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BOUNDARY VALUE PROBLEMS FOR THE GENERALIZED TIME-FRACTIONAL DIFFUSION EQUATION OF DISTRIBUTED ORDER

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Dedicated to Professor Ivan Dimovski
on the occasion of his 75th anniversary

Abstract

In the paper, boundary value problems for the generalized time-fractional diffusion equation of distributed order over an open bounded domain $G \times [0,T],\ G \in I\!R$ are considered. Both some uniqueness and existence results are presented. To show the uniqueness of the solution of the problem, an appropriate maximum principle for the generalized time-fractional diffusion equation of distributed order is formulated and proved. The maximum principle is based on an extremum principle for the Caputo-Dzherbashyan fractional derivative that was earlier introduced by the author. The existence of the solution of the problem is illustrated by constructing a formal solution using the Fourier method of variables separation. The initial-boundary-problems for the generalized time-fractional diffusion equation of distributed order are shown to be from the class of the well-posed problems in the Hadamard sense.

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1. Introduction

Differential equations of fractional order have been successfully employed for modeling of the so called anomalous phenomena (see e.g. [3], [6]-[10], [15], [16], [18], [19], [21] and references there) during the last few decades. A lot of attention both in applied and mathematical papers was given to the so called time-fractional diffusion equation that is obtained from the diffusion equation by replacing the first order time derivative by a fractional derivative of order α with $0 < \alpha < 1$. In [22], the Green function for the time-fractional diffusion equation was shown to be a probability density with the mean square displacement proportional to t^{α}. This fact stimulated appearance of a still growing number of publications, where the time-fractional diffusion equation is employed as a suitable mathematical model for the so called sub-diffusion processes.

On the other hand, in some recent publications (see e.g. [1]-[3], [20], [23] and references there) the sub-diffusion processes with the mean square displacement with a logarithmic growth have been introduced. The usual way to model such processes is to employ the so called time-fractional diffusion equation of distributed order. ¿From the mathematical viewpoint, such equations were investigated e.g. in [11], [17], and [24]. In these papers, mainly the case of the initial-value problems on the unbounded domains for the equations with the constant coefficients was considered. However, in the applications one has mainly to deal with the initial-boundary-problems for the equations with the variable coefficients. In this paper, a first attempt to attack such problems is undertaken. The employed technique follows the lines of the recent papers [13], [14] by the author, where the case of the initial-boundary-value problems for the generalized time-fractional diffusion equation with the variable coefficients and over an open bounded n-dimensional domain was considered. This equation is obtained from the classical diffusion equation by replacing the first-order time derivative by a fractional derivative of order α $(0 < \alpha \leq 1)$ and the Laplace operator by a more general linear second-order differential operator with the variable coefficients. In this connection, the paper [25] should be mentioned, where even more general linear and quasilinear evolutionary partial integro-differential equations of second order were considered. In particular, the global boundedness of appropriately defined weak solutions and a maximum principle for the weak solutions of such equations were established in [25] by employing a different technique compared to the one used in the papers [13], [14]. For the numerical methods for the fractional differential equations of distributed order we refer the reader to the paper [5].

In the recent paper [4], the case of the multi-term time-fractional diffusion-wave equation with the constant coefficients was considered. This equation can be interpreted as a particular case of the time-fractional diffusion-wave equation of distributed order with a linear combination of the Dirac δ-functions as the weight function. In the paper [4], solutions of the corresponding initial-boundary-value problems for the multi-term equations were formally represented in the form of the Fourier series via the multivariate Mittag-Leffler function introduced in [12]. No proofs for the convergence of the series (i.e. that the obtained formal solutions are in fact solutions) and for the uniqueness of the solution were given in [4]. The results presented in this paper can serve in particular as a theoretical background for the formal presentations in [4].

