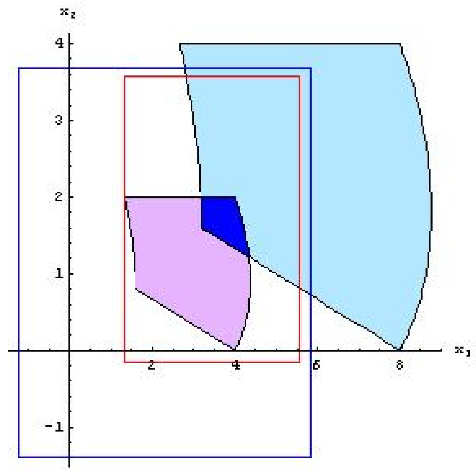
Parametric AE Solution Sets: Properties and Estimations



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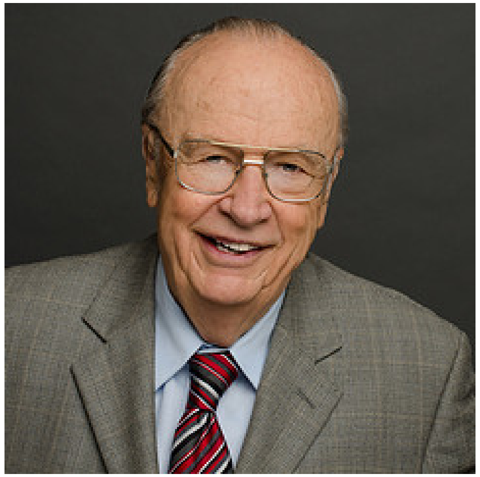
Parametric Problems



"Unless you are able to handle dependent data,

you will never gain interest of the engineers."

Ivo Babuška, talking to J. Rohn, 1992.

Outline

I Parametric AE Solution Sets (Σ^p_{AE}): Definition
Application example
Characterization, Properties

II Parametric AE Solution Sets (Σ^p_{AE}): Outer estimation
Inner estimation

III Unbounded Σ^p_{tol}

Parametric Linear Systems

Consider the linear algebraic system

$$A(p) \cdot x \ = \ b(p),$$

where

$$A(p) := A_0 + \sum_{k=1}^{K} A_k p_k, \qquad b(p) := b_0 + \sum_{k=1}^{K} b_k p_k$$

$$A_i \in \mathbb{R}^{n \times m}, \ b_i \in \mathbb{R}^n, \quad i = 0, \ldots, K$$

the uncertain parameters p_k vary within given intervals

$$p \ \in \ [p] = ([\underline{p}_1, \overline{p}_1], \ldots, [\underline{p}_K, \overline{p}_K])^\top.$$

Parametric AE Solution Sets

For $\quad \mathcal{A} \cup \mathcal{E} = \{1, \ldots, K\}, \quad \mathcal{A} \cap \mathcal{E} = \emptyset,$

$$\Sigma^{p}_{AE} := \{x \in \mathbb{R}^n \mid (\forall p_{\mathcal{A}} \in [p_{\mathcal{A}}])(\exists p_{\mathcal{E}} \in [p_{\mathcal{E}}])(A(p)x = b(p))\}\,.$$

AE terminology is after S. Shary.

The quantification of the parameters concerns the solution set, not the system.

For a given $A(p)x = b(p)$, $p \in [p] \in \mathbb{IR}^K$, there are 2^K parametric solution sets Σ^{p}_{AE}.

Parametric AE Solution Sets — special cases

$$\Sigma^p_{uni}\left(A(p), b(p), [p]\right) \quad := \quad \{x \in \mathbb{R}^n \mid \exists p \in [p],\; A(p)x = b(p)\}$$

$$\begin{aligned}\Sigma^p_{tol} \quad &= \quad \Sigma\left(A(p_{\mathcal{A}}), b(p_{\mathcal{E}}), [p]\right)\\ &:= \quad \{x \in \mathbb{R}^n \mid (\forall p_{\mathcal{A}} \in [p_{\mathcal{A}}])(\exists p_{\mathcal{E}} \in [p_{\mathcal{E}}])(A(p_{\mathcal{A}})x = b(p_{\mathcal{E}}))\}\end{aligned}$$

$$\begin{aligned}\Sigma^p_{cont} \quad &= \quad \Sigma\left(A(p_{\mathcal{E}}), b(p_{\mathcal{A}}), [p]\right)\\ &:= \quad \{x \in \mathbb{R}^n \mid (\forall p_{\mathcal{A}} \in [p_{\mathcal{A}}])(\exists p_{\mathcal{E}} \in [p_{\mathcal{E}}])(A(p_{\mathcal{E}})x = b(p_{\mathcal{A}}))\}\end{aligned}$$

Example

Consider the Lyapunov matrix equation

$$\boldsymbol{A}\boldsymbol{X} + \boldsymbol{X}\boldsymbol{A}^\top = \boldsymbol{F},$$

A common approach is to transform a matrix equation into linear system

$$\boldsymbol{P}\boldsymbol{x} = \boldsymbol{f},$$

where $\boldsymbol{P} = \boldsymbol{I}_n \otimes \boldsymbol{A} + \boldsymbol{A} \otimes \boldsymbol{I}_n, \quad \boldsymbol{x} = \mathrm{vec}(\boldsymbol{X}), \quad \boldsymbol{f} = \mathrm{vec}(\boldsymbol{F}).$

If $\boldsymbol{A} \in [\boldsymbol{A}]$, $\boldsymbol{F} \in [\boldsymbol{F}]$, or $\boldsymbol{A}$, $\boldsymbol{F}$ have linear uncertainty structure,

in both cases, $\boldsymbol{P}$ has a linear uncertainty structure.

