq \Sigma_{tol}(A_3, [b]).$$

The last inclusion chain is presented in Figure 5.2(a), where $\Sigma_{tol}([B], [b])$ is the innermost white polyhedron, $\Sigma_{tol}([A], [b])$ is the polyhedron in light gray, and $\Sigma_{tol}(A_3, [b])$ is the parallelogram with black corners.

Now, consider parametric systems involving the same matrices $[A]$, A_1 , A_2 , A_3 and a right-hand-side vector depending on a 2nd class parameter, that is, $b(q) = (q_1, q_1 - q_2)^\top$, where $q_1, q_2 \in [-1, 2]$ and $b([q]) = [b]$. For the tolerable solution sets of these systems we have the following inclusion relations:

$$(5.6) \quad \begin{aligned} \Sigma_{tol}(V, b(q)) &\stackrel{(5.4)}{\subseteq} \Sigma_{tol}(A_3, b(q)) \stackrel{Cor5.7}{\subseteq} \Sigma_{tol}(A_3, [b]), \\ \Sigma_{tol}(V, b(q)) &\stackrel{Cor5.7}{\subseteq} \Sigma_{tol}(V, [b]) \stackrel{(5.3)}{\subseteq} \Sigma_{tol}(A_3, [b]), \end{aligned}$$

where $V \in \{[A], A_1, A_2\}$.

In Figure 5.2(b) $\Sigma_{tol}(A_3, [b])$ is the black parallelogram, $\Sigma_{tol}(A_3, b(q))$ is the parallelogram in gray, and $\Sigma_{tol}(V, b(q))$ is the innermost white polyhedron.

The next theorem gives a better description of the shape of the parametric tolerable solution set than Theorem 5.9.

THEOREM 5.11. *The parametric tolerable solution set is a convex polyhedron.*

Proof. First we consider the special case where all E -parameters $q = (q_1, \dots, q_{m_2})$ are of 1st class. Then, defining $[b] := b([q])$, by Theorem 5.10 we have

$$\begin{aligned} \Sigma_{tol}(A(p), b(q), [p], [q]) &= \Sigma_{tol}(A(p), b([q]), [p]) \\ &= \{x \in \mathbb{R}^n \mid (\forall p \in [p])(A(p)x \in [b])\} \\ &= \left\{ x \in \mathbb{R}^n \mid (\forall p \in [p]) \left(A_{\bullet\bullet 0}x + \sum_{\mu=1}^{m_1} (A_{\bullet\bullet \mu}x)p_\mu \in [b] \right) \right\}. \end{aligned}$$

Define $\mathcal{L} := \{\lambda = (\lambda_1, \dots, \lambda_{m_1}) \mid \lambda_\mu \in \{-, +\}, \mu = 1, \dots, m_1\}$. The relation (4.3)

implies $\bigwedge_{\lambda \in \mathcal{L}} b_1 \leq f(p^\lambda) \leq b_2$ for a linear function $f(p)$ and $p \in [p] \in \mathbb{IR}^k$. Thus,

$$(5.7) \quad \Sigma_{tol}(A(p), b([q]), [p]) = \left\{ x \in \mathbb{R}^n \mid \bigwedge_{\lambda \in \mathcal{L}} b^- \leq A_{\bullet\bullet 0}x + \sum_{\mu=1}^{m_1} (A_{\bullet\bullet \mu}x)p_\mu^{\lambda_\mu} \leq b^+ \right\},$$

which proves the theorem since a convex polyhedron is expressed as the solution set for a system of linear inequalities.

If the parametric tolerable solution set involves 2nd class E -parameters, their elimination will generate cross inequalities with respect to these parameters. However, since all 2nd class E -parameters are involved in the right-hand side of the system, all cross inequalities with respect to these parameters will be linear involving additional (new) affine-linear dependencies between the parameters p . Then, the proof will continue the same way as for 1st class E -parameters above but with an enlarged matrix A' having $n + k$ rows and a vector $[b'] \in \mathbb{IR}^{n+k}$, where k is the number of cross inequalities. \square

The assertion of Theorem 5.11 and the left two relations in (5.3) are considered in [12], [13] for the special case of the row-independent parametric matrix and the right-hand side with independent components. Theorem 5.11 and relation (5.4) of Theorem 5.10 address the most general case of parametric tolerable solution sets. Note that (5.7) gives another description of the parametric tolerable solution set by $n2^{m_1+1}$ inequalities. This description is equivalent to the description given in Proposition 5.8 that contains only n absolute-value inequalities.

