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IN MEMORIUM OF CHARLES FOX

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CHARLES FOX was born on 17 March 1897, in London, England and was son of Morris and Fenny Fox. He studied in Sidney Sussex College, Cambridge in 1915. After two years, his studies were interrupted as he joined the British Expeditionary Forces in France and was wounded in action in 1918. After completing his studies in Cambridge, he joined Imperial College of Science, London in 1919, as a Lecturer in Mathematics. In 1928 he was awarded the D.Sc. degree of the University of London.

Charles Fox emigrated to Canada in 1949 to join the McGill University, Montreal, Canada as Associate Professor of Mathematics and was promoted there to the post of Professor of Mathematics in 1956. In 1961 he was elected a Fellow of the Royal Society of Canada. After retirement from the McGill University in 1967, he joined Sir George Williams University (now Concordia University) as a visiting Professor of Mathematics and continued his teaching near the age of 80. He breathed his last on 30 April 1977 at the age of 80.

His scientific works consist of Null Series and Null Integrals, Mathematics of Navigation, Integral Transforms, Integral Equations, and Generalized Hypergeometric Functions.

His first two papers [1], [2] are concerned with the investigation on null series and null integrals. His work on Mathematics of Navigation can be seen from his papers [20] and [43].

One of the major contributions of Fox involves the systematic investigation of the asymptotic expansion of the generalized hypergeometric function (now called *Wright function*, or by some authors as *Fox-Wright function*) defined by

$$ {}_pF_q\left((a_1,A_1),...,(a_p,A_p);(b_1,B_1),...,(b_q,B_q);z\right) = \sum_{r=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(a_i+A_ir)}{\prod_{j=1}^{q}\Gamma(b_j+B_jr)}\,\frac{z^r}{r!}\,, \qquad (1) $$

where $z \in \mathbb{C}$, $a_i, b_j \in \mathbb{C}$, $A_i, B_j \in \mathbb{R}$ $(i = 1,...,p; j = 1,...,q)$; $\sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i \geq -1$. His method is an improvement over that of E.W. Barnes [*Trans. Cambridge Philos. Soc.*, **20** (1907), 253-279], who discussed the asymptotic expansion of the ordinary generalized hypergeometric function ${}_pF_q(z)$. Fox's work is quoted in the Slater's book *"Generalized Hypergeometric Series"*, Cambridge University Press, 1961; and also in Luke's book *"Special Functions and Their Approximations"*, Vol. I, II, Academic Press, New York, 1969. *At present, the conventional notation for the function defined by (1)* is ${}_p\Psi_q(z)$ instead of ${}_pF_q(z)$ given above.

The contributions of Fox on the special functions, integral transforms and integral equations can be divided into three classes, based on: (i) Mellin transform theory; (ii) fractional integration operators; and (iii) application of the L and L^{-1} operators.

The *technique of the Mellin transform* is exhibited in most of his papers. In his first paper on the application of Mellin transform to integral equation, Fox [14] developed the solution of the following Fredholm type equation:

$$ F(x) = G(x) + \int_0^{\infty} H(ux)F(u)du, \qquad (2) $$

where $G(x)$ and $H(x)$ are known functions and $F(x)$ is to be determined. By the application of Mellin transform theory, Fox investigated certain properties in the form of theorems for the following integral transforms:

1. Iterated transforms [22];
2. Chain transforms [18], [24], [31];
3. Classification of kernels which possess integral transforms [26];
4. General unitary transforms [28];
5. Matrix integral transform [36].

Fox [23] also developed the inversion formula for the convolution transforms by the application of certain differential operators. It appears that after reading the paper of A. Erdélyi [*Rend. Sem. Mat. Univ. Politech. Torino* **10** (1950-51), 217-234] he became interested in the applications of the *Erdélyi-Kober operators of fractional integration*, defined by

$$ T[f(x);\alpha,\eta,m] := \frac{m}{\Gamma(\alpha)}\, x^{-\gamma-m\alpha-m-1} \int_0^{x} (x^m - t^m)^{\alpha-1} t^{\eta} f(t)dt, \qquad (3) $$

and

$$R[f(x);\alpha,\xi,m] := \frac{m}{\Gamma(\alpha)}\, x^{\xi} \int\limits_{x}^{\infty} (t^m - x^m)^{\alpha-1} t^{-\xi-m\alpha+m-1} f(t)dt, \tag{4}$$

which can be seen from his publications after 1958 onwards.