Therefore, a $\Sigma^{p}_{\boldsymbol{AE}}$ must be considered

depending on the context of the particular problem.

Example — Controllability

Sokolova S., Kuzmina, E., 2008.

Consider

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where $A \in [A] \in \mathbb{IR}^{n \times n}$, $B \in [B] \in \mathbb{IR}^{n \times m}$.

Let $[A]$ be assimptotically stable.

The interval object is completely controllable if and only if

$$\text{rank}[V] = n, \qquad [V] \subseteq \Sigma_{tol}([A], [B]),$$

where

$$\Sigma_{tol}([A], [B]) := \{V \in \mathbb{R}^{n \times n} \mid (\forall A \in [A])(\exists B \in [B])(AV + VA^{\top} = -BB^{\top})\}.$$

Example — Controllability

Controllability analysis reduces to

$$\text{finding} \qquad [\boldsymbol{v}] \subseteq \boldsymbol{\Sigma}_{tol}(\boldsymbol{P}(\boldsymbol{a}_{ij}), \boldsymbol{f}(\boldsymbol{f}_{ij}), [\boldsymbol{A}], [\boldsymbol{F}]),$$

where

$$\boldsymbol{P}(\boldsymbol{a}_{ij}) := \boldsymbol{I}_n \otimes \boldsymbol{A} + \boldsymbol{A} \otimes \boldsymbol{I}_n, \qquad \boldsymbol{a}_{ij} \in [\boldsymbol{a}_{ij}]$$

$$[\boldsymbol{v}] = \text{vec}([\boldsymbol{V}]), \qquad \boldsymbol{f}(\boldsymbol{f}_{ij}) := \text{vec}(\boldsymbol{F} = -\boldsymbol{B}\boldsymbol{B}^\top).$$

Parametric AE Solution Sets

Theorem 1.

$$\Sigma^p_{AE} = \bigcap_{p_{\mathcal{A}} \in [p_{\mathcal{A}}]} \bigcup_{p_{\mathcal{E}} \in [p_{\mathcal{E}}]} \{x \in \mathbb{R}^n \mid A(p_{\mathcal{A}}, p_{\mathcal{E}}) \cdot x = b(p_{\mathcal{A}}, p_{\mathcal{E}})\}\,.$$

Parametric AE Solution Sets

GOAL:

explicit representation of Σ^p_{AE} by means of inequalities

Why?

- exploring the solution set properties,

  which helps designing better (sharp, fast) numerical methods

- finding exact bounds,

  which helps in testing new numerical methods

The problem is related to Quantifier Elimination.

Explicit Description of Σ^p_{uni}

Fourier-Motzkin-like Elimination of $\mathcal{E}$-parameters

G. Alefeld, V. Kreinovich, G. Mayer, J. Comput. Appl. Math. 152, 2003.

Improved in: E. Popova, BIT Numerical Mathematics, 2011.

- uniform representation of the characterizing inequalities
- considerable reduction their number
- removing the dependency on the particular orthant
- proving some superfluous inequalities

Classification of the parameters

Definition 1. *A parameter is of* **1st class** *if it is involved in only one equation does not matter how many times.*

Definition 2. *A parameter is of* **2nd class** *if it is involved in more than one equation of the system.*

$$\begin{pmatrix} p_1 & 1 & 1 \\ p_2 & 2p_1 & p_2 + 1 \\ 1 & 1 & 3p_1 - 1 \end{pmatrix} \cdot x = \begin{pmatrix} p_3 - p_4 \\ p_1 - p_2/3 \\ p_3/2 \end{pmatrix}$$

Parametric AE Solution Sets

E. D. Popova, W. Krämer, *Characterization of AE Solution Sets to a Class of Parametric Linear Systems*, Compt. rend. Acad. bulg. Sci. 64(3):325-332, 2011.

Theorem 2. *If* $x \in \Sigma^{p}_{AE} \neq \emptyset$,

$$\sum_{\nu \in \mathcal{A}} (A_\nu x - b_\nu)[p_\nu] \subseteq b_0 - A_0 x + \sum_{\mu \in \mathcal{E}} (b_\mu - A_\mu x)[p_\mu].$$

equivallently

$$|A(\dot{p})x - b(\dot{p})| \quad \leq \quad \sum_{k=1}^{K} \delta_k |A_k x - b_k| \widehat{p}_k,$$

where $\delta_k := \{1 \text{ if } \mu \in \mathcal{E},\ -1 \text{ if } \mu \in \mathcal{A}\}$, $\quad \dot{p} := mid([p]),\ \widehat{p} := rad([p])$.

Parametric AE Solution Sets

Theorem 3. *Let* $\boldsymbol{A}(\boldsymbol{p})\boldsymbol{x} = \boldsymbol{b}(\boldsymbol{p})$ *involves only 1st class* $\boldsymbol{\mathcal{E}}$*-parameters. A point* $\boldsymbol{x} \in \mathbb{R}^n$ *belongs to* $\boldsymbol{\Sigma}^{\boldsymbol{p}}_{\boldsymbol{AE}}$, **if and only if**

$$\sum_{\nu \in \mathcal{A}} (A_\nu x - b_\nu)[p_\nu] \subseteq b_0 - A_0 x + \sum_{\mu \in \mathcal{E}} (b_\mu - A_\mu x)[p_\mu].$$

equivallently

$$|A(\dot{p})x - b(\dot{p})| \quad \leq \quad \sum_{k=1}^{K} \delta_k |A_k x - b_k| \widehat{p}_k,$$

where $\delta_k := \{1 \text{ if } \mu \in \mathcal{E}, \ -1 \text{ if } \mu \in \mathcal{A}\}$, $\qquad \dot{p} := mid([p]), \ \widehat{p} := rad([p])$.