5.2. 2D parametric AE solution sets. In [9] we studied the elimination of 2nd class existentially quantified parameters from two characterizing inequalities. The next theorem, giving an explicit description of the parametric AE solution sets to any two-dimensional (2D) linear system, follows from [9, Theorem 4.1] and Theorem 4.2.

THEOREM 5.12. *A parametric AE solution set (4.1) to a 2D linear system (1.1a)–(1.1c) is described by the inequalities*

$$(5.8) \quad |A(\hat{p})x - b(\hat{p})| \leq \sum_{\mu=1}^m \delta_\mu |A_{\bullet\bullet \mu}x - b_{\bullet \mu}| \hat{p}_\mu,$$

$$(5.9) \quad \left| \Delta_{0,i} + \sum_{\mu=1, \mu \neq i}^m \Delta_{\mu,i} \hat{p}_\mu \right| \leq \sum_{\mu=1, \mu \neq i}^m \delta_\mu |\Delta_{\mu,i}| \hat{p}_\mu, \quad i \in M,$$

where $\delta_\mu := \{1 \text{ if } \mu \in \mathcal{E}, -1 \text{ if } \mu \in \mathcal{A}\}$, M is the index set of the 2nd class existentially quantified parameters, $\Delta_{\alpha,\beta}(x) := f_{\alpha,1}(x)f_{\beta,2}(x) - f_{\alpha,2}(x)f_{\beta,1}(x)$, and $f_{\lambda,1}(x), f_{\lambda,2}(x)$ are the components of the coefficient vector $f_\lambda(x) := A_{\bullet\bullet \lambda}x - b_{\bullet \lambda}$ of the parameter p_λ for $\lambda \in \{\alpha, \beta\}$.

For a system of two equations the above theorem implies

- (i) any parametric AE solution set is described by $2 + m_1$ absolute-value inequalities, where m_1 is the number of 2nd class E -parameters;
- (ii) the maximal degree of the polynomial equations describing the boundary of a 2D parametric AE solution set is not greater than 2.

Remark 5.1. The elimination of a 2nd class \mathcal{E} -parameter from more than two inequality couples is done by combining every two inequality couples containing this parameter. Although Theorem 5.12 (resp., [9, Theorem 4.1]) describes the solution

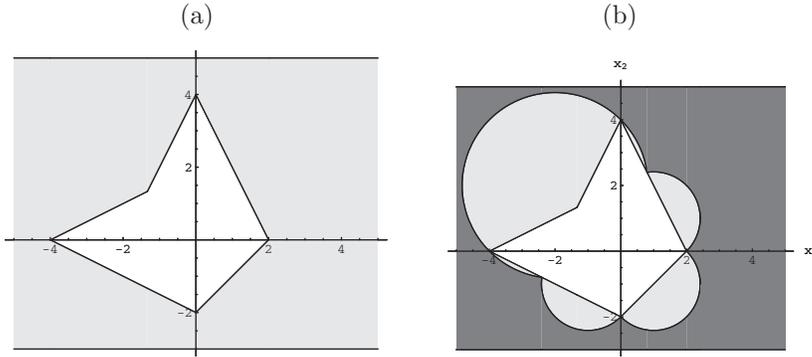


FIG. 5.3. The controllable nonparametric solution set (a) and the parametric controllable solution set (b) for the system from Example 5.3.

set of a 2D system, the proof of [9, Theorem 4.1] is general and constructive with respect to superfluous inequalities. Therefore [9, Theorem 4.1] (resp., Theorem 5.12) is applicable to any two characterizing inequality couples.

