Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 12, NUMBER 3 (2009) ISSN 1311-0454

OPERATIONAL METHOD FOR THE SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH GENERALIZED RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVES

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Abstract

The operational calculus is an algorithmic approach for the solution of initial-value problems for differential, integral, and integro-differential equations. In this paper, an operational calculus of the Mikusiński type for a generalized Riemann-Liouville fractional differential operator with types introduced by one of the authors is developed. The traditional Riemann-Liouville and Liouville-Caputo fractional derivatives correspond to particular types of the general one-parameter family of fractional derivatives with the same order. The operational calculus constructed in this paper is used to solve the corresponding initial value problem for the general n-term linear equation with these generalized fractional derivatives of arbitrary orders and types with constant coefficients. Special cases of the obtained solutions are presented.

Mathematics Subject Classification: 26A33, 33E12, 44A40, 45J05

Key Words and Phrases: operational calculus, fractional derivatives and integrals, Mittag-Leffler type functions, fractional differential equations, generalized Riemann-Liouville fractional derivative, Caputo fractional derivative

1. Introduction

In the 1950's, Jan Mikusiński proposed a new approach to develop an *operational calculus for the operator of differentiation* (see e.g. [35]). This algebraic approach was based on the interpretation of the Laplace convolution as a multiplication in the ring of the continuous functions on the real half-axis. The Mikusiński operational calculus was successfully used in ordinary differential equations, integral equations, partial differential equations and in the theory of special functions. It is worth mentioning that the Mikusiński scheme was extended by several mathematicians to develop operational calculi for differential operators with variable coefficients (see, for example, [8], [9], [32]), too. These operators are all particular cases of the so called *hyper-Bessel differential operator*

$$(B\,y)(x) = x^{-\beta} \prod_{i=1}^{n} \left(\gamma_i + \frac{1}{\beta} x \frac{d}{dx} \right) y(x). \tag{1}$$

An operational calculus for the operator (1) was constructed by Dimovski in [7]. Details on his construction, and extended theory of the operator (1) as a generalized fractional derivative can be found also in [25, Ch. 3]. New results in the field of operational calculus have been presented in the publications by Luchko and his co-authors (see e.g. [1], [11], [12], [15], [27]–[30], [43]), where the operational calculi for the Riemann-Liouville fractional derivative, for the Caputo fractional derivative and for the more general multiple Erdélyi-Kober fractional derivative have been constructed and applied for solution of the fractional differential equations and integral equations of the Abel type.

The theory and applications of fractional differential equations received in recent years considerable interest both in pure mathematics and in applications (see, for example, [2]–[6], [13]– [25], [28]–[34], [36]–[38], and [41]). There exist several different definitions of fractional differentiation. Whereas in mathematical treatises on fractional differential equations the Riemann-Liouville approach to the notion of the fractional derivative of order α ($n-1 < \alpha \le n \in \mathbb{N}$) is normally used, the Caputo fractional derivative often appears in applications. The (right-sided) *Riemann-Liouville fractional derivative* is defined by

$$(D^{\alpha} y)(x) := \frac{d^n}{dx^n} (J^{n-\alpha} y)(x), \;\; x > 0. \tag{2}$$

where

$$(J^{\alpha} y)(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} y(t)\, dt, \;\; \alpha > 0, \;\; x > 0, \tag{3}$$

$$(J^0 y)(x) := y(x), \; x > 0$$

is the (right-sided) *Riemann-Liouville fractional integral of order* α with lower limit 0.

The Riemann-Liouville fractional derivative is left-inverse (but not right-inverse) of the Riemann-Liouville fractional integral, which is a natural generalization of the Cauchy formula for the n-fold primitive of a function y. As to the initial value problems for fractional differential equations with fractional derivatives in the Riemann-Liouville sense, they should be given as (bounded) initial values of the fractional integral $J^{n-\alpha}y$ and of its integer derivatives of order $k = 1, 2, \ldots, m-1$.

An alternative definition of fractional derivative was introduced by Liouville [26, p.10] and rediscovered by *Caputo*, see e.g. [5],[6], in the framework of the theory of linear viscoelasticity:

$$(D_*^\alpha y)(x) := (J^{n-\alpha} y^{(n)})(t), \quad n-1 < \alpha \leq n \in \mathbb{N}, \; x > 0. \tag{4}$$

This definition allows to consider the initial-value problems for fractional differential equations with initial conditions that are expressed in terms of a given number of bounded values assumed by the field variable and its derivatives of integer order.

In [16] another new definition of the fractional derivative was suggested. The *generalized Riemann-Liouville fractional derivative (GRLFD) of order* α *and type* β is defined as

$$(D^{\alpha,\beta} y)(x) := (J^{\beta(n-\alpha)} \frac{d^n}{dx^n} (J^{(1-\beta)(n-\alpha)} y))(x), \qquad x > 0. \tag{5}$$

Here the order $\alpha \in \mathbb{R}$ obeys $n-1 < \alpha \leq n \in \mathbb{N}$ and the type $\beta \in \mathbb{R}$ obeys $0 \leq \beta \leq 1$. *The type* β *allows to interpolate continuously from the Riemann-Liouville case* $D^{\alpha,0} \equiv D^\alpha$ *to the Liouville-Caputo case* $D^{\alpha,1} \equiv D_*^\alpha$.