Parametric AE Solution Sets

E.Popova, *Explicit Description of AE Solution Sets to Parametric Linear Systems*, SIMAX 33(4):11721189.

If $A(p)x = b(p)$ involves 2nd class $\mathcal{E}$-parameters,
a point $x \in \mathbb{R}^n$ belongs to Σ^p_{AE}, **if and only if**

$$|A(\dot{p})x - b(\dot{p})| \quad \leq \quad \sum_{k=1}^{K} \delta_k |A_k x - b_k| \widehat{p}_k,$$

and "cross" inequalities

$$\left| w_\lambda(x) + \sum_{\mu \in \mathcal{E}} u_{\lambda,\mu}(x) \dot{p}_\mu + \sum_{\mu \in \mathcal{A}} v_{\lambda,\mu}(x) \dot{p}_\mu \right| \leq \sum_{\mu \in \mathcal{E}} |u_{\lambda,\mu}(x)| \widehat{p}_\mu - \sum_{\mu \in \mathcal{A}} |v_{\lambda,\mu}(x)| \widehat{p}_\mu,$$

$$\lambda \in \mathcal{T}$$

obtained by Fourier-Motzkin-like elimination of $\mathcal{E}$-parameters

$\delta_\mu := \{1$ if $\mu \in \mathcal{E}$, -1 if $\mu \in \mathcal{A}\}$, $\quad \dot{p} :=$ mid$([p])$, $\widehat{p} :=$ rad$([p])$.

Parametric AE Solution Sets

- The description of Σ^p_{AE} by F-M elimination of $\mathcal{E}$-parameters

  is feasible, much faster & compact than by Quantifier Elimination.

- We have Oettly-Prager-type description of Σ^p_{AE}

– explicit for some classes Σ^p_{AE} (symmetric, skew-symmetric, for 1st class $\mathcal{E}$-pars, 2D)

– algorithmic procedure in general

- For description & visualization of 2D Σ^p_{AE}, use

  `http://cose.math.bas.bg/webMathematica/webComputing/ParametricAESSet.jsp`

Further research is necessary on:

– the description of Σ^p_{uni} with fixed data dependencies,

– more conditions for sflu/red ineqs & formula for the degree of the poly.

Parametric AE Solution Sets — Properties

- The elimination of $\mathcal{A}$-parameters and 1st class $\mathcal{E}$-parameters

  does not introduce "cross" inequalities.

Corollary 1. *The infimum/supremum of a parametric $\boldsymbol{AE}$ solution set is attained at particular end-points of the intervals for the 1st class $\mathcal{E}$-parameters and for the $\mathcal{A}$-parameters.*

The boundary of $\boldsymbol{\Sigma}^{\boldsymbol{p}}_{\boldsymbol{AE}}$ is linear w.r.t. these parameters,

although $\boldsymbol{\Sigma}^{\boldsymbol{p}}_{\boldsymbol{AE}}$ may not depend linearly on these parameters.

Parametric AE Solution Sets — Properties

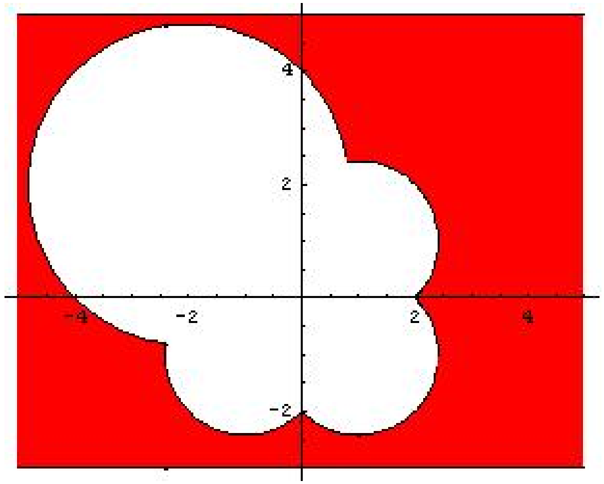
- The boundary of Σ^p_{AE} involving 2nd class $\mathcal{E}$-parameters

  may consist of polynomials of arbitrary degree.

$$\begin{pmatrix} p_1 & -p_2 \\ p_2 & p_1 \end{pmatrix} x = \begin{pmatrix} 2p_3 \\ 2p_3 \end{pmatrix}$$