Example 5.3. Consider the parametric linear system $A(p)x = b(q)$, where

$$A(p) = \begin{pmatrix} p_1 & -p_2 \\ p_2 & p_1 \end{pmatrix}, \quad b(q) = \begin{pmatrix} 2q \\ 2q \end{pmatrix}, \quad p_1 \in [-2, 2], p_2 \in [-1, 2], q \in [1, 2].$$

The parametric controllable solution set

$$\Sigma_{cont}^p = \Sigma_{cont}(A(p), b(q), [p], [q]) := \{x \in \mathbb{R}^n \mid (\forall q \in [q])(\exists p \in [p])(A(p)x = b(q))\}$$

is described by the inequalities

$$\begin{aligned} \left| 3 + \frac{x_2}{2} \right| &\leq -1 + 2|x_1| + \frac{3|x_2|}{2}, \quad |-3x_1 - 3x_2| \leq -|x_1 + x_2| + 2|x_1^2 + x_2^2|, \\ \left| 3 - \frac{x_1}{2} \right| &\leq -1 + 2|x_2| + \frac{3|x_1|}{2}, \quad \left| 3x_1 - 3x_2 - \frac{x_1^2}{2} - \frac{x_2^2}{2} \right| \leq -|x_1 - x_2| + 3\frac{|x_1^2 + x_2^2|}{2}. \end{aligned}$$

The left two inequalities above are the so-called end-point inequalities which describe the nonparametric controllable solution set $\Sigma_{cont}(A([p]), b([q]))$. Since $A(p)$ is the zero matrix for $p = (0, 0)^T \in [p]$, both controllable solution sets (the parametric one and its corresponding nonparametric one) are unbounded. $\Sigma_{cont}(A([p]), b([q]))$ is presented in gray in Figure 5.3(a), and both the parametric (in dark gray) and the nonparametric controllable solution sets are presented in Figure 5.3(b).

The webComputing service framework [8] is expanded by a program (<http://cose.math.bas.bg/webMathematica/webComputing/ParametricAESSet.jsp>) for generating an explicit description of 2D parametric AE solution sets and for their graphical visualization.

5.3. Parametric controllable solution sets. For two nonempty disjoint index sets \mathcal{A} and \mathcal{E} , the general parametric controllable solution set is defined by

$$\begin{aligned} \Sigma_{cont}(A(p_{\mathcal{E}}), b(q_{\mathcal{A}}), [p_{\mathcal{E}}], [q_{\mathcal{A}}]) \\ := \{x \in \mathbb{R}^n \mid (\forall q_{\mathcal{A}} \in [q_{\mathcal{A}}])(\exists p_{\mathcal{E}} \in [p_{\mathcal{E}}])(A(p_{\mathcal{E}})x = b(q_{\mathcal{A}}))\}. \end{aligned}$$

It follows from Theorem 4.2 that the explicit description of a parametric controllable solution set can be easily derived from the explicit description of the united parametric solution set for a system with the same parametric matrix and a right-hand-side vector $[b] = b([q_A])$ with independent components. So far we know the explicit description of the united parametric solution set for systems with a symmetric or skew-symmetric matrix [6], as well as for arbitrary 2D parametric matrices [9] or systems involving only 1st class E -parameters [10]. The next theorem is obtained by applying Theorem 4.2 to the explicit description of the united parametric solution set for a system with a skew-symmetric matrix from [6].

THEOREM 5.13. *The controllable solution set to a system with a skew-symmetric matrix and independent right-hand-side vector, that is,*

$$\Sigma_{cont}^{skew} := \{x \in \mathbb{R}^n \mid \forall b \in [b], \exists A^{skew} \in [A], A^{skew}x = b\},$$

is described by

$$\begin{aligned} & \left| \dot{A}x - \dot{b} \right| \leq \hat{A}x - \hat{b}, \\ & \left| \sum_{i=1}^n M_{i \bullet} x_i (u_i + v_i) \right| \leq \sum_{i,j=1}^n |x_i x_j (u_i - v_j)| \hat{a}_{ij} - \sum_{i=1}^n |x_i (u_i + v_i)| \hat{b}_i \\ & \forall u, v \in \{0, 1\}^n \setminus \{0\}, u \preceq_{lex} v, \end{aligned}$$

where $M = \dot{A}x - \dot{b}$.