Some properties and applications of the GRLFD are given in [17]. In particular, fractional stationarity, fractional relaxation and fractional diffusion equations with fractional time derivatives $D^{\alpha,\beta}$ of order α and type β were investigated. For fixed order the type of the fractional derivative was found to determine the type of initial conditions. Explicit solutions to the corresponding fractional differential equations with general type were given. As expected, *they interpolate smoothly between the results for ordinary Riemann-Liouville* (*type* $\beta = 0$) *and Liouville-Caputo* (*type* $\beta = 1$) *fractional derivatives.* For fractional diffusion with fractional time derivatives of order $0 < \alpha \leq 1$ and type $0 \leq \beta \leq 1$ it was shown that there does not exist a probabilistic interpretation for the unknown function whenever $0 \leq \beta < 1$. These findings extended earlier results [19] on the absence of a probabilistic interpretation for Riemann-Liouville diffusion (type $\beta = 0$) and disproved claims in the literature according to which the Liouville-Caputo type $\beta = 1$ is the only type relevant for, or consistent with physical applications. Later GRLFD's appeared also in the theoretical modeling of broadband dielectric relaxation spectroscopy for glasses [20]. More recently, equations involving GRLFD's with constant coefficients were investigated in [42].

In this paper, some basic elements of the operational calculus of the Mikusiński type for GRLFD's are presented with special emphasis on an operational method for solving linear differential equations of fractional order with constant coefficients. The plan of the rest of the paper is as follows. In Section 2, some important properties of the operators $D^{\alpha,\beta}$ needed for the further discussions are given. Section 3 is devoted to the construction of the operational calculus of the Mikusiński type for GRLFD's. In Section 4, the operational calculus is applied to deduce the solution of the general n-term linear equation with $D^{\alpha,\beta}$ derivatives of arbitrary α, β and constant coefficients. Special cases of the obtained solutions are presented.

2. Some properties of the generalized fractional derivative with types

The Riemann-Liouville, the Liouville-Caputo, and the generalized fractional derivatives ((2), (4), and (5), respectively) are defined as certain compositions of the Riemann-Liouville fractional integral (3) and ordinary derivatives. It is clear that these operators play a decisive role in the development of the corresponding operational calculi and there should be some common elements in the operational calculi for all three fractional derivatives. The operational calculus for the Riemann-Liouville fractional derivative was presented in [8], [11], [15] and the operational calculus for the Liouville-Caputo fractional derivative in [27], [28]. In this section, some theorems from there will be cited without proofs.

We begin by defining the function space $C_\gamma,\ \gamma \in \mathbb{R}$, which was introduced for the first time by Dimovski in his papers devoted to the operational calculus for the hyper-Bessel differential operator (see e.g. [7]).

Definition 1. A real or complex-valued function y, is said to belong to the space $C_\gamma,\ \gamma \in \mathbb{R}$, if there exists a real number p, $p > \gamma$, such that

$$y(t) = t^p y_1(t),\ t > 0$$

with a function $y_1 \in C[0, \infty)$.

Clearly, C_γ is a vector space and the set of spaces C_γ is ordered by inclusion according to

$$C_\gamma \subset C_\delta \Leftrightarrow \gamma \geq \delta. \tag{6}$$

THEOREM 1. *The Riemann-Liouville fractional integral* $J^\alpha,\ \alpha \geq 0$, *is a linear map of the space* $C_\gamma,\ \gamma \geq -1$, *into itself, that is,*

$$J^\alpha : C_\gamma \to C_{\alpha+\gamma} \subset C_\gamma.$$

For the proof of the theorem see e.g. [29].

It is well known, that the operator J^α, $\alpha > 0$ has a convolution representation in the space C_γ, $\gamma \geq -1$:

$$(J^\alpha y)(x) = (h_\alpha \circ y)(x),\ h_\alpha(x) := x^{\alpha-1}/\Gamma(\alpha), \quad y \in C_\gamma. \tag{7}$$

Here

$$(g \circ f)(x) = \int_0^x g(x-t)f(t)\,dt,\ x > 0$$

is *the Laplace convolution.* Moreover, the following properties of the Riemann-Liouville fractional integral are well known:

$$(J^\delta J^\eta y)(x) = (J^\eta J^\delta y)(x),\ y \in C_\gamma,\ \gamma \geq -1,\ \delta \geq 0,\ \eta \geq 0.$$

$$(J^\delta J^\eta y)(x) = (J^{\delta+\eta} y)(x),\ y \in C_\gamma,\ \gamma \geq -1,\ \delta \geq 0,\ \eta \geq 0. \tag{8}$$

In particular, it follows from (8) that

$$(\underbrace{J^\alpha \ldots J^\alpha}_{n} y)(x) = (J^{n\alpha} y)(x),\ y \in C_\gamma,\ \gamma \geq -1,\ \alpha \geq 0,\ n \in \mathbb{N}. \tag{9}$$

The GRLFD (5) is not defined on the whole space C_γ. Let us introduce a subspace of C_γ, which is suitable for dealing with GRLFD's.

DEFINITION 2. A function $y \in C_{-1}$ is said to be in the space $\Omega^\mu_{-1},\ \mu \geq 0$, if $D^{\alpha,\beta} y \in C_{-1}$ for all $0 \leq \alpha \leq \mu,\ 0 \leq \beta \leq 1$.

REMARK 1. For $\beta = 0$, i.e. for the Riemann-Liouville fractional derivative, the space Ω^μ_{-1} from Definition 2 coincides with the function space introduced in [29], [43].

Obviously, Ω^μ_{-1} is a vector space and $\Omega^0_{-1} \equiv C_{-1}$. The space Ω^μ_{-1} contains in particular all functions z that can be represented in the form $z(x) = x^\gamma y(x)$ with $\gamma \geq \mu$ and y being an analytical function on the real half-axis.

The following result plays a very important role for the applications of the operational calculus for $D^{\alpha,\beta}$ to solution of differential equations with these generalized derivatives.

