THE COMMUTANT OF THE RIEMANN-LIOUVILLE OPERATOR OF FRACTIONAL INTEGRATION

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Abstract

Characterization results for the continuous linear operators $M : C[a, b] \to C[a, b]$ and $N : L^1[a, b] \to L^1[a, b]$ commuting with a fixed Riemann-Liouville operator for integration of fractional order

$$I^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} (x-t)^{\alpha-1} f(t) \, dt, \; \alpha > 0,$$

in the corresponding space are found.

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1. Introduction

It is well known that every two Riemann-Liouville operators $I^{\beta}_{a+}$ and $I^{\gamma}_{a+}$ with $\beta, \gamma > 0$ commute (cf. Samko, Kilbas, Marichev [6], pp. 33-34, (2.17) and (2.21)), i.e. $I^{\beta}_{a+}I^{\gamma}_{a+} = I^{\gamma}_{a+}I^{\beta}_{a+}$.

Our aim here is to characterize all continuous linear operators $M : C[a, b] \to C[a, b]$ and $N : L^1[a, b] \to L^1[a, b]$ which commute with a fixed Riemann-Liouville operator
\[ I_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \, \alpha > 0, \] (1)
in \(C[a,b]\) or \(L^1[a,b]\).

Further, we will consider elaborately only the commutation in the space \(C = C[a,b]\), since the considerations for the space \(L^1[a,b]\) are analogous.

As a basic tool we will use some properties of the Duhamel convolution algebra \((C, \ast)\) with the multiplication operation
\[ (f \ast g)(x) = \int_a^x f(x + a - t)g(t)dt. \] (2)

As it is well known, the bilinear operation (2) is commutative and associative one, and the Riemann-Liouville operator (1) can be represented as a convolution operator \(\left\{ \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} \right\} \ast\), i.e.:
\[ I_{a+}^\alpha f(x) = \left\{ \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} \right\} \ast f(x). \] (3)

The operator \(I_{a+}^\alpha\) is a multiplier of the Duhamel convolution algebra in the sense of the following definition.

**DEFINITION 1.** (cf. Larsen [4], p.13). A mapping \(M : C \rightarrow C\) is said to be a multiplier of the algebra \((C, \ast)\), if the identity
\[ (Mf) \ast g = f \ast Mg \] (4)
holds for all \(f, g \in C\).

As it is shown in [4], pp. 13-14, the set of all multipliers of the convolution algebra \((C, \ast)\) form a commutative subalgebra of the algebra of all continuous linear operators \(A : C \rightarrow C\). It contains the identity operator (which is not a convolution operator).

**COROLLARY.** Let \(M : C \rightarrow C\) be a multiplier of the convolution algebra \((C, \ast)\). Then, for all \(f, g \in C\) it holds the identity
\[ M(f \ast g) = (Mf) \ast g. \]

The assertion follows from the commutativity of the algebra of multipliers of \((C, \ast)\). Indeed, if \(f\) denotes the multiplier operator \((f \ast)g = f \ast g\), then it commutes with \(M\), i.e. \(M(f \ast) = (f \ast)M\). The identity \([M(f \ast)]g = [(f \ast)M]g\) holding for arbitrary \(g \in C\) can be written as...
THE COMMUTANT OF THE RIEMANN-LIOUVILLE \ldots 445

\[ M(f \ast g) = f \ast Mg. \]

**Theorem 1.** Let \( M : C \to C \) be an arbitrary continuous linear operator, commuting with \( I^\alpha_{a+} \), i.e. such that \( MI^\alpha_{a+} = I^\alpha_{a+}M \). Then \( M \) is a multiplier of the convolution algebra \((C, \ast)\).

**Proof.** We use the continuity of operation (2) with respect to both operands, i.e. the fact that \( f_n \to f \) and \( g_n \to g \) imply \( f_n \ast g_n \to f \ast g \) for arbitrary \( f \) and \( g \), along with the assumed continuity of the operator \( M \).

We start with the evident identity

\[ M\{1\} \ast \{1\} = \{1\} \ast M\{1\}. \]

We take arbitrary non-negative integers \( m \) and \( n \) and apply the operator \((I^\alpha_{a+})^m + (I^\alpha_{a+})^n = I^m_{a+} + I^n_{a+}\) to both sides of the above identity. Thus we get

\[ (I^m_{a+} M\{1\}) \ast (I^n_{a+} \{1\}) = (I^m_{a+} \{1\}) \ast (I^n_{a+} M\{1\}) \]

which can be written as

\[ M(I^m_{a+} \{1\}) \ast (I^n_{a+} \{1\}) = (I^m_{a+} \{1\}) \ast (I^n_{a+} M\{1\}), \]

using the commutation relation \( MI^\alpha_{a+} = I^\alpha_{a+}M \). But

\[ I^\beta_{a+} \{1\} = \frac{(x - a)^\beta}{\Gamma(\beta + 1)} \]

for arbitrary \( \beta > 0 \).