My personal association with Charles Fox goes to the year 1965, when I joined the McGill University, Montreal, Canada as a post-doctoral Fellow of National Research Council of Canada *under his supervision and worked with him for one year, during the session 1965-1966.*

In 1961, *Fox* [32] *introduced the H-function* in the theory of special functions, which is a generalization of the MacRobert's E-function, Wright's generalized hypergeometric function (1) and the Meijer's G-function. In that paper, he investigated the most generalized Fourier kernel associated with the H-function and established many properties and special cases of this kernel. The H-function is defined in terms of Mellin-Barnes type integral [32, p.408] and represented in the following manner:

$$H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_1,A_1),...,(a_p,A_p) \\ (b_1,B_1),...,(b_q,B_q) \end{matrix} \right] = \frac{1}{2\pi i} \int\limits_{\mathcal{L}} \chi(\xi)\, z^{-\xi} d\xi, \tag{5}$$

where $i = (-1)^{1/2}$,

$$\chi(\xi) = \frac{\left[\prod_{j=1}^{m} \Gamma(b_j + B_j\xi)\right] \left[\prod_{i=1}^{n} \Gamma(1 - a_j - A_j\xi)\right]}{\left[\prod_{j=m+1}^{q} \Gamma(1 - b_j - B_j\xi)\right] \left[\prod_{i=n+1}^{p} \Gamma(a_j + A_j\xi)\right]}, \tag{6}$$

and an empty product is always interpreted as unity; $m,n,p,q \in \mathbb{N}_0$ with $0 \le n \le p$, $1 \le m \le q$, $A_i, B_j \in \mathbb{R}_+$, $a_i, b_j \in \mathbb{R}$ or $\mathbb{C}$ $(i = 1,...,p; j = 1,...,q)$ such that

$$A_i(b_j + k) \ne B_j(a_i - l - 1),\ k,l \in \mathbb{N}_0;\ i = 1,...,n; j = 1,...,m,$$

$\mathcal{L}$ is a suitable contour which separates all the poles of $\Gamma(b_j + B_j\xi), j = 1,...,m$, from those of $\Gamma(1 - a_i - A_i\xi), i = 1,...,n$. The integral in (5) converges absolutely and defines an analytic function in the sector

$$|\arg(z)| < \frac{1}{2}\pi\varphi, \text{ where } \varphi = \sum_{i=1}^{n} A_i - \sum_{i=n+1}^{p} A_i + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j > 0, \tag{7}$$

the point $z = 0$ being tacitly excluded.

It has been shown by Fox [32, p. 408] that a special case of the H-function

$$H_{2p,2q}^{q,p}(x) = H_{2p,2q}^{q,p}\left[x \,\middle|\, \begin{matrix} (1-a_i,A_i)_{1,p}, (a_i - A_i, A_i)_{1,p} \\ (b_j,B_j)_{1,q}, (1-b_j-B_j,B_j)_{1,q} \end{matrix} \right], \quad x > 0, \tag{8}$$

behaves as a *symmetrical Fourier kernel* in $L_2(\mathbb{R}_+)$ under certain conditions on the parameters. He was the first person to study the properties of the spaces $L_2(\mathbb{R}_+)$ and proved that ([32, Theorem 5]) if $f(x) \in L_2(\mathbb{R}_+)$ and the following conditions are satisfied

$$D = 2\sum_{j=1}^{q} B_j - 2\sum_{i=1}^{p} A_i > 0;\ \Re(a_i) > \frac{A_i}{2},\ \Re(b_j) > \frac{B_j}{2},\ i = 1, ..., p; j = 1, ..., q,$$

then

$$g(x) = \frac{d}{dx}\int_0^\infty H_1(xt) f(t)\,\frac{dt}{t},$$

defines almost everywhere a function $g(x) \in L_2(\mathbb{R}_+)$, and

$$f(x) = \frac{d}{dx}\int_0^\infty H_1(xt) g(t)\frac{dt}{t}$$

holds almost everywhere, and the Parseval theorem

$$\int_0^\infty |f(x)|^2 dx = \int_0^\infty |g(x)|^2 dx \tag{9}$$

holds, where $H_1(x) = \int_0^x H_{2p,2q}^{q,p}(t)dt$. Fox [32] further investigated the asymptotic expansion at infinity of this kernel for $x > 0$ by following a method which was originated in his paper on the Fourier kernels [9].