If q_ν are 1st class parameters for all $\nu \in \mathcal{A}$, then

$$\Sigma_{cont}(A(p\mathcal{E}), b([q_A]), [p\mathcal{E}]) = \Sigma_{cont}(A(p\mathcal{E}), b(q_A), [p\mathcal{E}], [q_A]).$$

However, in the general case of 2nd class universally quantified parameters we have an inclusion.

THEOREM 5.14. *If there are two equations α, β of the parametric system which involve simultaneously an existentially quantified parameter p_k and a universally quantified parameter q_l such that*

$$(5.10) \quad \text{sign}(f_{k\beta} b_{\alpha,l}) = \text{sign}(f_{k\alpha} b_{\beta,l}) \neq 0,$$

where $f_{k\lambda}(x) = A_{\lambda \bullet k} x$, $\lambda \in \{\alpha, \beta\}$, then

$$\Sigma_{cont}(A(p\mathcal{E}), b([q_A]), [p\mathcal{E}]) \subset \Sigma_{cont}(A(p\mathcal{E}), b(q_A), [p\mathcal{E}], [q_A]).$$

Proof. By Theorem 5.12,³ the description of $\Sigma_{cont}(A(p\mathcal{E}), b(q_A), [p\mathcal{E}], [q_A])$ involves the inequality (relative to (5.9))

$$(5.11) \quad \left| \Delta_{0,k} + \sum_{\mu \in \mathcal{E}, \mu \neq k} \Delta_{\mu,k} \dot{p}_\mu + \sum_{\nu \in \mathcal{A}} \Delta_{\nu,k} \dot{q}_\nu \right| \leq \sum_{\mu \in \mathcal{E}, \mu \neq k} |\Delta_{\mu,k}| \hat{p}_\mu - \sum_{\nu \in \mathcal{A}} |\Delta_{\nu,k}| \hat{q}_\nu,$$

while in the description of $\Sigma_{cont}(A(p\mathcal{E}), b([q_A]), [p\mathcal{E}])$ the corresponding inequality is

$$(5.12) \quad \left| \Delta_{0,k} + \sum_{\mu \in \mathcal{E}, \mu \neq k} \Delta_{\mu,k} \dot{p}_\mu + \sum_{\lambda \in \{\alpha, \beta\}} \Delta_{\lambda,k} \dot{b}(\dot{q}) \right| \leq \sum_{\mu \in \mathcal{E}, \mu \neq k} |\Delta_{\mu,k}| \hat{p}_\mu - |f_{k\beta}(x)| \hat{b}_\alpha([q_A]) - |f_{k\alpha}(x)| \hat{b}_\beta([q_A]),$$

³See Remark 5.1.

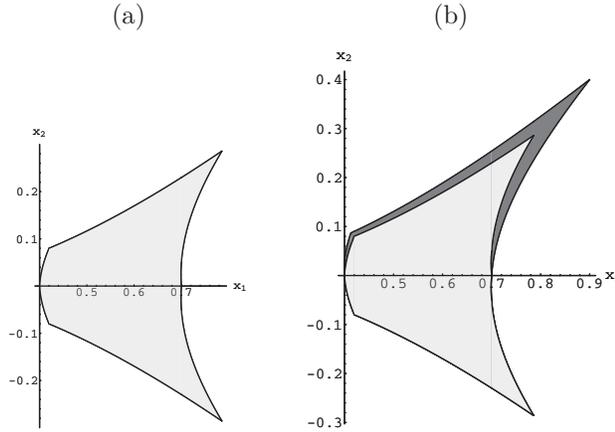


FIG. 5.4. The parametric controllable solution sets for the system from Example 5.4: (a) $\Sigma_{cont}(A(p), b([q]), [p])$ and (b) $\Sigma_{cont}(A(p), b(q), [p], [q])$ represented by $\Sigma_{cont}(A(p), b([q]), [p])$ in gray, which is expanded by the dark gray spike in the 1st orthant.

where $\hat{b}_\lambda([q_A]) = \sum_{\nu \in \mathcal{A}} |b_{\nu\lambda}| \hat{q}_\nu$. The left side of (5.11) is equal to the left side of (5.12). The relations (5.10) and $|u - v| \leq |u| + |v|$, where in the latter we have strong inequality for $\text{sign}(u) = \text{sign}(v) \neq 0$ and equality otherwise, imply that the right side of (5.11) is greater than the right side of (5.12), which proves the theorem. \square

