Improved Enclosure for Some Parametric Solution Sets with Linear Shape

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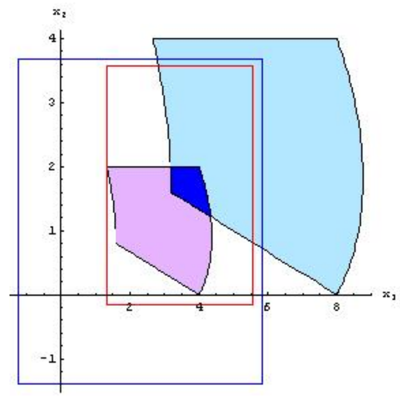

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Outline

- Parametric linear systems, Σ^p_{uni}, outer estimation
- Sufficient conditions for linear boundary of Σ^p_{uni}
- Generalization of Neumaier & Pownuk's method
- Implications, Applications, Examples
- Conclusions

Parametric Linear Systems

Consider the linear algebraic system

$$A(p) \cdot x \;=\; b(p),$$

where

$$A(p) := A_0 + \sum_{k=1}^{K} p_k A_k, \qquad b(p) := b_0 + \sum_{k=1}^{K} p_k b_k$$

$$A_i \in \mathbb{R}^{n \times m},\; b_i \in \mathbb{R}^n, \quad i = 0, \ldots, K$$

the uncertain parameters p_k vary within given intervals

$$p \;\in\; [p] = ([\underline{p}_1, \overline{p}_1], \ldots, [\underline{p}_K, \overline{p}_K])^\top.$$

Parametric Linear Systems

$$\begin{aligned} \Sigma^p_{uni} &= \Sigma^p_{uni}\,(A(p), b(p), [p]) \\ &:= \{x \in \mathbb{R}^n \mid \exists p \in [p],\ A(p)\cdot x = b(p)\} \end{aligned}$$

In worst-case analysis of uncertain systems

the GOAL is outer interval estimation $[u]$ of Σ^p_{uni}.

that is: Find interval vector $[u]$, such that $\Sigma^p_{uni} \subseteq [u]$.

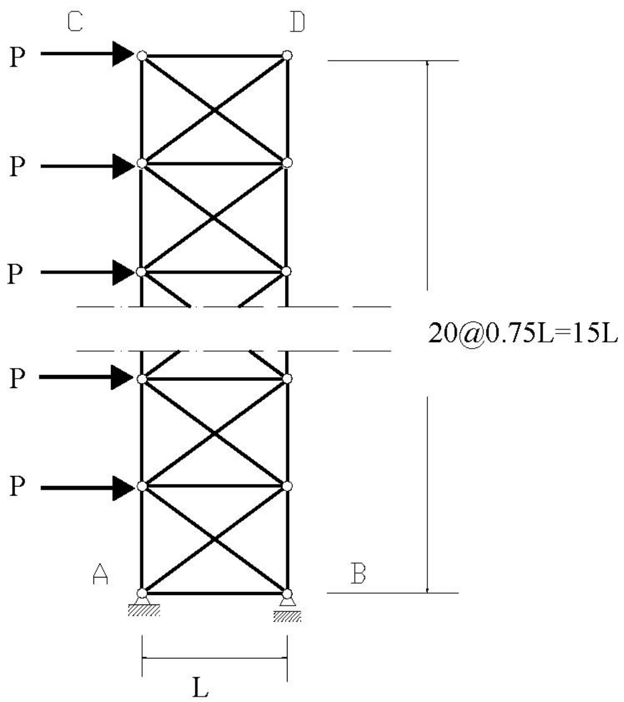
The best parametric method

influenced and motivated by

R.L.Muhanna, R.L.Mullen, H.Zhang, Interval FE as a basis for generalized models of uncertainty in engineering mechanics, in Proc. of REC'2004.