The rest of the paper is organized as follows. In the second section, the maximum and minimum principles formulated and proved earlier by the author for the generalized time-fractional diffusion equation are extended for the case of the time-fractional diffusion equation of distributed order. To prove the maximum and minimum principles, an appropriate extremum principle for the Caputo-Dzherbashyan fractional derivative is employed. In the third section, the maximum and minimum principles are applied to show the uniqueness of the solution of the initial-boundary-value problem for the time-fractional diffusion equation of distributed order. This solution - if it exists - depends continuously on the data given in the problem. Finally, in the last section the existence of the solution is illustrated by constructing a formal solution using the Fourier method of variables separation.

2. Maximum principle

In this section, the maximum and minimum principles for the generalized time-fractional diffusion equation of distributed order are formulated and proved. These principles are the main tool for establishing the uniqueness of solution of the boundary value problems for the generalized time-fractional diffusion equation of distributed order over an open bounded domain $G \times (0,T),\ G \subset \mathbb{R}^n$ in the next section. The generalized time-fractional diffusion equation of distributed order is obtained from the classical diffusion equation by replacing the first-order time derivative by a fractional derivative of distributed order and the Laplace operator by a more general liner second-order differential operator:

$$\mathbb{D}_t^{w(\alpha)}(u(x,t)) = L_x(u(x,t)) + F(x,t),\ (x,t) \in \Omega_T := G \times (0,T),\ G \subset \mathbb{R}^n, \tag{1}$$

where

$$L_x(u) := \operatorname{div}(p(x)\operatorname{grad} u) - q(x)u, \tag{2}$$

$$p \in C^1(\bar{G}), q \in C(\bar{G}),\ 0 < p(x), 0 \le q(x), x \in \bar{G}, \tag{3}$$

the distributed order derivative $\mathbb{D}_t^{w(\alpha)}$ is defined by

$$\mathbb{D}_t^{w(\alpha)} f(t) = \int_0^1 (D_t^\alpha f)(t)\, w(\alpha)\, d\alpha \tag{4}$$

with the Caputo-Dzherbashyan fractional derivative

$$(D_t^\alpha f)(t) := (I^{1-\alpha}\frac{d\,f}{d\,\tau})(t),\quad 0 < \alpha \le 1, \tag{5}$$

I^α being the fractional Riemann-Liouville integral

$$(I^\alpha f)(t) := \begin{cases} \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, & 0 < \alpha < 1, \\ f(t), & \alpha = 0, \end{cases}$$

and with a continuous non-negative weight function $w:\ [0,1] \to \mathbb{R}$ that is not identically equal to zero on the interval $[0,1]$, such that

$$0 \le w(\alpha),\ w \not\equiv 0,\ \alpha \in [0,1],\ \int_0^1 w(\alpha)\, d\,\alpha\ =\ W\ > 0, \tag{6}$$

and the domain G with the boundary S is open and bounded in $\mathbb{R}^n$.

The operator L_x from the equation (1) is a linear elliptic differential operator of the second order

$$L_x(u) = \sum_{k=1}^n \left(p(x)\frac{\partial^2 u}{\partial x_k^2} + \frac{\partial p}{\partial x_k}\frac{\partial u}{\partial x_k} \right) - q(x)u,$$

that can be represented in the form

$$L_x(u) = p(x)\triangle u\ +\ (\operatorname{grad} p,\, \operatorname{grad} u) - q(x)u, \tag{7}$$

$\triangle$ being the Laplace operator.

Evidently, the equation (1) possesses in general an infinite number of solutions. In the real world situations that are modeled with the equation (1), certain conditions that describe an initial state of the corresponding process and the observations of its visible parts ensure the deterministic character of the process. In this paper, the initial-boundary-value problem

$$u\big|_{t=0} = u_0(x), \; x \in \bar{G}, \tag{8}$$

$$u\big|_S = v(x,t), \; (x,t) \in S \times [0,T] \tag{9}$$

for the equation (1) is considered. Here S denotes as usual the boundary of the domain G and $\bar{G}$ its closure.

Following [13], [14], the notion of the classical solution of the problem (1), (8), (9) is first introduced.