$p_1 \in [-2, 2], p_2 \in [-1, 2], \qquad p_3 \in [1, 2]$

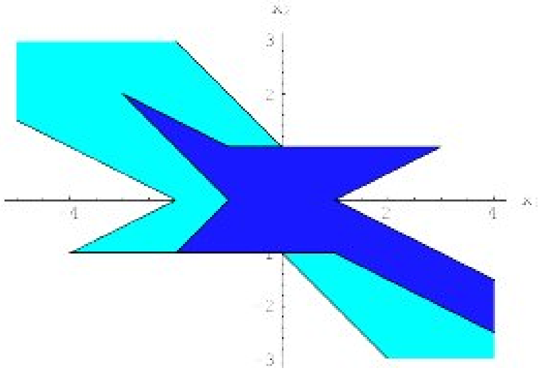
$\Sigma_{\forall p_3 \exists p_1, p_2}$

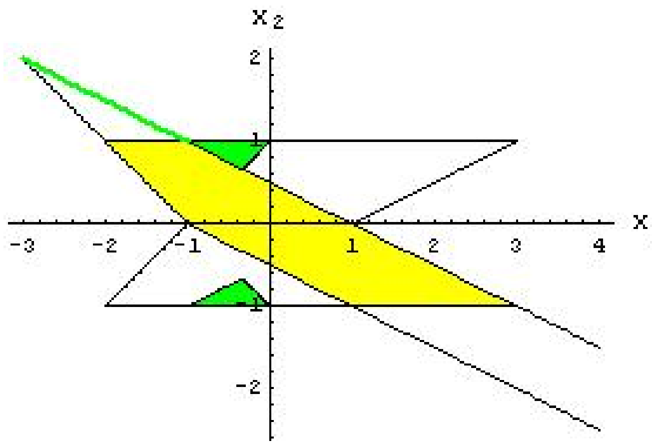


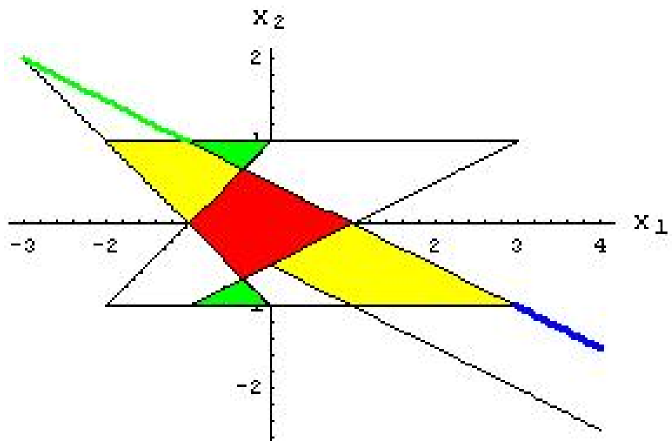
However Σ^p_{AE} is not convex even in a single orthant.

Examples

$$\begin{pmatrix} p_1 & p_1+1 \\ p_2+1 & -2p_4 \end{pmatrix} x \;=\; \begin{pmatrix} p_3 \\ -3p_2+1 \end{pmatrix}, \quad p_1, p_2 \in [0,1],\ p_3, p_4 \in [-1,1]$$







$\Sigma_{\exists p_{1\dots4}} \subseteq \Sigma_{\substack{\exists\exists\ \exists \\ \exists\exists\ \exists}}$

$\Sigma_{\forall p_1 \exists p_{2\dots4}}$ – bounded

$\Sigma_{\forall p_3 \exists p_1, p_2, p_4}$ – unbounded

$\Sigma_{\forall p_2 \exists p_1, p_3, p_4}$ – disconnected

$\Sigma_{\forall p_4 \exists p_1, p_2, p_3}$ – bounded

$\Sigma_{\forall p_1, p_2 \exists p_3, p_4}$ – segment

$\Sigma_{\forall p_2, p_4 \exists p_2, p_4}, \dots$ – empty

Parametric Tolerable Solution Set — Properties

$$\begin{aligned}\Sigma^{p}_{tol} &= \Sigma\left(A(p_{\mathcal{A}}), b(p_{\mathcal{E}}), [p]\right) \\ &:= \{x \in \mathbb{R}^n \mid (\forall p_{\mathcal{A}} \in [p_{\mathcal{A}}])(\exists p_{\mathcal{E}} \in [p_{\mathcal{E}}])(A(p_{\mathcal{A}})x = b(p_{\mathcal{E}}))\}\end{aligned}$$

Theorem 4. $\Sigma\left(A(p_{\mathcal{A}}), b(p_{\mathcal{E}}), [p]\right)$ *is a* **convex polyhedron**.

I. Sharaya & S. Shary prove it for some special cases.

Inclusion Relations

$$\cdots \subseteq \Sigma_{class}(A(u), b(u), [u]) \subseteq \Sigma_{class}(A(v), b(v), [v]) \subseteq \cdots$$

class $\in$ {uni, tol, cont, fixed$\mathcal{A}$-pars}

for given $A(p), b(p), [p]$, there are unique $A([p]), b([p])$

however, for given $[A], [b]$ there are infinitely many choices of $p, [p], A(p), b(p)$

such that $A([p]) = [A], \quad b([p]) = [b].$

Inclusion Relations

Lemma 1. *For*

$$f(p) = \alpha_0 + \alpha p_{i_1} + f_0(p \backslash \{p_{i_1}, p_{i_2}\}), \qquad g(p) = \beta_0 + \beta p_{i_2} + g_0(p \backslash \{p_{i_1}, p_{i_2}\})$$

we can define

$$\begin{aligned} \tilde{f}(q) &:= q_1 + q_2 + f_0(p \setminus \{p_{i_1}, p_{i_2}\}) \\ \tilde{g}(q) &:= q_1 + q_3 + g_0(p \setminus \{p_{i_1}, p_{i_2}\}), \end{aligned}$$

where $q_1 \in [q_1]$ *is arbitrary*, $\dot{q}_2 = \alpha_0 + \alpha \dot{p}_{i_1} - \dot{q}_1$, $\hat{q}_2 = |\alpha| \hat{p}_{i_1} - \hat{q}_1$,