THEOREM 2. *Let $y \in \Omega_{-1}^{\alpha}$, $n-1 < \alpha \le n \in \mathbb{N}$. Then the Riemann-Liouville fractional integral (3) and the generalized fractional derivative (5) are connected by the relation*

$$(J^{\alpha} D^{\alpha,\beta} y)(x) = y(x) - y_{\alpha,\beta}(x),\ x > 0, \tag{10}$$

where

$$y_{\alpha,\beta}(x) := \sum_{k=0}^{n-1} \frac{x^{k-n+\alpha-\beta\alpha+\beta n}}{\Gamma(k-n+\alpha-\beta\alpha+\beta n+1)} \lim_{x\to 0+} \frac{d^k}{dx^k} (J^{(1-\beta)(n-\alpha)} y)(x),\ x > 0. \tag{11}$$

P r o o f. For $n-1 < \alpha \le n \in \mathbb{N}$ and $0 \le \beta \le 1$, the generalized derivative (5) can be represented as a composition of the Riemann-Liouville fractional integral (3) and the Riemann-Liouville fractional derivative (2) as follows:

$$(D^{\alpha,\beta} y)(x) = (J^{\beta(n-\alpha)} \frac{d^n}{dx^n} (J^{(1-\beta)(n-\alpha)} y))(x) = (J^{\beta(n-\alpha)} D^{\alpha+\beta n-\alpha\beta} y)(x). \tag{12}$$

Using the formula (8) we get now

$$(J^{\alpha} D^{\alpha,\beta} y)(x) = (J^{\alpha} J^{\beta(n-\alpha)} D^{\alpha+\beta n-\alpha\beta} y)(x) = (J^{\alpha+\beta n-\alpha\beta} D^{\alpha+\beta n-\alpha\beta} y)(x).$$

The formula (10) follows now from the known formula for the composition of the Riemann-Liouville fractional integral and the Riemann-Liouville fractional derivative (see e.g. [29], [43]). ■

3. Operational calculus for fractional derivatives with types

The formula (10) shows that the generalized derivative of order α and type β always corresponds to the Riemann-Liouville fractional integral of order α. The type β influences the form of the initial conditions that should appear while formulating the initial-value problems for the differential equations. That is why the main part of the operational calculus for $D^{\alpha,\beta}$ follows the lines of the construction of the operational calculus for the Riemann-Liouville or for the Liouville-Caputo fractional derivatives presented in [11], [27]–[30], [43].

As in the case of Mikusiński's type operational calculus for the Riemann-Liouville or for the Liouville-Caputo fractional derivatives, we have the following theorem:

THEOREM 3. *The space C_{-1} with the operations of the Laplace convolution $\circ$ and ordinary addition becomes a commutative ring $(C_{-1}, \circ, +)$ without divisors of zero.*

This ring can be extended to the field $\mathcal{M}_{-1}$ of convolution quotients by following the lines of the classical Mikusiński operational calculus [35]:

$$\mathcal{M}_{-1} := C_{-1} \times (C_{-1} \setminus \{0\})/\sim,$$

where the equivalence relation $(\sim)$ is defined, as usual, by

$$(f, g) \sim (f_1, g_1) \Leftrightarrow (f \circ g_1)(t) = (g \circ f_1)(t).$$

For the sake of convenience, the elements of the field $\mathcal{M}_{-1}$ can be formally considered as convolution quotients f/g. The operations of addition and multiplication are then defined in $\mathcal{M}_{-1}$ as usual:

$$\frac{f}{g} + \frac{f_1}{g_1} := \frac{f \circ g_1 + g \circ f_1}{g \circ g_1} \tag{13}$$

and

$$\frac{f}{g} \cdot \frac{f_1}{g_1} := \frac{f \circ f_1}{g \circ g_1}\,. \tag{14}$$

THEOREM 4. *The space $\mathcal{M}_{-1}$ with the operations of addition (13) and multiplication (14) becomes a commutative field $(\mathcal{M}_{-1}, \cdot, +)$.*

The ring C_{-1} can be embedded into the field $\mathcal{M}_{-1}$ by the map $(\alpha > 0)$:

$$f \mapsto \frac{h_\alpha \circ f}{h_\alpha}\,,$$

with, by (7), $h_\alpha(x) = x^{\alpha-1}/\Gamma(\alpha)$.

In the field $\mathcal{M}_{-1}$, the operation of multiplication with a scalar λ from the field $\mathbf{R}$ (or $\mathbf{C}$) can be defined by the relation

$$\lambda \frac{f}{g} := \frac{\lambda f}{g}, \ \frac{f}{g} \in \mathcal{M}_{-1}.$$

Because the space C_{-1} is a vector space, the space $\mathcal{M}_{-1}$ can be shown to be a vector space, too. Since the constant function $f(x) \equiv \lambda,\ x > 0$ belongs to the space C_{-1}, we have to distinguish the operation of multiplication with a scalar in the vector space $\mathcal{M}_{-1}$ and the operation of multiplication with a constant function in the field $\mathcal{M}_{-1}$. In this last case we get

$$\{\lambda\} \cdot \frac{f}{g} = \frac{\lambda h_{\alpha+1}}{h_\alpha} \cdot \frac{f}{g} = \{1\} \cdot \frac{\lambda f}{g}\,. \tag{15}$$

Whereas the space C_{-1} consists of the conventional functions, the majority of the elements of the field $\mathcal{M}_{-1}$ are not reduced to the functions from the ring C_{-1} and, consequently, can be considered to be the generalized functions or the so called hyperfunctions. In particular, let us consider the element $I = \frac{h_\alpha}{h_\alpha}$ of the field $\mathcal{M}_{-1}$ that is the identity of this field with respect to the operation of multiplication:

$$I \cdot \frac{f}{g} = \frac{h_\alpha \circ f}{h_\alpha \circ g} = \frac{f}{g}\,.$$

The last formula shows that the identity element I of the field $\mathcal{M}_{-1}$ plays the role of the Dirac δ-function in the conventional theory of the generalized functions.