Definition 2.1. A classical solution of the problem (1), (8), (9) is called a function u defined in the domain $\bar{\Omega}_T := \bar{G} \times [0,T]$ that belongs to the space $CW_T(G) := C(\bar{\Omega}_T) \cap W_t^1((0,T)) \cap C_x^2(G)$ and satisfies both the equation (1) and the initial and boundary conditions (8)-(9). By $W_t^1((0,T))$ the space of the functions $f \in C^1((0,T])$ such that $f' \in L((0,T))$ is denoted.

If a classical solution to the initial-boundary-value problem (8), (9) for the equation (1) exists, then the functions F, u_0 and v given in the problem have to belong to the spaces $C(\Omega_T)$, $C(\bar{G})$ and $C(S \times [0,T])$, respectively. In the further discussions, we always suppose these inclusions to be valid.

The main focus of the paper is to prove the uniqueness of the solution of the problem (1), (8), (9). The method used to do this is based on the appropriate maximum and minimum principles for the equation (1). In the proof of the maximum principle, the following extremum principle for the Caputo-Dzherbashyan fractional derivative (5) plays an essential role:

Theorem 2.1. *Let a function $f \in W_t^1((0,T)) \cap C([0,T])$ attain its maximum over the interval $[0,T]$ at the point $\tau = t_0$, $t_0 \in (0,T]$. Then the Caputo-Dzherbashyan fractional derivative of the function f is non-negative at the point t_0 for any α, $0 < \alpha \leq 1$:*

$$0 \leq (D^\alpha f)(t_0), \; 0 < \alpha \leq 1. \tag{10}$$

For the proof of the theorem we refer the reader to the paper [13]. To illustrate the extremum principle for the Caputo-Dzherbashyan fractional derivative a simple example is presented.

Let us consider a family of functions in the form

$$f(t) := -at^2 + bt + c, \; 0 < a, \; 0 < b \leq 2a, \; c \in \mathbb{R} \tag{11}$$

on the closed interval $[0,1]$. The conditions on the parameters of the function f ensure the existence of the maximum point $t = b/(2a)$ that belongs to the interval $[0,1]$. The function f is evidently a $C^1([0,1])$ function and thus fulfils all conditions of the theorem.

Let us now evaluate the Caputo-Dzherbashyan fractional derivative of the function f at the maximum point $t = b/(2a)$. Simple calculations lead to the following formulae for $0 < \alpha \le 1$:

$$(D^\alpha f)(t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\left(-\frac{2at}{2-\alpha} + b\right),$$

$$(D^\alpha f)(t)\Big|_{t=\frac{b}{2a}} = \frac{\left(\frac{b}{2a}\right)^{1-\alpha} b}{\Gamma(2-\alpha)}\frac{1-\alpha}{2-\alpha}.$$

Thus

$$(D^\alpha f)(t)\Big|_{t=\frac{b}{2a}} > 0,\ 0 < \alpha < 1,\ \ (D^\alpha f)(t)\Big|_{t=\frac{b}{2a}} = 0,\ \alpha = 1,$$

that is in accordance with the statement of the theorem.

The maximum principle for the generalized time-fractional diffusion equation (1) of distributed order is given by the following theorem:

Theorem 2.2. *Let a function $u \in CW_T(G)$ be a solution of the generalized time-fractional diffusion equation (1) of distributed order in the domain Ω_T and $F(x,t) \le 0,\ (x,t) \in \Omega_T$. Then either $u(x,t) \le 0,\ (x,t) \in \bar{\Omega}_T$ or the function u attains its positive maximum on the part $S_G^T := (\bar{G} \times \{0\}) \cup (S \times [0,T])$ of the boundary of the domain Ω_T, i.e.,*

$$u(x,t) \le \max\{0, \max_{(x,t)\in S_G^T} u(x,t)\},\ \ \forall (x,t) \in \bar{\Omega}_T. \tag{12}$$