- A.Neumaier, A.Pownuk, Linear systems with large uncertainties, with applications to truss structures, Reliable Computing 13, 149-172, 2007.

exploited the structure of the parameter dependencies



FEM Analysis of Truss Structures

$$K(p)x = b$$

however, the el. stiffness matrices

$$(K_e)_{mn} = \frac{E_e A_e}{L_e} a_m a_n = a_m p_e a_n$$

thus

$$K(p) = B^\top Diag(p) B.$$

Neumaier & Pownuk's method: Main Theorem

Let $A(p)x = b(q)$ is representable as

$$(A_0 + LDR)x \;=\; b_0 + Fq \qquad (1)$$

for some $L \in \mathbb{R}^{m\times K_1}$, $R \in \mathbb{R}^{K_1\times n}$, $F \in \mathbb{R}^{n\times K_2}$ and $D = \mathrm{Diag}(p)$.

Let $D_0 \in \mathbb{R}^{K_1\times K_1}$ be such that $C := (A_0 + LD_0R)^{-1}$ exists,
and put $d = (D_0 - D)y$, where $y = Rx$.
If there are vectors $w \geq 0$, $w' > 0$ and w'' such that

$$w' \leq w - |D_0 - D||RCL|w, \qquad w'' \geq |D_0 - D||RCb_0 + RCFq|,$$

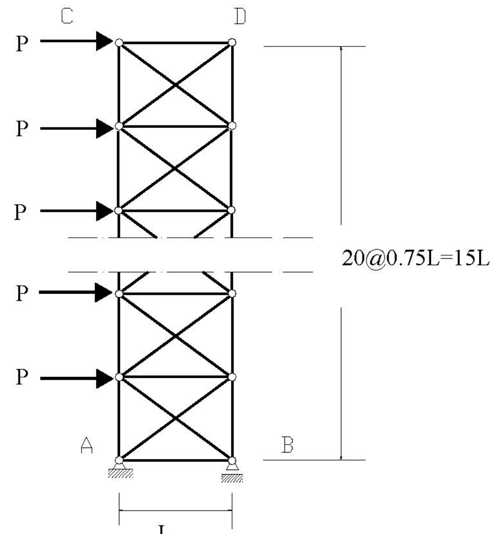
then

$$d \in [d] := [-\alpha w, \alpha w], \qquad \alpha = \max_i \frac{w_i''}{w_i'},$$

and the solution x of (1) is related to y and d by the equations

$$x \;=\; Cb_0 + CFq + CLd,$$
$$y \;=\; RCb_0 + RCFq + RCLd.$$

Neumaier & Pownuk's method for Σ^p_{uni}



Advantages:

- does not require strong regularity of $\boldsymbol{A}(\boldsymbol{p})$ on $[\boldsymbol{p}]$, while all other parametric methods do
- large parameter uncertainties
- scalable to high dimensions: $> \mathbf{5000}$ variables, $> \mathbf{10000}$ int. parameters (application to problems untractable so far)

Restrictions:

not every parametric matrix has the representation

$$(\boldsymbol{A}_0 + \boldsymbol{L}\text{Diag}(\boldsymbol{p})\boldsymbol{R})\boldsymbol{x} \ = \ \boldsymbol{b}_0 + \boldsymbol{F}\boldsymbol{q}$$

required is knowledge of $\boldsymbol{L} \in \mathbb{R}^{m \times \boldsymbol{K}_1}$, $\boldsymbol{R} \in \mathbb{R}^{\boldsymbol{K}_1 \times n}$ such that $\boldsymbol{A}(\boldsymbol{p}) = \boldsymbol{A}_0 + \boldsymbol{L}\boldsymbol{D}\boldsymbol{R}$.

the matrix and the r.h.side vector are independent.

Neumaier & Pownuk's method

$$(A_0 + LDR)x \;=\; b_0 + Fq$$

restriction:

required is knowledge of $L \in \mathbb{R}^{m \times K_1}$, $R \in \mathbb{R}^{K_1 \times n}$ such that $A(p) = A_0 + LDR$

GOAL:

- How to find such a representation, if it exists?