Another hyperfunction, i.e. an element of the field $\mathcal{M}_{-1}$ that cannot be represented as a conventional function from the space C_{-1} that will play an important role in the applications of the operational calculus for the generalized fractional derivative is given by

Definition 3. The algebraic inverse of the Riemann-Liouville fractional integral operator J^α is said to be the element S_α of the field $\mathcal{M}_{-1}$, which is reciprocal to the element h_α in the field $\mathcal{M}_{-1}$, that is,

$$S_\alpha = \frac{I}{h_\alpha} \equiv \frac{h_\alpha}{h_\alpha \circ h_\alpha} \equiv \frac{h_\alpha}{h_{2\alpha}}\,, \tag{16}$$

where (and in what follows) $I = \frac{h_\alpha}{h_\alpha}$ denotes the identity element of the field $\mathcal{M}_{-1}$ with respect to the operation of multiplication.

The Riemann-Liouville fractional integral J^α can be represented as a multiplication (convolution) in the ring C_{-1} (with the function h_α, see (7)). Since the ring C_{-1} is embedded into the field $\mathcal{M}_{-1}$ of convolution quotients, this fact can be rewritten as follows:

$$(J^\alpha y)(x) = \frac{I}{S_\alpha} \cdot y\,. \tag{17}$$

As to the generalized fractional derivative $D^{\alpha,\beta}$, there exists no convolution representation in the ring C_{-1} for it, but it is reduced to the operator of multiplication in the field $\mathcal{M}_{-1}$.

Theorem 5. *Let a function y be from the space Ω^α_{-1}, $n-1 < \alpha \le n$, $n \in \mathbb{N}$. Then the generalized fractional derivative $D^{\alpha,\beta} y$ can be represented as multiplication in the field $\mathcal{M}_{-1}$ of convolution quotients:*

$$(D^{\alpha,\beta} y)(x) = S_\alpha \cdot y - S_\alpha \cdot y_{\alpha,\beta}\,, \tag{18}$$

$$y_{\alpha,\beta}(x) := \sum_{k=0}^{n-1} \frac{x^{k-n+\alpha-\beta\alpha+\beta n}}{\Gamma(k-n+\alpha-\beta\alpha+\beta n+1)} \lim_{x\to 0+} \frac{d^k}{dx^k}(J^{(1-\beta)(n-\alpha)} y)(x),\ x > 0. \tag{19}$$

P r o o f. To prove the formula (18), we just use the embedding of the ring C_{-1} into the field $\mathcal{M}_{-1}$ and then multiply the relation (10) with the algebraic inverse of the Riemann-Liouville fractional integral operator - the element S_α. The obtained relation is exactly the formula (18). ∎

Using Theorem 5, the differential equations with GRLFD's can be reduced to algebraic equations in the field $\mathcal{M}_{-1}$ of convolution quotients. These equations can be solved by following the standard techniques. The obtained solutions belong then to the field $\mathcal{M}_{-1}$, i.e. are the hyperfunctions and not the conventional functions. The application of other methods to solution of the same equations (say, the Laplace transform method) teaches us that in many cases the solutions of the differential equations with GRLFD's are still conventional functions, i.e. we have to look for mechanisms to represent some suitable hyperfunctions in terms of conventional functions. To do so, let us first define some power functions in the field $\mathcal{M}_{-1}$.

Formula (9) means that for $\alpha > 0,\ n \in \mathbb{N}$

$$h_\alpha^n(x) := \underbrace{h_\alpha \circ \ldots \circ h_\alpha}_{n} = h_{n\alpha}(x).$$

This relation can be extended to an arbitrary positive real power exponent:

$$h_\alpha^\lambda(x) := h_{\lambda\alpha}(x),\ \lambda > 0. \tag{20}$$

For any $\lambda > 0$, the inclusion $h_\alpha^\lambda \in C_{-1}$ holds true and the following relations can be easily proved ($\beta > 0,\ \gamma > 0$):

$$h_\alpha^\beta \circ h_\alpha^\gamma = h_{\alpha\beta} \circ h_{\alpha\gamma} = h_{(\beta+\gamma)\alpha} = h_\alpha^{\beta+\gamma}, \tag{21}$$

$$h_{\alpha_1}^\beta = h_{\alpha_2}^\gamma \ \Leftrightarrow\ \alpha_1\beta = \alpha_2\gamma. \tag{22}$$

The above relations motivate the following definition of a power function of the element S_α with an arbitrary real power exponent λ:

$$S_\alpha^\lambda = \begin{cases} h_\alpha^{-\lambda}, & \lambda < 0, \\ I, & \lambda = 0, \\ \frac{I}{h_\alpha^\lambda}, & \lambda > 0. \end{cases} \tag{23}$$

For any $\alpha, \beta \in \mathbb{R}$, it follows from this definition and the relations (21) and (22) that

$$S_\alpha^\beta \cdot S_\alpha^\gamma = S_\alpha^{\beta+\gamma}, \tag{24}$$

$$S_{\alpha_1}^\beta = S_{\alpha_2}^\gamma \ \Leftrightarrow\ \alpha_1\beta = \alpha_2\gamma. \tag{25}$$