Hence,

\[ M\{(x - a)^m\} \ast \{(x - a)^n\} = \{(x - a)^m\} \ast M\{(x - a)^n\} \]

for arbitrary \( m, n \in \mathbb{N}_0 \). From Weierstrass’ approximation theorem it follows that

\[ \text{span}\{(x - a)^m\}_{m=0}^{\infty} = C[a, b]. \]

It remains to use the continuity of the convolution product, and thus to obtain the identity

\[ Mf \ast g = f \ast Mg \]

for arbitrary \( f, g \in C[a, b] \). Hence, \( M \) is a multiplier of the convolution algebra \((C, \ast)\). \( \blacksquare \)
Theorem 1 reduces the problem of characterizing of the continuous linear operators \( M : C[a, b] \rightarrow C[a, b] \) commuting with the Riemann-Liouville operator \( I_{a+}^\alpha \) to the problem of characterizing the multipliers of the convolution algebra \((C, \ast)\). The solution of the last problem is given by the following theorem.

**Theorem 2.** (Dimovski [2], pp. 6-7) The following assertions are equivalent:

i) \( M \) is a multiplier of the convolution algebra \((C, \ast)\);

ii) a) \( m(t) = M\{1\} \in C \) is a function with bounded variation on \([a, b]\);

b) It holds the integral representation

\[
Mf(x) = \frac{d}{dx} \int_{a}^{x} m(x + a - t)f(t)dt
\]  

(6)

for arbitrary \( f \in C[a, b]\).

**Remark.** In [2] it is considered the case \( a = 0 \), but the result for arbitrary \( a \in \mathbb{R} \) could be obtained by a simple translation.

2. Characterization of the commutant of \( I_{a+}^\alpha \) on \( C[a, b] \)

Theorems 1 and 2 give a complete characterization of the commutant of a fixed Riemann-Liouville operator on \( C[a, b] \), which we state as the following theorem.

**Theorem 3.** A continuous linear operator \( M : C[a, b] \rightarrow C[a, b] \) commutes with a fixed Riemann-Liouville operator \( I_{a+}^\alpha \) in \( C[a, b] \) iff:

a) the function \( m(t) = M\{1\} \) is a function with bounded variation on \([a, b]\);

and

b) \[
Mf(x) = \frac{d}{dx} \int_{a}^{x} m(x + a - t)f(t)dt .
\]

This representation can be written in the equivalent form

\[
Mf(x) = m(a)f(x) + \int_{a}^{x} f(x + a - t)dm(t),
\]  

(7)

where the integral is to be understood in the sense of Stieltjes.

The characterization, given by Theorem 3, can be stated in a more transparent form as the following theorem.
Theorem 4. The commutant of an arbitrary Riemann-Liouville operator is a commutative algebra, isomorphic to the convolution algebra $\text{BV} \cap C$ of the functions of $C[a, b]$ with bounded variation, where the multiplication operation is
\[
(f \tilde{*} g)(x) = \frac{d}{dx} \int_{a}^{x} f(x + a - t)g(t)dt.
\]
(8)
The isomorphism is given by the mapping $M \mapsto M\{1\} = m(t)$ and it is both algebraic and topological isomorphism.

The fact that (8) is an inner operation in $\text{BV} \cap C$ is proved in Mikusiński and Ryll-Nardzewski [5].

3. Characterization of the commutant of $I_{a+}^\alpha$ on $L^1[a, b]$

We use almost the same approach as in the case of $C[a, b]$. Instead of Weierstrass’ approximation theorem, we rely on the analogous theorem which assert that each function of $L^1[a, b]$ can be approximated by polynomials in the $L^1$-metrix. An analogue of Theorem 1 holds with the only difference that for the function $n(t) = N\{1\}$, it is asserted that it is a function with essentially bounded variation, i.e. that it is equal almost everywhere to a function with bounded variation on $[a, b]$ (cf. Dixmier [3]). In [3] it is proposed a characterization of the commutant of the integration operator $I = I_{0+}^1$ in $L^1[0, 1]$.

Theorem 5. The following assertions are equivalent:

i) A continuous linear operator $N : L^1[a, b] \rightarrow L^1[a, b]$ commutes with the Riemann-Liouville operator $I_{a+}^\alpha$ on $L^1[a, b]$;

ii) a) $n(x) = N\{1\} \in L^1[a, b]$ is a function with essentially bounded variation on $[a, b]$;

b) the operator $N$ admits the representation
\[
Nf(x) = \frac{d}{dx} \int_{a}^{x} n(x + a - t)f(t)dt,
\]
(9)
where the integral is to be understood in the Lebesgue sense.

The only non-trivial point is the proof of the fact that if $n$ is a function with essentially bounded variation, and $f$ is a Lebesgue integrable function, then the convolution product $n \ast f$ is an absolutely continuous function on $[a, b]$. Such result cannot be seen in Mikusiński and Ryll-Nardzewski [5]. It is proved by N. Bozhinov in his book [1], p.137.
In the same way as in the case of $C[a, b]$ the characterization result, given by Theorem 5, can be stated in an equivalent form:

**Theorem 6.** The commutant of $I_{\alpha}^a +$ in $L^1[a, b]$ is a commutative algebra, isomorphic to the convolution algebra $\text{(ess BV, } \tilde{\ast})$ of the functions of $L^1[a, b]$ with essentially bounded variation with the convolution product

$$
(f \tilde{\ast} g)(x) = \frac{d}{dx} \int_a^x f(x + a - t)g(t)dt.
$$

(10)
as the multiplication. The algebraic and topological isomorphism is given by the mapping $N \mapsto n(x) = N\{1\}$.

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**References**