The H-function already existed in the so-called *Bateman-Erdélyi project* [A. Erdélyi et al., *Higher Transcendental Functions*, Vol. **1**, Section 1.19, p. 49, Equation (1), McGraw-Hill, New York, 1953; see also Transl. in Russian, Nauka, Moscow, 1973, p. 64]. *This fact was brought to the notice of Fox in January 1966 by the author.* This is the reason that he did not pursue any further research work on this H-function beyond the paper [39]. In this paper, *Fox derived a formal solution of dual integral equations involving special H-functions, in terms of modified Erdélyi-Kober operators.*

In passing, it may be remarked here that according to S. Pincherle, who first discovered the so-called *Mellin-Barnes integrals* in 1888, these integrals are based on the duality principle between linear differential equations and linear difference equations with rational coefficients. These integrals are studied in one form or the other, by E.W. Barnes in 1908, H. Mellin [*Math. Ann.* **68** (1910), 305-337], A.L. Dixon and W.L. Ferrar [*Quart. J. Math.* **7** (1936), 81-96] and S. Bochner [*Ann. Math.* (2) **53** (1951), 332-363]. *A* *systematic study of this function is, however, made by Fox* in connection with the investigation of the generalized symmetrical Fourier kernel. According to the author, the Fourier kernels are useful in the characterization of probability distributions.

The work by Fox on statistical distributions can be seen in the two papers [29], [40].

The reciprocities

$$g(x) = \int_0^\infty k(xt)f(t)dt\,,\ f(x) = \int_0^\infty k(xt)g(t)dt\,, \tag{10}$$

usually known as the *Generalized Fourier Integral transform*, can be established if certain convergence conditions and the functional equation $K(s)$ $K(1-s) = 1$ are satisfied, where $K(s)$ is the Mellin transform of $k(x)$. As a result of his consistent interest in the Fourier kernels, Fox [35] has shown that the reciprocities can be obtained with the help of fractional integration operators (3) and (4), which are analogous to the Fourier transforms above but which develop from the much more general functional equation

$$K(s)K(1-s) = \prod_{i=1}^{n} \frac{\Gamma[\alpha_i + (\eta_i + 1 - s)/m_i]\Gamma[\alpha_i + (\eta_i + s)/m_i]}{\Gamma[(\eta_i + 1 - s)/m_i]\Gamma[(\eta_i + s)/m_i]}\,. \tag{11}$$

Analogues of the Parseval theorem and discontinuous integrals associated with Fourier transforms are also obtained.

In [38], Fox reduced the *modified Meijer transform*, defined by

$$g(x) = \int_0^\infty (xt)^\nu\, K_\nu(xt)f(t)dt, \tag{12}$$

in terms of the Laplace transform and derived its inversion formula.

The technique of the Laplace transform L and its inverse L^{-1} is exhibited by Fox in two of his papers [41], [42]. In one of the papers, Fox [42] obtained the inversion formula for the Varma transform, defined by

$$g(t) = \int_0^\infty (tu)^{m-1/2} \exp(-\frac{1}{2}tu)\, W_{k,m}(tu)f(u)du, \quad t > 0, \tag{13}$$

in the form: $f(x) = x^{-2m}\, L^{-1}\left\{t^{-k-m+1/2}\, L\left[x^{m-k+1/2}\, L^{-1}\{g(t)\}\right]\right\}$.

Finally, it is interesting to observe that Fox also derived *many results for miscellaneous topics of interest, such as*: orthocenter, potential function, polar equation of a curve, Hankel's theorem, magic matrices, Pascal line, etc. In this connection, one can refer to his papers [6], [8], [10], [11], [12], [13], [15], [17], [19], [25], [27] and [30].

It is not out of place to mention that due to occurrence of the *Mittag-Leffler functions* in fractional differential equations, the importance of the H-function is considerably increased in recent years for workers in Science and Engineering, specially in the areas of fractional reaction, fractional diffusion and anomalous diffusion problems in complex systems. The trend of research in many branches of applied sciences and technology is changing towards applications of the H-function (and its particular case, the Mittag-Leffler function) in their areas of research. The first paper in which the solution of a fractional diffusion equation is obtained in terms of the H-function is published by W. Wyss [*J. Math. Phys.* **27**, No. 11 (1986), 2782-2785]. *Thus, Fox will always be remembered for this rediscovery of the H-function.*

He was a dedicated, honest and effective teacher, who made significant contributions in the field of his specialization and inspired and encouraged research workers in doing independent research work.

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Publications of Charles Fox

A: **Book**: *An Introduction to the Calculus of Variations*, Oxford University Press, First Edition 1953; Reprinted 1954 and 1963; Reprinted again in 1988, Dover Publications, 1988.

B: Research Papers:

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