For the application of the operational calculus to solution of differential equations with GRLFD's it is important to identify those hyperfunctions from the field $\mathcal{M}_{-1}$, which can be represented by conventional functions, i.e. as the elements of the ring C_{-1}. One useful class of such representations is given by the following theorem (see e.g. [15], [27], [28]):

THEOREM 6. *Let the multiple power series*

$$\sum_{i_1,\dots,i_n=0}^{\infty} a_{i_1,\dots,i_n} z_1^{i_1} \times \dots \times z_n^{i_n}, \quad z_1,\dots,z_n \in \mathbb{C},\ a_{i_1,\dots,i_n} \in \mathbb{C}$$

be convergent at a point $z_0 = (z_{10},\dots,z_{n0})$ with all $z_{k0} \neq 0,\ k = 1,\dots,n$. Then the hyperfunction

$$z(S_\alpha) := S_\alpha^{-\beta} \sum_{i_1,\dots,i_n=0}^{\infty} a_{i_1,\dots,i_n} (S_\alpha^{-\alpha_1})^{i_1} \times \dots \times (S_\alpha^{-\alpha_n})^{i_n}$$

with $\beta > 0,\ \alpha_i > 0,\ i = 1,\dots,n$ can be represented as an element of the ring C_{-1}:

$$z(S_\alpha) = \sum_{i_1,\dots,i_n=0}^{\infty} a_{i_1,\dots,i_n} h_{(\beta+\alpha_1 i_1+\dots+\alpha_n i_n)\alpha}(x),$$

where $h_\alpha(x)$ is given by (7).

The proof of the theorem can be found in [15]. Theorem 6 is a source of a number of important operational relations (for more operational relations we refer to [11], [15], and [29]). In the discussions below the relation

$$\frac{I}{S_\alpha - \rho} = x^{\alpha-1} E_{\alpha,\alpha}(\rho x^\alpha), \tag{26}$$

with $\rho \in \mathbb{R}$ (or $\rho \in \mathbb{C}$) plays an important role. Here $E_{\alpha,\beta}$ is the *generalized Mittag-Leffler function* defined by (see [10, Vol.3])

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}\ , \quad \alpha > 0,\ |z| < \infty.$$

Equation (26) can formally be obtained from the geometric series as

$$\frac{I}{S_\alpha - \rho} = \frac{I}{\frac{I}{h_\alpha} - \rho} = \frac{h_\alpha}{I - \rho h_\alpha} = \sum_{k=0}^{\infty} \rho^k h_\alpha^{k+1} = \sum_{k=0}^{\infty} \frac{\rho^k x^{(k+1)\alpha-1}}{\Gamma(\alpha k + \alpha)} = x^{\alpha-1} E_{\alpha,\alpha}(\rho x^\alpha).$$

The m-fold convolution of the right-hand side of the relation (26) gives the following operational relation:

$$\frac{I}{(S_\alpha - \rho)^m} = x^{\alpha m-1} E_{\alpha,m\alpha}^m(\rho x^\alpha),\ m \in \mathbb{N}, \tag{27}$$

where

$$E_{\alpha,\beta}^m(z) := \sum_{k=0}^{\infty} \frac{(m)_k z^k}{k!\Gamma(\alpha k + \beta)},\ \alpha > 0,\ |z| < \infty,\ (m)_k = \prod_{i=0}^{k-1}(m+i),$$

is a function introduced in [40] (see also [23]).

Let $\beta > 0,\ \alpha_i > 0,\ i = 1, \ldots, n.$ Then

$$\frac{S_\alpha^{-\beta}}{I - \sum_{i=1}^n \lambda_i S_\alpha^{-\alpha_i}} = x^{\beta\alpha - 1} E_{(\alpha_1\alpha, \ldots, \alpha_n\alpha), \beta\alpha}(\lambda_1 x^{\alpha_1\alpha}, \ldots, \lambda_n x^{\alpha_n\alpha}) \tag{28}$$

with the Mittag-Leffler function

$$E_{(a_1,\ldots,a_n),b}(z_1, \ldots, z_n) := \sum_{k=0}^{\infty} \sum_{\substack{l_1+\cdots+l_n=k \\ l_1 \geq 0, \ldots, l_n \geq 0}} (k; l_1, \ldots, l_n) \frac{\prod_{i=1}^n z_i^{l_i}}{\Gamma(b + \sum_{i=1}^n a_i l_i)} \tag{29}$$

and the multinomial coefficients

$$(k; l_1, \ldots, l_n) := \frac{k!}{l_1! \times \cdots \times l_n!}\ .$$

Remark 2. In [27], the function (29) was called the "multivariate Mittag-Leffler function". Because of the close relation of this function to the multinomial expansion and the multinomial coefficients, we find it more appropriate to call it the "*multinomial Mittag-Leffler function*".

4. Fractional differential equations with types

In this section, the constructed operational calculus is applied to solve linear fractional differential equations with generalized derivatives and constant coefficients.

First, some simple fractional differential equations are considered. We begin with the initial value problem (n–$1 < \alpha \leq n$, $n \in \mathbb{N}$, $0 \leq \beta \leq 1, \lambda \in \mathbb{R}$)

$$\begin{cases} (D^{\alpha,\beta} y)(x) - \lambda y(x) = g(x), \\ \lim_{x \to 0+} \frac{d^k}{dx^k} (J^{(1-\beta)(n-\alpha)} y)(x) = c_k \in \mathbb{R}, \quad k = 0, \ldots, n-1. \end{cases} \tag{30}$$

The function g is assumed to lie in C_{-1} and the unknown function y is to be determined in the space Ω_{-1}^α.