To prove the theorem, the method of contradiction is used like in the classical case of the parabolic PDEs that corresponds to the weight function $w(\alpha) = \delta(\alpha - 1)$, δ being the Dirac function. First we assume that the statement of the theorem does not hold true, i.e., $\exists (x_0, t_0),\ x_0 \in G,\ 0 < t_0 \le T$ with the property

$$u(x_0,t_0) > \max_{(x,t)\in S_G^T}\{0,\ u(x,t)\} = M > 0. \tag{13}$$

To arrive to a contradiction, the auxiliary function

$$f(x,t) := u(x,t) + \frac{\epsilon}{2}\frac{T-t}{T},\ (x,t) \in \bar{\Omega}_T$$

with $\epsilon := u(x_0,t_0) - M > 0$ is introduced. ¿From the conditions of the theorem and the assertion made at the beginning of the proof, the following inequalities can be obtained:

$$f(x,t) \le u(x,t) + \frac{\epsilon}{2},\ (x,t) \in \bar{\Omega}_T,$$

$$f(x_0,t_0) \ge u(x_0,t_0) = \epsilon + M \ge \epsilon + u(x,t)$$
$$\ge \epsilon + f(x,t) - \frac{\epsilon}{2} \ge \frac{\epsilon}{2} + f(x,t),\ (x,t) \in S_G^T,$$

that means that the function f cannot attain its maximum on the part S_G^T of the boundary of the domain Ω_T. Because f is a continuous function, there exists a maximum point of f over the closed domain $\bar{\Omega}_T$ that is denoted by (x_1, t_1). Then $x_1 \in G,\ 0 < t_1 \le T$ and the inequality

$$f(x_1,t_1) \ge f(x_0,t_0) \ge \epsilon + M > \epsilon \tag{14}$$

holds true. Moreover, Theorem 2.1 and the necessary conditions for the existence of a maximum of the function f in on open domain Ω_T ensure that

$$\begin{cases} (D^\alpha f)(t_1) \ge 0,\ 0 < \alpha < 1 \\ (D^\alpha f)(t_1) = f(x_1,t_1) > \epsilon > 0,\ \alpha = 0, \\ (D^\alpha f)(t_1) = 0,\ \alpha = 1, \\ \operatorname{grad} f|_{(x_1,t_1)} = 0, \\ \triangle f|_{(x_1,t_1)} \le 0. \end{cases} \tag{15}$$

To have a contradiction, the operator $\mathbb{D}_t^{w(\alpha)} u - L_x(u) - F$ for the solution u of the time-fractional diffusion equation (1) of distributed order at the point (x_1,t_1) is estimated. The function u satisfies the relation

$$u(x,t) = f(x,t) - \frac{\epsilon}{2}\frac{T-t}{T},\quad (x,t) \in \bar{\Omega}_T, \tag{16}$$

according to the definition of the function f. Then

$$\mathbb{D}_t^{w(\alpha)} u = \mathbb{D}_t^{w(\alpha)} f - \int_0^1 \left(D_t^\alpha \left(-\frac{\epsilon}{2}\frac{T-\tau}{T} \right) \right)(t)\, w(\alpha)\, d\alpha \tag{17}$$
$$= \mathbb{D}_t^{w(\alpha)} f + \frac{\epsilon}{2T} \int_0^1 \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\, w(\alpha)\, d\alpha.$$

To get the expression above, the well-known formulae

$$(D^\alpha \tau^\beta)(t) = \frac{\Gamma(1+\beta)}{\Gamma(1-\alpha+\beta)} t^{\beta-\alpha},\ 0 < \beta,\ 0 < \alpha \le 1,$$
$$(D^\alpha C)(t) \equiv 0,\ C \text{ being a constant},$$

for the Caputo-Dzherbashyan fractional derivative were used.