Example 5.4. Consider the parametric linear system $A(p)x = b(q)$, where

$$A(p) = \begin{pmatrix} 2p_1 & p_2 \\ p_2 & 2p_1 \end{pmatrix}, \quad b(q) = \begin{pmatrix} \frac{3}{2} + q \\ q \end{pmatrix}, \quad p_1 \in [1, 2], \quad p_2 \in [-1, 1], \quad q \in \left[-\frac{1}{10}, \frac{1}{10}\right].$$

Since the coefficients of q in the two components of $b(q)$ are positive and likewise the coefficients of p_1 and p_2 , $\Sigma_{cont}(A(p), b([q]), [p]) \subseteq \Sigma_{cont}(A(p), b(q), [p], [q])$ in the first orthant ($x_1 \geq 0, x_2 \geq 0$), while in the fourth orthant ($x_1 \geq 0, x_2 \leq 0$) $\Sigma_{cont}(A(p), b([q]), [p]) = \Sigma_{cont}(A(p), b(q), [p], [q])$. The cross inequalities describing the two parametric controllable solution sets with eliminated p_1 are

$$3|x_2| \leq -|x_1|/5 - |x_2|/5 + |-2x_1^2 + 2x_2^2| \leq -|2x_1 - 2x_2|/10 + |-2x_1^2 + 2x_2^2|,$$

where the expression in the middle presents the right-hand side of the characterizing inequality for $\Sigma_{cont}(A(p), b([q]), [p])$, the expression to the right presents the right-hand side of the characterizing inequality for $\Sigma_{cont}(A(p), b(q), [p], [q])$. Eliminating p_2 we obtain

$$|-3x_1/2 + 3x_1^2 - 3x_2^2| \leq -|x_1|/10 - |x_2|/10 + |x_1^2 - x_2^2| \leq -|x_1 + x_2|/10 + |x_1^2 - x_2^2|.$$

Both parametric controllable solution sets are presented in Figure 5.4. For their interval hull we have

$$\begin{aligned} \square \Sigma_{cont}(A(p), b([q]), [p]) &= \left(\left[\frac{2}{5}, \frac{11}{14} \right], \left[-\frac{2}{7}, \frac{2}{7} \right] \right)^\top, \\ \square \Sigma_{cont}(A(p), b(q), [p], [q]) &= \left(\left[\frac{2}{5}, \frac{9}{10} \right], \left[-\frac{2}{7}, \frac{2}{5} \right] \right)^\top. \end{aligned}$$

It follows from Corollary 4.4 that the parametric controllable solution set has the same shape as the parametric united solution set for a system with the same

parametric matrix and a right-hand-side vector $[b] = b([q])$. In the special case when $A(p)$ involves only 1st class parameters, the parametric controllable solution set has linear shape.

6. Conclusion. The description of parametric AE solution sets by Fourier–Motzkin type parameter elimination is feasible and much faster and more compact than by quantifier elimination or other techniques. The description of a parametric AE solution set is simpler and usually involves fewer characterizing inequalities than the description of the corresponding united parametric solution set for the same system. Knowing the explicit description of a united parametric solution set, we can easily obtain the explicit description of any parametric AE solution set for the same system. Unfortunately, so far we know the explicit description of the united parametric solution set to only a few systems with fixed data dependencies. Therefore more research is necessary in this direction.

Many AE solution sets for a given parametric system are empty sets. The inequalities describing a parametric AE solution set present necessary and sufficient conditions for the solution set to be nonempty. If we do not know the so-called cross inequalities obtained by the elimination of 2nd class existentially quantified parameters, then the well-known end-point characterizing inequalities present a necessary condition for the parametric AE solution set to be nonempty. We proved various inclusion relations between different parametric AE solution sets corresponding to a nonparametric system. Knowing the description of a parametric AE solution set, we know the maximal degree of the polynomial equations describing the solution set boundary. We proved that all parametric tolerable solution sets are convex polyhedrons. We hope that the explicit description of parametric AE solution sets will facilitate exploring more properties and developing new numerical methods for the parametric AE solution sets.

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