Making use of the relation (18), the initial value problem (30) can be reduced to the following algebraic equation in the field $\mathcal{M}_{-1}$ of convolution quotients:

$$S_\alpha \cdot y - \lambda y = S_\alpha \cdot y_{\alpha,\beta} + g,$$

$$y_{\alpha,\beta}(x) = \sum_{k=0}^{n-1} c_k \frac{x^{k-n+\alpha-\beta\alpha+\beta n}}{\Gamma(k - n + \alpha - \beta\alpha + \beta n + 1)}\ .$$

This linear equation can be easily solved in the field $\mathcal{M}_{-1}$:

$$y = y_g + y_h = \frac{I}{S_\alpha - \lambda} \cdot g + \frac{S_\alpha}{S_\alpha - \lambda} \cdot y_{\alpha,\beta} \,. \tag{31}$$

The right-hand side of this relation can be interpreted as a function from the space Ω_{-1}^{α}, i.e., as a classical solution of the initial value problem (30).

It follows from the operational relation (26) and the embedding of the ring C_{-1} into the field $\mathcal{M}_{-1}$, that the first term in the formula (31), y_g (solution of the inhomogeneous fractional differential equation (30) with zero initial conditions), can be represented in the form

$$y_g(x) = \int_0^x (x-t)^{\alpha-1} E_{\alpha,\alpha}(\lambda(x-t)^\alpha) g(t)\, dt. \tag{32}$$

As to the second term, y_h, in (31) it is a solution of the homogeneous fractional differential equation (30) with the given initial conditions and we have

$$y_h(x) = \sum_{k=0}^{n-1} c_k u_k(x),\ \ u_k(x) = \frac{S_\alpha}{S_\alpha - \lambda} \cdot \{\frac{x^{k-n+\alpha-\beta\alpha+\beta n}}{\Gamma(k-n+\alpha-\beta\alpha+\beta n+1)}\} \,. \tag{33}$$

Making use of the relation

$$\frac{x^{k-n+\alpha-\beta\alpha+\beta n}}{\Gamma(k-n+\alpha-\beta\alpha+\beta n+1)} = h_{k-n+\alpha-\beta\alpha+\beta n+1}(x)$$

$$= h^{\alpha}_{(k-n+\alpha-\beta\alpha+\beta n+1)/\alpha}(x) = \frac{I}{S_\alpha^{(k-n+\alpha-\beta\alpha+\beta n+1)/\alpha}} \,, \tag{34}$$

the formula (24), and the operational relation (28), we get the representation of the functions $u_k(x),\ k = 0, \ldots, n-1$ in terms of the generalized Mittag-Leffler function:

$$u_k(x) = \frac{S_\alpha}{S_\alpha - \lambda} \cdot \{\frac{x^{k-n+\alpha-\beta\alpha+\beta n}}{\Gamma(k-n+\alpha-\beta\alpha+\beta n+1)}\}$$

$$= \frac{S_\alpha^{-(k-n+\alpha-\beta\alpha+\beta n+1)/\alpha}}{I - \lambda S_\alpha^{-1}} = x^{k-(1-\beta)(n-\alpha)} E_{\alpha,k+1-(1-\beta)(n-\alpha)}(\lambda x^\alpha).$$

For $n = 1$ this solution was first given in [17]. Putting now the two parts of the solution together, we get the final form of the solution of the initial-value problem (30):

$$y(x) = y_g(x) + y_h(x) = \int_0^x (x-t)^{\alpha-1} E_{\alpha,\alpha}(\lambda(x-t)^\alpha) g(t)\, dt \tag{35}$$

$$+ \sum_{k=0}^{n-1} c_k x^{k-(1-\beta)(n-\alpha)} E_{\alpha,k+1-(1-\beta)(n-\alpha)}(\lambda x^\alpha).$$

The proof of the fact, that the solution y belongs to the space Ω_{-1}^{α} is straightforward (see [28]), and we omit it here.

Whereas the solution of the inhomogeneous fractional differential equation (30) with zero initial conditions - the function y_g - only depends on the order α of the derivative, the solution of the homogeneous equation - the function y_h - differs for different values of the type β of the derivative. In particular, the part y_h of the solution takes the form

$$y_h(x) = \sum_{k=0}^{n-1} c_k u_k(x),\ u_k(x) = x^k\, E_{\alpha,k+1}(\lambda x^\alpha)$$

and

$$y_h(x) = \sum_{k=0}^{n-1} c_k u_k(x),\ u_k(x) = x^{k-n+\alpha}\, E_{\alpha,k+1-n+\alpha}(\lambda x^\alpha)$$

for the Liouville-Caputo fractional derivative ($\beta = 1$) and for the Riemann-Liouville fractional derivative ($\beta = 0$), respectively.

Next, we consider the linear differential equation

$$\sum_{i=1}^{n} \lambda_i \left(D^{\alpha_i,\beta_i} y\right)(x) - \lambda y\,(x) = g\,(x) \tag{36}$$

with the initial conditions

$$\lim_{x\to 0+} \frac{d^k}{dx^k}(J^{(1-\beta_i)(n-\alpha_i)} y)(x) = c_k \in \mathbb{R}, \tag{37}$$

where $i = 1, 2, ..., n$; $k = 0, ..., n–1$, $n–1 < \alpha_i \le n$, $n \in \mathbb{N}$; $0 \le \beta_i \le 1$; $\lambda, \lambda_i \in \mathbb{R}$ and the ordering $\alpha_1 > \alpha_2 > ... > \alpha_n > 0$ is assumed without loss of generality. Then the following algebraic equation in the field $\mathcal{M}_{-1}$ of convolution quotients is obtained

$$\sum_{i=1}^{n} \lambda_i \left(S^{\alpha_i} y - S^{\alpha_i} y_{\alpha_i,\beta_i}\right) - \lambda y = g. \tag{38}$$