The conditions (6) and the formulae (3), (7), (14)-(17) lead now to the following chain of equalities and inequalities:

$$(\mathbb{D}_t^{w(\alpha)} u - \operatorname{div}(p \operatorname{grad} u) + qu - F)|_{(x_1,t_1)}$$

$$= I\!D_t^{w(\alpha)} f|_{(x_1,t_1)} + \frac{\epsilon}{2T}\int_0^1 \frac{t_1^{1-\alpha}}{\Gamma(2-\alpha)}\, w(\alpha)\, d\alpha \ - p\triangle f|_{(x_1,t_1)}$$

$$-(\mathrm{grad}\, p|_{x_1}, \mathrm{grad}\, f|_{(x_1,t_1)}) + q(x_1)\left(f(x_1,t_1) - \frac{\epsilon}{2}\frac{T-t_1}{T}\right) - F(x_1,t_1)$$

$$\geq \frac{\epsilon}{2T}\int_0^1 \frac{t_1^{1-\alpha}}{\Gamma(2-\alpha)}\, w(\alpha)\, d\alpha \ + q(x_1)\epsilon\left(1 - \frac{T-t_1}{2T}\right) > 0,$$

that contradicts the condition of the theorem saying that the function u is a solution of the equation (1). The obtained contradiction completes the proof of the theorem. ■

Following the lines of the proof of the maximum principle, the following minimum principle can be established if we substitute $-u$ instead of u in the reasoning above:

THEOREM 2.3. *Let a function $u \in C(\bar{\Omega}_T) \cap W_t^1((0,T)) \cap C_x^2(G)$ be a solution of the generalized time-fractional diffusion equation (1) of distributed order in the domain Ω_T and $F(x,t) \geq 0,\ (x,t) \in \Omega_T$. Then either $u(x,t) \geq 0,\ (x,t) \in \bar{\Omega}_T$ or the function u attains its negative minimum on the part $S_G^T = (\bar{G} \times \{0\}) \cup (S \times [0,T])$ of the boundary of the domain Ω_T, i.e.,*

$$u(x,t) \geq \min\{0, \min_{(x,t)\in S_G^T} u(x,t)\}, \quad \forall (x,t) \in \bar{\Omega}_T. \tag{18}$$

3. Uniqueness of the solution

In this section, the maximum and minimum principles are used to show the uniqueness of solution of the initial-boundary-value problem (8)-(9) for the generalized time-fractional diffusion equation (1) of distributed order. Moreover, we prove that the solution - if it exists - continuously depends on the problem data.

The main result of the section is given in the following theorem:

THEOREM 3.1. *The initial-boundary-value problem (8)-(9) for the generalized time-fractional diffusion equation (1) of distributed order possesses at most one solution in the sense of Definition 2.1. This solution - if it exists - continuously depends on the data given in the problem and the estimate*

$$\|u - \tilde{u}\|_{C(\bar{\Omega}_T)} \leq \max\{\epsilon_0, \epsilon_1\} + \frac{T^\alpha}{W\,\Gamma(1+\alpha)}\,\epsilon \tag{19}$$

holds true for the solutions u and $\tilde{u}$ of the problem (1), (8)-(9) with the data F, u_0, v and $\tilde{F}$, $\tilde{u_0}$, $\tilde{v}$, respectively, that satisty the conditions

$$\|F - \tilde{F}\|_{C(\bar{\Omega}_T)} \leq \epsilon,$$

$$\|u_0 - \tilde{u}_0\|_{C(\bar{G})} \le \epsilon_0, \ \|v - \tilde{v}\|_{C(S\times[0,T])} \le \epsilon_1.$$

The constant W is determined by the weight function w of the distributed order derivative (4):

$$W := \int_0^1 w(\alpha)\, d\alpha\, > 0.$$

In the proof, the a priory estimate

$$\|u\|_{C(\bar{\Omega}_T)} \le \max\{M_0,\ M_1\} + \frac{M}{W}\frac{T^\alpha}{\Gamma(1+\alpha)} \tag{20}$$

for the norm of a solution u of the problem (1), (8)-(9) plays the main role. In the estimate, the constants are defined as follows:

$$M := \|F\|_{C(\bar{\Omega}_T)},\ M_0 := \|u_0\|_{C(\bar{G})},\ M_1 := \|v\|_{C(S\times[0,T])}. \tag{21}$$