- If the representation is possible,
  generalize the method for dependencies between A and b

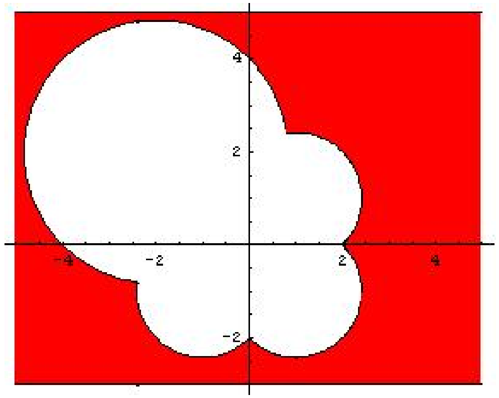
$$A(p)x = b(p, q)$$

Shape of Σ^p_{uni}

The shape of $\boldsymbol{\Sigma^p_{uni}}$ is determined by

the degree of the polynomials that define the boundary of $\boldsymbol{\Sigma^p_{uni}}$.

- In general, the shape of $\boldsymbol{\Sigma^p_{uni}}$ is nonlinear.



- For a large class of applied problems the shape of $\boldsymbol{\Sigma^p_{uni}}$ is linear.

Σ^p_{uni} with Linear Shape

basing on E. Popova, BIT Numerical Mathematics 52(1):179-200, 2012.

Theorem 1. *Σ^p_{uni} has linear shape w.r.t. a parameter p_k, $k \in \{1, \ldots, K\}$, if*

- *the nonzero elements of the vector $A_k x - b_k$ are linearly dependent,*
- *equivalently:* $\quad rank((A_k | b_k)) = 1$
- *equivalently:* *denoting $g_k(x)$ the polynomial GCD of $A_k x - b_k$*
  $g_k(x) =$*const if* $A_k = 0$ *and* $g_k(x) \neq$*const otherwise.*

Σ_{uni}^{p} with Linear Shape

Important Implication

Corollary 1. *The infimum/supremum of* Σ_{uni}^{p} *is attained at particular end-points of the intervals for the parameters that contribute linearly to the boundary.*

Theorem 1 gives only sufficient conditions for linear boundary of Σ_{uni}^{p}.

Σ_{uni}^{p} with Linear Shape

consider the parametric linear algebraic system in the form

$$A(p)x \;=\; b(p,q)$$

$$A(p) := A_0 + \sum_{k=1}^{k_1} p_k A_k$$

$$b(p,q) := b_0 + \sum_{k=1}^{k_1} p_k b_k + \sum_{k=k_1+1}^{K} q_k b_k.$$

Conversion Theorem

Theorem 2. *Let* $g_k(x) := GCD(A_k x)$ *be* $g_k(x) \neq$*const*

for every p_k, $k \in \{1, \ldots, k_1\}$.

Define

$$L := (l_1 | \ldots | l_{k_1}) \in \mathbb{R}^{n \times k_1}, \qquad l_k := A_k x / g_k(x) \in \mathbb{R}^n$$

$$R := (r_1 | \ldots | r_{k_1})^\top \in \mathbb{R}^{k_1 \times n}, \qquad r_k := \left(\frac{\partial g_k(x)}{\partial x_1}, \ldots, \frac{\partial g_k(x)}{\partial x_n}\right)^\top \in \mathbb{R}^n$$

$$F := (b_{k_1+1} | \ldots | b_K) \in \mathbb{R}^{n \times (K - k_1)}$$

If for every $k \in \{1, \ldots, k_1\}$, *there exists* $t_k \in \mathbb{R}$

such that $t_k l_k = b_k = \partial b(p,q) / \partial p_k$,

then Σ_{uni}^{p} *has linear shape* *and*

Equivalent Representation

Corollary 2. *If all parameters satisfy previous Theorem, then the system*

$$\boldsymbol{A}(\boldsymbol{p})\boldsymbol{x} \;=\; \boldsymbol{b}(\boldsymbol{p}, \boldsymbol{q})$$

has the equivalent representation

$$(\boldsymbol{A}_0 + \boldsymbol{L}\boldsymbol{D}\boldsymbol{R})\boldsymbol{x} \;=\; \boldsymbol{b}_0 + \boldsymbol{L}\boldsymbol{D}\boldsymbol{t} + \boldsymbol{F}\boldsymbol{q}$$

where $\boldsymbol{D} = Diag(\boldsymbol{p})$.