$\dot{q}_3 = \beta_0 + \beta \dot{p}_{i_2} - \dot{q}_1$, $\hat{q}_3 = |\beta| \hat{p}_{i_2} - \hat{q}_1$,

such that $\quad f([p]) = \tilde{f}([q]), \qquad g([p]) = \tilde{g}([q]).$

Inclusion Relations

Theorem 5. *For two parameter vectors* $\boldsymbol{u} \in [\boldsymbol{u}] \in \mathbb{IR}^{m_1}$, $\boldsymbol{v} \in [\boldsymbol{v}] \in \mathbb{IR}^{m_2}$, *such that* $\boldsymbol{A}([\boldsymbol{u}]) = \boldsymbol{A}([\boldsymbol{v}]) = [\boldsymbol{A}]$, $\boldsymbol{b}([\boldsymbol{u}]) = \boldsymbol{b}([\boldsymbol{v}]) = [\boldsymbol{b}]$ *and*

$\boldsymbol{A}(\boldsymbol{u})$, $\boldsymbol{b}(\boldsymbol{u})$ *are obtained from* $\boldsymbol{A}(\boldsymbol{v})$, $\boldsymbol{b}(\boldsymbol{v})$ *by successive application of Lemma 1,*

similarly $\boldsymbol{A}(\boldsymbol{v})$, $\boldsymbol{b}(\boldsymbol{v})$ *are obtained from* $[\boldsymbol{A}]$, $[\boldsymbol{b}]$, *then*

$$\cdots \subseteq \Sigma_{uni}(A(u), b(u), [u]) \subseteq \Sigma_{uni}(A(v), b(v), [v]) \subseteq \cdots \subseteq \Sigma_{uni}([A], [b]).$$

Corollary 2. *Theorem 5 is applicable to parametric* $\boldsymbol{AE}$ *solution sets which have the same structure of the dependencies between the* $\boldsymbol{\mathcal{A}}$*-parameters and the same domain* $[\boldsymbol{p}_{\mathcal{A}}]$.

Inclusions — Parametric Tolerable Solution Set

Theorem 6. *Let* $\boldsymbol{A}_{ri}([\boldsymbol{u}]) = \boldsymbol{A}_{rd}([\boldsymbol{v}]) \subseteq [\boldsymbol{A}]$.
If $\boldsymbol{q} \in [\boldsymbol{q}]$ *involves only 1st class parameters, then*

$$\boldsymbol{\Sigma}_{tol}([\boldsymbol{A}], \boldsymbol{b}([\boldsymbol{q}])) \subseteq \boldsymbol{\Sigma}_{tol}(\boldsymbol{A}([\boldsymbol{u}]), \boldsymbol{b}([\boldsymbol{q}])) = \boldsymbol{\Sigma}_{tol}(\boldsymbol{A}_{ri}(\boldsymbol{u}), [\boldsymbol{u}], \boldsymbol{b}([\boldsymbol{q}])) \subseteq \boldsymbol{\Sigma}_{tol}(\boldsymbol{A}_{rd}(\boldsymbol{v}), [\boldsymbol{v}], \boldsymbol{b}([\boldsymbol{q}])).$$

If $\boldsymbol{A}(\boldsymbol{v})$ *involves more dependencies than* $\boldsymbol{A}(\boldsymbol{u})$ *and* $\boldsymbol{A}([\boldsymbol{u}]) = \boldsymbol{A}([\boldsymbol{v}])$, *then*

$$\boldsymbol{\Sigma}_{tol}(\boldsymbol{A}(\boldsymbol{u}), \boldsymbol{b}(\boldsymbol{q}), [\boldsymbol{u}], [\boldsymbol{q}]) \quad \subseteq \quad \boldsymbol{\Sigma}_{tol}(\boldsymbol{A}(\boldsymbol{v}), \boldsymbol{b}(\boldsymbol{q}), [\boldsymbol{v}], [\boldsymbol{q}]).$$

Special cases for $\boldsymbol{A}_{ri}(\boldsymbol{u})$ are considered by Sharaya (2008), Sharaya & Shary, RC (2011).

Inclusions — Parametric Controllable Solution Set

Theorem 7.

$$\Sigma_{cont}(A(p_{\mathcal{E}}), b([q_{\mathcal{A}}]), [p_{\mathcal{E}}]) \subseteq \Sigma_{cont}(A(p_{\mathcal{E}}), b(q_{\mathcal{A}}), [p_{\mathcal{E}}], [q_{\mathcal{A}}]).$$

Theorem 7 can be combined with the inclusion theorem for Σ^{p}_{uni}.

Examples demonstrating the combination of Inclusion Theorems are given in Popova, SIMAX.

Outer and Inner Estimations

$$[\boldsymbol{v}] \;\subseteq\; \boldsymbol{\Sigma}^{\boldsymbol{p}}_{\boldsymbol{AE}} \;\subseteq\; [\boldsymbol{u}]$$

Outer and Inner Estimations: 1) End-point Approach

For a given index set I, define the set $\mathcal{B}_I$ of end-points (vertices) of $[p_\mathcal{I}]$.