Let us prove the a priory estimate (20). Like in the classical case of the parabolic PDEs, first an auxiliary function f in the form

$$f(x,t) := u(x,t) - \frac{M}{W}\frac{t^\alpha}{\Gamma(1+\alpha)},\ (x,t) \in \bar{\Omega}_T$$

is introduced. If u is a solution of the problem (1), (8)-(9), then it can be easy checked that f is a solution of the same problem with the functions $F_1(x,t) := F(x,t) - M - q(x)\frac{M}{W}\frac{t^\alpha}{\Gamma(1+\alpha)}$, $v_1(x,t) := v(x,t) - \frac{M}{W}\frac{t^\alpha}{\Gamma(1+\alpha)}$ instead of F and v, respectively. The function F_1 satisfies the estimate $F_1(x,t) \le 0,\ (x,t) \in \bar{\Omega}_T$ because $|F(x,t)| \le M$, M being defined as $\|F\|_{C(\bar{\Omega}_T)}$. We can apply now the maximum principle to the solution f and obtain thus the estimate

$$f(x,t) \le \max\{M_0, M_1\},\ (x,t) \in \bar{\Omega}_T, \tag{22}$$

where the constants $M_0,\ M_1$ are defined as in (21). This estimate can be rewritten for the function u in the form

$$u(x,t) = f(x,t) + \frac{M}{W}\frac{t^\alpha}{\Gamma(1+\alpha)} \tag{23}$$

$$\le \max\{M_0, M_1\} + \frac{M}{W}\frac{T^\alpha}{\Gamma(1+\alpha)},\ (x,t) \in \bar{\Omega}_T.$$

The estimate

$$u(x,t) \ge -\max\{M_0, M_1\} - \frac{M}{W}\frac{T^\alpha}{\Gamma(1+\alpha)},\ (x,t) \in \bar{\Omega}_T$$

follows from the minimum principle given in Theorem 2.3 and the same reasoning as the one presented above if the auxiliary function f is defined as

$$f(x,t) := u(x,t) + \frac{M}{W}\frac{t^\alpha}{\Gamma(1+\alpha)},\ (x,t) \in \bar{\Omega}_T.$$

The last two estimates lead to the inequality (20).

We prove now the uniqueness of solution of the problem (1), (8)-(9). The homogeneous problem (1), (8)-(9) with zero initial and boundary conditions, i.e. the problem with the data $F \equiv 0$, $u_0 \equiv 0$, and $v \equiv 0$ has one trivial solution $u(x,t) \equiv 0$, $(x,t) \in \bar{\Omega}_T$ that is unique due to the estimate (20). Because the problem under consideration is a linear one, the uniqueness of solution of the problem (1), (8)-(9) in the general case follows from the uniqueness of the homogeneous problem with with zero initial and boundary conditions.

Finally, the inequality (19) is obtained from the estimate (20) for the function $u-\tilde{u}$ that is a solution of the problem (1), (8)-(9) with the functions $F-\tilde{F}$, $u_0-\tilde{u}_0$, and $v-\tilde{v}$ instead of the functions F, u_0, and v, respectively. ■

4. Existence of the solution

In the previous section, the uniqueness of solution of the problem (1), (8)-(9) was established. The problem of the existence of solution can be handled following the lines of the paper [14], where the case of the generalized time-fractional diffusion equation was considered. In the paper [14], the notions of the generalized and the formal solutions were introduced. The formal solution of the initial-boundary-problem for the generalized time-fractional diffusion equation was constructed by the Fourier method of variables separation. Under some appropriate conditions, the formal solution can be shown to be a generalized solution that can be interpreted as a classical solution under some additional conditions. In this paper, we just present a formal solution of the initial-boundary-problem (8)-(9) for the generalized time-fractional diffusion equation (1) of distributed order. The next steps, namely, to show that the formal solution can be interpreted as a generalized solution and this last one as a classical solution follow the lines presented in [14] and are omitted here.

In the further discussions, we restrict ourselves to the equation (1) with the initial condition (8) and zero boundary condition

$$u\big|_S = 0,\ (x,t) \in S \times [0,T]. \tag{24}$$

The general case can be reduced to this one following the standard technique employed for the linear parabolic PDEs.