Neumaier & Pownuk's method: Generalization

Let $A(p)x = b(q)$ is representable as

$$(A_0 + LDR)x = b_0 + LDt + Fq \qquad (2)$$

for some $L \in \mathbb{R}^{n\times k_1}$, $R \in \mathbb{R}^{k_1\times n}$, $F \in \mathbb{R}^{n\times(K-k_1)}$, $t \in \mathbb{R}^{k_1}$, $D = \mathrm{Diag}(p)$.

Let $D_0 \in \mathbb{R}^{k_1\times k_1}$ be such that $C := (A_0 + LD_0R)^{-1}$ exists, and put $d = (D_0 - D)(y - t)$, where $y = Rx$.
If there are vectors $w \geq 0$, $w' > 0$ and w'' such that

$$w' \leq w - |D_0 - D||RCL|w, \quad w'' \geq |D_0 - D||RCb_0 + RCFq + RCLD_0t - t|,$$

then

$$d \in [d] := [-\alpha w, \alpha w], \qquad \alpha = \max_i \frac{w''_i}{w'_i},$$

and the solution x of (2) is related to y and d by the equations

$$\begin{aligned} x &= Cb_0 + CFq + CL(D_0t + d), \\ y &= RCb_0 + RCFq + RCL(D_0t + d). \end{aligned}$$

Example

Consider

$$\begin{pmatrix} p_1 + p_2 & p_1 - p_2 \\ p_1 - p_2 & p_1 + p_2 \end{pmatrix} x = \begin{pmatrix} 2p_2 + 3p_1 - 1 \\ -2p_2 + 3p_1 + 3 \end{pmatrix}, \qquad p \in \begin{pmatrix} [\frac{1}{2}, \frac{3}{2}] \\ [\frac{1}{2}, \frac{3}{2}] \end{pmatrix}$$

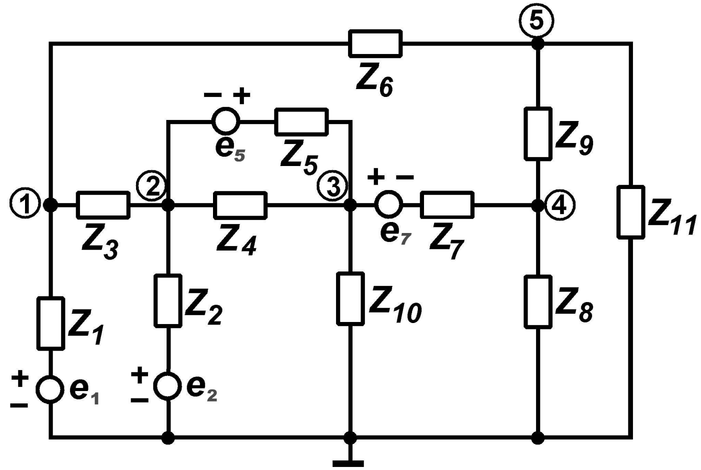
The original NP-method gives $x_{NP} = ([-2.51, 6.51], [-2.51, 6.51])^\top$,

while all other methods fail.

The generalized NP-method gives $x_{gNP} = ([0.5, 3.51], [0.5, 3.51])^\top$.

x_{NP} overestimates x_{gNP} by **66.6%**.

Electrical Circuits



$Z_k \pm 10\%$.

Find bounds for the node voltages V_i.

By loop analysis:

$$\begin{pmatrix} \frac{1}{Z_1}+\frac{1}{Z_3}+\frac{1}{Z_6} & -\frac{1}{Z_3} & 0 & 0 & -\frac{1}{Z_6} \\ -\frac{1}{Z_3} & \frac{1}{Z_2}+\frac{1}{Z_3}+\frac{1}{Z_4}+\frac{1}{Z_5} & -\frac{1}{Z_4}-\frac{1}{Z_5} & 0 & 0 \\ 0 & -\frac{1}{Z_4}-\frac{1}{Z_5} & \frac{1}{Z_4}+\frac{1}{Z_5}+\frac{1}{Z_7}+\frac{1}{Z_{10}} & -\frac{1}{Z_7} & 0 \\ 0 & 0 & -\frac{1}{Z_7} & \frac{1}{Z_7}+\frac{1}{Z_8}+\frac{1}{Z_9} & -\frac{1}{Z_9} \\ -\frac{1}{Z_6} & 0 & 0 & -\frac{1}{Z_9} & \frac{1}{Z_6}+\frac{1}{Z_9}+\frac{1}{Z_{11}} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{pmatrix} = \begin{pmatrix} \frac{e_1}{Z_1} \\ \frac{e_2}{Z_2}-\frac{e_5}{Z_5} \\ \frac{e_5}{Z_5}+\frac{e_7}{Z_7} \\ -\frac{e_7}{Z_7} \\ 0 \end{pmatrix}$$