This linear equation can be easy solved in the field $\mathcal{M}_{-1}$:

$$y = y_g + y_h = \frac{I}{\sum\limits_{i=1}^{n} \lambda_i S^{\alpha_i} - \lambda} \cdot g + \sum_{j=1}^{n} \frac{\lambda_j S^{\alpha_j}}{\sum\limits_{i=1}^{n} \lambda_i S^{\alpha_i} - \lambda} \cdot y_{\alpha_j,\beta_j}, \tag{39}$$

where

$$y_{\alpha_j,\beta_j} = \sum_{k=0}^{n-1} c_k \frac{x^{k-n+\alpha_j-\beta_j\alpha_j+\beta_j n}}{\Gamma(k-n+\alpha_j-\beta_j\alpha_j+\beta_j n+1)}. \tag{40}$$

From (28) we get

$$\frac{I}{\sum\limits_{i=1}^{n}\lambda_i S^{\alpha_i}-\lambda}=\frac{S^{-\alpha_1}}{\lambda_1+\sum\limits_{i=2}^{n}\lambda_i S^{\alpha_i-\alpha_1}-\lambda S^{-\alpha_1}}=\frac{1}{\lambda_1}\frac{S^{-\alpha_1}}{I-\sum\limits_{i=2}^{n}\left(-\frac{\lambda_i}{\lambda_1}\right)S^{\alpha_i-\alpha_1}-\frac{\lambda}{\lambda_1}S^{-\alpha_1}}=$$
$$\frac{1}{\lambda_1}x^{\alpha_1-1}E_{(\alpha_1,(\alpha_1-\alpha_2),(\alpha_1-\alpha_3),\ldots,(\alpha_1-\alpha_n)),\alpha_1}\left(-\frac{\lambda}{\lambda_1}x^{\alpha_1},-\frac{\lambda_2}{\lambda_1}x^{(\alpha_1-\alpha_2)},\ldots,-\frac{\lambda_n}{\lambda_1}x^{(\alpha_1-\alpha_n)}\right).$$

Hence, by (32) we obtain

$$y_g=\frac{1}{\lambda_1}\int\limits_0^x(x-t)^{\alpha_1-1}E_{(a_1,\ldots,a_n),\alpha_1}\left(c_1(x-t)^{a_1},\ldots,c_n(x-t)^{a_n}\right)g(t)\,dt$$

with

$$a_1=\alpha_1, c_1=-\frac{\lambda}{\lambda_1}, a_i=\alpha_1-\alpha_i, c_i=-\frac{\lambda_i}{\lambda_1},\ i=2,\ldots,n. \tag{41}$$

Applying the relations (34) and (28) we get

$$\begin{aligned} y_h &= \sum_{j=1}^{n}\lambda_j\frac{S^{\alpha_j}}{\sum\limits_{i=1}^{n}\lambda_i S^{\alpha_i}-\lambda}\left[\sum_{k=0}^{n-1}c_k\frac{x^{k-n+\alpha_j-\beta_j\alpha_j+\beta_j n}}{\Gamma(k-n+\alpha_j-\beta_j\alpha_j+\beta_j n+1)}\right] \qquad (42)\\ &= \sum_{j=1}^{n}\lambda_j\frac{S^{\alpha_j}}{\sum\limits_{i=1}^{n}\lambda_i S^{\alpha_i}-\lambda}\left(\sum_{k=0}^{n-1}c_k S^{-(k-n+\alpha_j-\beta_j\alpha_j+\beta_j n+1)}\right)\\ &= \sum_{j=1}^{n}\sum_{k=0}^{n-1}\lambda_j c_k\frac{S^{-(k-n-\beta_j\alpha_j+\beta_j n+1)}}{\sum\limits_{i=1}^{n}\lambda_i S^{\alpha_i}-\lambda}\\ &= \frac{1}{\lambda_1}\sum_{j=1}^{n}\sum_{k=0}^{n-1}\lambda_j c_k\frac{S^{-(k-n-\beta_j\alpha_j+\alpha_1+\beta_j n+1)}}{I-\sum\limits_{i=2}^{n}\left(-\frac{\lambda_i}{\lambda_1}\right)S^{\alpha_i-\alpha_1}-\frac{\lambda}{\lambda_1}S^{-\alpha_1}}\\ &= \frac{1}{\lambda_1}\sum_{j=1}^{n}\sum_{k=0}^{n-1}\lambda_j c_k x^{k-n-\beta_j\alpha_j+\alpha_1+\beta_j n}E_{(a_1,\ldots,a_n),b}\left(c_1x^{a_1},\ldots,c_nx^{a_n}\right)\end{aligned}$$

with $b=k-n-\beta_j\alpha_j+\alpha_1+\beta_j n+1$ and the coefficients $a_1,\ldots,a_n$, and $c_1,\ldots,c_n$ given by (41).

If $\beta_j=0$, $j=1,2,..,n$, the solution coincides with the solution of the *linear n-term differential equation with Riemann-Liouville fractional derivatives* (see e.g. [27]):

$$y = y_g + y_h,$$

where
$$y_h = \frac{1}{\lambda_1}\sum_{j=1}^{n}\sum_{k=0}^{n-1}\lambda_j c_k x^{k-n+\alpha_1} E_{(a_1,\ldots,a_n),b}\left(c_1 x^{a_1},\ldots,c_n x^{a_n}\right)$$

with $b = k - n + \alpha_1 + 1$ and the coefficients $a_1, \ldots, a_n$, and $c_1, \ldots, c_n$ given by (41).