Theorem 8. *It holds*

$$\Sigma^p_{AE} = \bigcap_{\tilde{p}_\mathcal{A} \in \mathcal{B}_\mathcal{A}} \Sigma(A(\tilde{p}_\mathcal{A}, p_\mathcal{E}), b(\tilde{p}_\mathcal{A}, p_\mathcal{E}), [p_\mathcal{E}]).$$

Corollary 1. *For* $\Sigma^p_{AE} \neq \emptyset$,

$$\Box\Sigma^p_{AE} \subseteq \bigcap_{\tilde{p}_\mathcal{A} \in \mathcal{B}_\mathcal{A}} \Box\Sigma(A(\tilde{p}_\mathcal{A}, p_\mathcal{E}), b(\tilde{p}_\mathcal{A}, p_\mathcal{E}), [p_\mathcal{E}]).$$

$[v] \subseteq \Sigma(A(\tilde{p}_\mathcal{A}, p_\mathcal{E}), b(\tilde{p}_\mathcal{A}, p_\mathcal{E}), [p_\mathcal{E}]) \subseteq [u]$ by any parametric solver for Σ^p_{uni}.

Outer and Inner Estimations: 2nd Approach

based on the characterization

$$|A(\dot{p})x - b(\dot{p})| \leq \sum_{\mu=1}^{K} \delta_\mu |A_\mu x - b_\mu| \widehat{p}_\mu,$$

where $\delta_\mu := \{1 \text{ if } \mu \in \mathcal{E}, \ -1 \text{ if } \mu \in \mathcal{A}\}$, $\dot{p} :=$ mid$([p])$, $\widehat{p} :=$ rad$([p])$.

Outer Estimation $\quad \Sigma^{p}_{AE} \subseteq [u]$

E. D. Popova, M. Hladík, *Outer Enclosures to Parametric AE Solution Set*, to appear in Soft Computing.

Theorem 9. *(Bauer–Skeel generalization) Let* $A(\dot{p})$ *be regular and define*

$$C := A^{-1}(\dot{p}), \qquad x^* := Cb(\dot{p}), \qquad M := \sum_{k=1}^{K} |CA_k|\hat{p}_k.$$

If $\rho(M) < 1$, *then every* $x \in \Sigma^{p}_{AE}$ *satisfies*

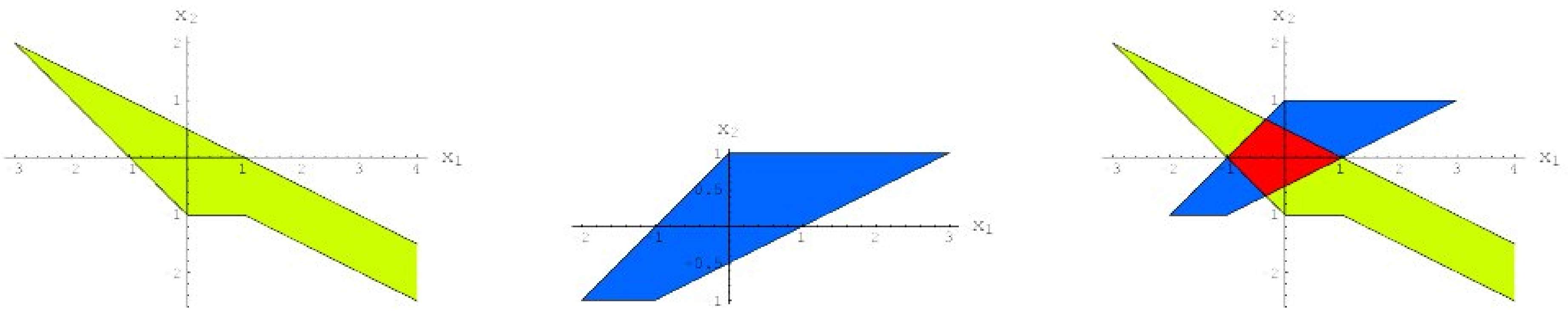
$$|x - x^*| \leq (I - M)^{-1}\left(\sum_{k\in\mathcal{E}} |C(A_k x^* - b_k)|\hat{p}_k - \sum_{k\in\mathcal{A}} |C(A_k x^* - b_k)|\hat{p}_k\right).$$

Outer Estimations — Properties for $\Sigma^p_{tol}(A(p_{\mathcal{A}}), b(p_{\mathcal{E}}), [p])$

- Bauer-Skeel method — gives worse enclosures
- End-Point Approach — gives the best enclosures, but not always the hull

$$\begin{pmatrix} p_1 & p_1+1 \\ p_2+1 & -2p_4 \end{pmatrix} x = \begin{pmatrix} p_3 \\ -3p_2+1 \end{pmatrix}, \quad p_1, p_2 \in [0,1], \quad p_3, p_4 \in [-1,1].$$

$$\Sigma^p_{\forall p_4 \exists p_{123}}$$



$$\Sigma^p_{\exists p_{123}}(A(\underline{p}_4)) \bigcap \Sigma^p_{\exists p_{123}}(A(\overline{p}_4)) = \Sigma^p_{\forall p_4 \exists p_{123}} \subset \square \Sigma^p_{\exists p_{123}}(A(\overline{p}_4)).$$

Outer Estimation of $\Sigma^p_{tol}(A(p_{\mathcal{A}}), b(p_{\mathcal{E}}), [p])$ — LP Approach

E. D. Popova, M. Hladík, *Outer Enclosures to Parametric AE Solution Set*, to appear in Soft Computing.

Proposition 1. *For every* $x \in \Sigma^p_{tol}$ *there are* $y^k \in \mathbb{R}^n$, $k \in \mathcal{A}$, *such that*

$$A(\dot{p})x + \sum_{k\in\mathcal{A}} \hat{p}_k y^k \le \sum_{k\in\mathcal{E}} |b_k|\hat{p}_k + b(\dot{p}),$$

$$-A(\dot{p})x + \sum_{k\in\mathcal{A}} \hat{p}_k y^k \le \sum_{k\in\mathcal{E}} |b_k|\hat{p}_k - b(\dot{p}),$$

$$A_k x \le y^k, \; -A_k x \le y^k, \quad \forall k \in \mathcal{A}.$$

Proposition 1 gives $\square\Sigma^p_{tol}$ for systems involving only 1st class $\mathcal{E}$-parameters.