Main Application: Sharpen the Bounds

Corollary 3. *The infimum/supremum of Σ^p_{uni} is attained at particular end-points of the intervals for the parameters that satisfy Theorem 1.*

Which endpoints of $[p_k]$ form the lower/upper bounds of $\square\Sigma^p_{uni}$ is determined by:

- solving derivative parametric systems, see

  E.Popova, Computer-Assisted Proofs in Solving Linear Parametric Problems, SCAN 2006, IEEE Computer Society Press, 2007.

APPLICATION:

Fix the values of the parameters, that satisfy Theorem 1, at appropriate endpoints of their intervals

and find sharper bounds for the respective Σ^p_{uni} involving the remaining less number of interval parameters.

Parametric AE Solution Sets

$$\Sigma^p_{AE} := \{x \in \mathbb{R}^n \mid (\forall p_{\mathcal{A}} \in [p_{\mathcal{A}}])(\exists p_{\mathcal{E}} \in [p_{\mathcal{E}}])(A(p_{\mathcal{A}}, p_{\mathcal{E}})x = b(p_{\mathcal{A}}, p_{\mathcal{E}}))\}$$

E.Popova, *Explicit Description of AE Solution Sets to Parametric Linear Systems*, SIMAX 33, 2012.

- The boundary of Σ^p_{AE} is linear w.r.t. all $\mathcal{A}$-parameters.
  (the shape of Σ^p_{AE} is determined by the $\mathcal{E}$-parameters)

implies the APPLICABILITY of

- the sufficient conditions for linear boundary w.r.t. $\mathcal{E}$-parameters

- the Conversion Theorem for these parameters

to Σ^p_{AE}.

Outer Estimation $\Sigma^p_{AE} \subseteq [u]$

E. D. Popova, M. Hladík, *Outer Enclosures to Parametric AE Solution Set*, Soft Computing 17, 2013.

For a given index set $\mathcal{A}$, define the set $\mathcal{B}_{\mathcal{A}}$ of end-points (vertices) of $[p_{\mathcal{A}}]$.

Theorem 3. *For* $\Sigma^p_{AE} \neq \emptyset$,

$$\square\Sigma^p_{AE} \subseteq \bigcap_{\tilde{p}_{\mathcal{A}} \in \mathcal{B}_{\mathcal{A}}} \square\Sigma_{uni}(A(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}}), b(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}}), [p_{\mathcal{E}}]).$$

$\Sigma_{uni}(A(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}}), b(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}}), [p_{\mathcal{E}}]) \subseteq [u]$ by any parametric solver for Σ^p_{uni}.

The above end-point approach implies the APPLICABILITY of the generalized Neumaier & Pownuk method.

Conclusions

The sufficient conditions for linear boundary w.r.t. some $\mathcal{E}$-parameters:

- expand our knowledge about the shape of Σ^p_{uni}, Σ^p_{AE}
- allow reducing the dependencies (which cause overestimation)

$$(A_0 + LDiag(p)R)\, x(p,q) \;=\; b_0 + LDiag(p)t + Fq$$

- the above allow:
  - expanding the applicability of NP-method to other problem domains
  - generalizing the NP-method for $A(p)x = b(p,q)$
    preserving its scalability
  - obtain exact or sharper bounds for Σ^p_{uni}, Σ^p_{AE} accounting for
    the linear boundary w.r.t. some parameters.

A detailed discussion can be found in:

Popova, E.: Improved Enclosure for Some Parametric Solution Sets with Linear Shape, **Computers and Mathematics with Applications**, online: April 2014.

THANK YOU!