If $\beta_j = 1$, $j = 1, 2, .., n$, the solution coincides with the solution of the *linear n-term differential equation with the Caputo fractional derivatives* (see e.g. [27]):

$$y = y_g + y_h,$$

where
$$y_h = \frac{1}{\lambda_1}\sum_{j=1}^{n}\sum_{k=0}^{n-1}\lambda_j c_k x^{k+\alpha_1-\alpha_j} E_{(a_1,\ldots,a_n),b}\left(c_1 x^{a_1},\ldots,c_n x^{a_n}\right)$$

with $b = k + \alpha_1 - \alpha_j + 1$ and the coefficients $a_1, \ldots, a_n$, and $c_1, \ldots, c_n$ given by (41).

For $n = 2$ and $n = 3$, the initial value problem (37) for the equation (36) was solved in [42] using the Laplace transform method.

If $\alpha_i = \alpha$, $i = 1, 2, .., n$, the equation (36) is reduced to the equation

$$\sum_{i=1}^{n}\lambda_i\left(D^{\alpha,\beta_i}y\right)(x) - \lambda y(x) = g(x)$$

with the initial conditions

$$\lim_{x\to 0+}\frac{d^k}{dx^k}(J^{(1-\beta_i)(n-\alpha)}y)(x) = c_k \in \mathbb{R}, \quad i = 1, 2, \ldots, n,\ k = 0, \ldots, n-1,$$

$$0 < \alpha < 1,\ 0 \le \beta_i \le 1, \lambda, \lambda_i \in \mathbf{R}, \quad i = 1, 2, .., n,\ \Lambda = \sum_{i=1}^{n}\lambda_i \neq 0.$$

Applying the operational method, we get the following algebraic equation in the field $\mathcal{M}_{-1}$ of convolution quotients:

$$\sum_{i=1}^{n}\lambda_i\left(S^\alpha y - S^\alpha y_{\alpha,\beta_i}\right) - \lambda y = g.$$

This linear equation can be easy solved in the field $\mathcal{M}_{-1}$:

$$\begin{aligned} y &= y_g + y_h \\ &= \frac{I}{\Lambda S^\alpha - \lambda}g + \frac{S^\alpha}{\Lambda S^\alpha - \lambda}\sum_{j=1}^{n}\lambda_j y_{\alpha,\beta_j}. \end{aligned}$$

By using the operational relation

$$\frac{I}{\Lambda S^{\alpha}-\lambda}=\frac{1}{\Lambda}x^{\alpha-1}E_{\alpha,\alpha}\left(\frac{\lambda}{\Lambda}x^{\alpha}\right),$$

we get

$$y_g=\frac{1}{\Lambda}\int\limits_0^x (x-t)^{\alpha-1}E_{\alpha,\alpha}\left(\frac{\lambda}{\Lambda}(x-t)^{\alpha}\right)g\left(t\right)dt.$$

On the other hand, the operational relations (34) and (28) lead to the representations

$$y_h=\frac{S^{\alpha}}{\Lambda S^{\alpha}-\lambda}\sum_{i=1}^{n}\sum_{k=0}^{n-1}\lambda_i c_k\frac{x^{k-n+\alpha-\beta_i\alpha+\beta_i n}}{\Gamma(k-n+\alpha-\beta_i\alpha+\beta_i n+1)}$$

$$=\sum_{i=1}^{n}\sum_{k=0}^{n-1}\lambda_i c_k\frac{S^{-(k-n-\beta_i\alpha+\beta_i n+1)}}{\Lambda S^{\alpha}-\lambda}=\sum_{i=1}^{n}\sum_{k=0}^{n-1}\frac{\lambda_i}{\Lambda}c_k\frac{S^{-(k-n-\beta_i\alpha+\beta_i n+\alpha+1)}}{I-\frac{\lambda}{\Lambda}S^{-\alpha}}$$

$$=\frac{1}{\Lambda}\sum_{i=1}^{n}\sum_{k=0}^{n-1}\lambda_i c_k x^{k-n-\beta_i\alpha+\beta_i n+\alpha}E_{\alpha,k-n-\beta_i\alpha+\beta_i n+\alpha+1}\left(\frac{\lambda}{\Lambda}x^{\alpha}\right).$$

Let now $\alpha_i=(n-i+1)\,\alpha$, $i=1,2,..,n$, where $0<\alpha<1$. Then the solution of the above problem can be represented in terms of the *Prabhakar type Mittag-Leffler function* $E^{m}_{\alpha,\beta}\left(t\right)$:

$$y_g=\frac{I}{\sum\limits_{i=1}^{n}\lambda_i S_{\alpha}^{n-i+1}-\lambda}g=\left[\sum_{j=1}^{p}\sum_{m=1}^{n_j}\frac{c_{jm}}{\left(S_{\alpha}-\gamma_j\right)^{m}}\right]g\,,$$

$$n_1+n_2+...+n_p=n.$$

The operational relation (27) leads to the expression

$$y_g=\int\limits_0^t u_{\delta}\left(\tau\right)g\left(t-\tau\right)d\tau,$$

where

$$u_{\delta}\left(t\right)=\sum_{j=1}^{p}\sum_{m=1}^{n_j}c_{jm}t^{\alpha m-1}E^{m}_{\alpha,\alpha m}\left(\gamma_j t^{\alpha}\right).$$

Acknowledgement

The third author was supported by 3 months postdoctoral fellowship of DAAD during 2008/09 and ICP-University of Stuttgart.

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Received: December 17, 2008

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