SCAN'2012

Outer Estimation — Properties for $\Sigma^p_{con}(A(p_{\mathcal{E}}), b(p_{\mathcal{A}}), [p])$

E. D. Popova, M. Hladík, *Outer Enclosures to Parametric AE Solution Set*, to appear in Soft Computing.

Proposition.

The enclosure of $\boldsymbol{\Sigma^p_{con}}$ computed by the parametric AE-Bauer-Skeel method

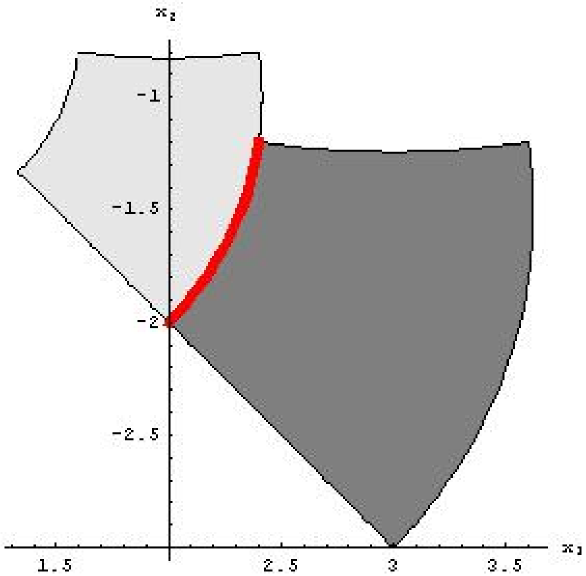
is always a subset

of the enclosure obtained by the end-point approach.

Outer Estimation — Properties for $\Sigma^p_{con}(A(p_{\mathcal{E}}), b(p_{\mathcal{A}}), [p])$

$$A(p) = \begin{pmatrix} p_1 & -p_2 \\ p_2 & p_1 \end{pmatrix}, \quad b(q) = \begin{pmatrix} 2q \\ 2q \end{pmatrix}, \qquad p_1 \in [0, \frac{1}{2}], p_2 \in [1, \frac{3}{2}], q \in [1, \frac{3}{2}]$$

$$\Sigma^p_{con} = \Sigma(A(p), b(1), [p]) \cap \Sigma(A(p), b(3/2), [p])$$



Outer Estimation — Properties for $\Sigma^{p}_{con}(A(p_{\mathcal{E}}), b(p_{\mathcal{A}}), [p])$

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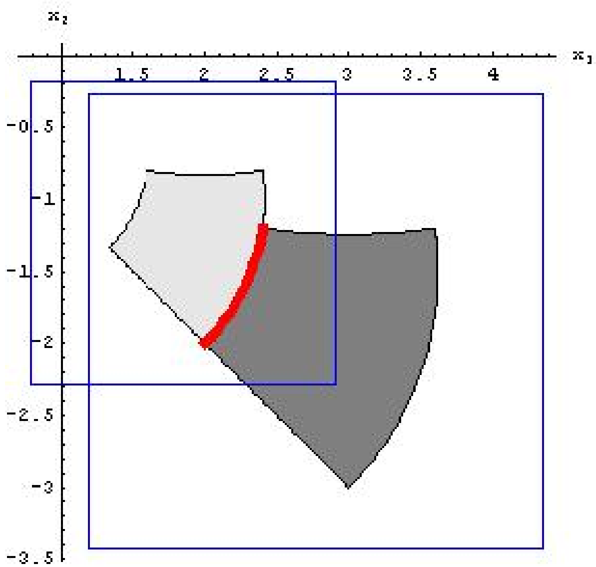
$$\Sigma^{p}_{con} = \Sigma(A(p), b(1), [p]) \cap \Sigma(A(p), b(3/2), [p])$$

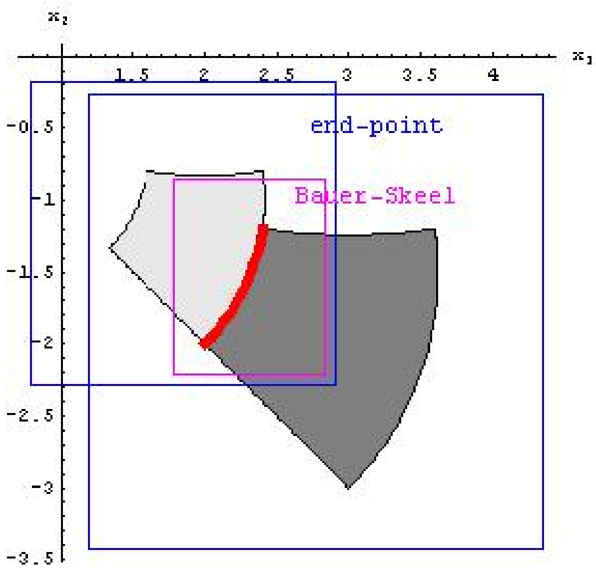
End-Point Approach:



Outer Estimation — Properties for $\Sigma^p_{con}(A(p_\mathcal{E}), b(p_\mathcal{A}), [p])$

$$A(p) = \begin{pmatrix} p_1 & -p_2 \\ p_2 & p_1 \end{pmatrix}, \quad b(q) = \begin{pmatrix} 2q \\ 2q \end{pmatrix}, \qquad p_1 \in [0, \frac{1}{2}], p_2 \in [1, \frac{3}{2}], q \in [1, \frac{3}{2}]$$

$$\Sigma^p_{con} = \Sigma(A(p), b(1), [p]) \cap \Sigma(A(p), b(3/2), [p])$$



Methods for Outer Estimation of Σ^p_{AE}

- The end-point approach has high computational complexity,

  however, it allows applying only methods for Σ^p_{uni}, and

  to attack large scale Σ^p_{tol}.

- The parametric B-S method is in real arithmetic,
  - its self-verified analogue requires Kaucher arithmetic;
  - B-S requires strong regularity of the parametric matrix & fails otherwise.

- There is a large room for further research.

Inner Estimation: $[v] \subseteq \Sigma_{tol}(A(p_{\mathcal{A}}), [b], [p_{\mathcal{A}}])$

$\ldots$

Inner Estimation:

S.Shary, 1996: The "end-point" approach provides $[v] \subseteq \Sigma_{tol}([A],[b])$

with comp. complexity $O(2^{n^2})$

By a complicated search-like algorithm he reduces the comp. complexity to $O(2^n)$.

Since $\Sigma_{tol}([A],[b]) = \Sigma_{tol}(A_{ri}(p),[b]),\qquad [A] = A_{ri}([p])$

consider $A_{ri}(p) = A^0 + \sum_{\nu=1}^{n} A^{\nu} p_{\nu}$,

where $A^0 = \mathrm{mid}([A])$, $A^{\nu} = \mathrm{rad}([A]_{\bullet\nu})$, $p_{\nu} \in [-1,1]$, $\nu = 1,\ldots,n$

and apply the "end-point" approach to

the parametric system with comp. complexity $O(2^n)$.

Inner Estimation: Application to Controllability

Consider

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = \boldsymbol{A}\boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{B}\boldsymbol{u}(\boldsymbol{t})$$

where $\boldsymbol{A} \in [\boldsymbol{A}] \in \mathbb{IR}^{n\times n}$, $\boldsymbol{B} \in [\boldsymbol{B}] \in \mathbb{IR}^{n\times m}$.

Let $[\boldsymbol{A}]$ be assimptotically stable.

The interval object is completely controllable if and only if

$$\text{rank}[\boldsymbol{V}] = \boldsymbol{n}, \qquad [\boldsymbol{V}] \subseteq \boldsymbol{\Sigma}_{tol}([\boldsymbol{A}], [\boldsymbol{B}]),$$

where

$$\boldsymbol{\Sigma}_{tol}([\boldsymbol{A}], [\boldsymbol{B}]) := \{\boldsymbol{V} \in \mathbb{R}^{n\times n} \mid (\forall \boldsymbol{A} \in [\boldsymbol{A}])(\exists \boldsymbol{B} \in [\boldsymbol{B}])(\boldsymbol{A}\boldsymbol{V} + \boldsymbol{V}\boldsymbol{A}^\top = -\boldsymbol{B}\boldsymbol{B}^\top)\}.$$

Inner Estimation: Application to Controllability

For

$$\mathrm{mid}([A]) = \begin{pmatrix} -1 & -1 & 2 \\ 3 & -2 & -5 \\ -2 & 1 & -5 \end{pmatrix}, \qquad \mathrm{rad}([a_{ij}]) = 3/100,$$

$$[B] = ([\tfrac{31}{4}, \tfrac{41}{4}], [-\tfrac{37}{4}, -\tfrac{27}{4}], [\tfrac{103}{4}, \tfrac{113}{4}])^\top$$

we obtain

$$[V] = \begin{pmatrix} [109.599, 110.685] & [-16.9308, -15.844] & [25.6931, 26.7799] \\ [-16.9308, -15.844] & [92.951, 94.0378] & [-41.2174, -40.1306] \\ [25.6931, 26.7799] & [-41.2174, -40.1306] & [53.8834, 54.9702] \end{pmatrix}$$

Parametric Tolerable Solution Set — Unboundedness

inspired by the work of I. Sharaya (2006, 07, ...) on unbounded nonparametric AE SSets

- a criterion for unbounded Σ^p_{tol}
- a more precise structure of Σ^p_{tol}

which imply

- new conditions for $\Sigma^p_{tol} \neq \emptyset$

and allow

- inner & outer estimations of unbounded Σ^p_{tol}

by methods for bounded Σ^p_{tol}.

General Conclusions

- Explicit Description of Σ^p_{AE} helps understanding their properties.

  key problem is the description of Σ^p_{uni}: several open problems

- Methods are available for $\Sigma^p_{AE} \neq \emptyset$, connected, further research on methods for:
  - disconnected Σ^p_{AE},
  - efficient estimation of Σ^p_{uni}

- Searching for best estimation of Σ^p_{AE}, one has to consider

  the inclusion relations & the properties of the methods.

- We have to pay more attention to the applications.

- These initial results open **a Large Room for Further Research**.