The formal solution of the equation (1) satisfying the boundary condition (24) is constructed in the form of the Fourier series

$$u(x,t) = \sum_{i=1}^{\infty} T_i(t)\, X_i(x), \tag{25}$$

where $X_i,\ i = 1, 2, \ldots$ are the eigenfunctions corresponding to the eigenvalues $\lambda_i,\ i = 1, 2, \ldots$ of the eigenvalue problem

$$L_x(X) = \lambda\, X, \tag{26}$$

$$X\big|_S = 0,\ x \in S \tag{27}$$

for the operator L_x. Due to the conditions (3), the operator L_x is a positive definite and self-adjoint linear operator. The theory of the eigenvalue problems for such operators is well-known. In particular, the eigenvalue problem (26) - (27) possesses a counted number of the positive eigenvalues $0 < \lambda_1 \le \lambda_2 \le \ldots$ with the finite multiplicity and - if the boundary S of G is a smooth surface - any function $f \in \mathcal{M}_{\mathcal{L}}$ can be represented through its Fourier series in the form

$$f(x) = \sum_{i=1}^{\infty} (f, X_i)\, X_i(x), \tag{28}$$

where $X_i \in \mathcal{M}_{\mathcal{L}}$ are the eigenfunctions corresponding to the eigenvalues λ_i:

$$L_x(X_i) = \lambda_i\, X_i,\ i = 1, 2, \ldots. \tag{29}$$

By $\mathcal{M}_{\mathcal{L}}$, the space of the functions f that satisfy the boundary condition (27) and the inclusions $f \in C^1(\bar{\Omega}_T) \cap C^2(G),\ L_x(f) \in L^2(G)$ is denoted. Of course, the exact form of the eigenfunctions $X_i,\ i = 1, 2, \ldots$ and the eigenvalues $\lambda_i,\ i = 1, 2, \ldots$ can be determined only in some special cases of the operators L_x and the domain G with the boundary S.

To determine the functions $T_i(t),\ i = 1, 2, \ldots,$ the formal solution (25) is substituted into the equation (1). Supposing that the source function $F(x,t)$ belongs to the space $\mathcal{M}_{\mathcal{L}}$ for any $t \in [0, T]$ and is represented through its Fourier series

$$F(x,t) = \sum_{i=1}^{\infty} f_i(t)\, X_i(x), \tag{30}$$

we get the following fractional differential equation of distributed order for the unknown functions $T_i,\ i = 1, 2, \ldots$:

$$(\mathbb{D}_t^{w(\alpha)} T_i)(t) = \lambda_i T_i(t) + f_i(t),\ i = 1, 2, \ldots. \tag{31}$$

If the function u_0 from the initial condition (8) is from the space $\mathcal{M}_{\mathcal{L}}$, too, and its Fourier series is given by

$$u_0(x) = \sum_{i=1}^{\infty} u_{0i}\, X_i(x), \tag{32}$$

then the representation (25) of the solution u leads to the initial conditions for the functions $T_i,\ i = 1, 2, \ldots$ in the form

$$T_i(0) = u_{0i},\ i = 1, 2, \ldots. \tag{33}$$

To determine the unknown functions $T_i,\ i = 1, 2, \ldots$, the initial-value problem (33) for the fractional differential equation (31) of distributed order has to be solved. In the paper [11], the existence of the solution of such problems was shown. Moreover, some integral representations of the solution as well as its properties including asymptotics are given there, too. The interested reader is referred to [11] for details.

Summarizing, the formal solution of the initial-boundary-problem (8), (24) for the generalized time-fractional diffusion equation (1) of distributed order is given by the representation (25) through the eigenfunctions of the eigenvalue problem (26)-(27) and the solutions of the initial-value problems (33) for the fractional differential equations (31) of distributed order. Under certain conditions and following the method presented in [14], the formal solution (25) can be proved to be a generalized solution of the problem that can be interpreted as the classical solution under some additional conditions. Together with the results presented in Section 3, this means that the initial-boundary-value problem (8), (24) for the time-fractional diffusion equation (1) of distributed order is a well-posed problem in the Hadamard sense.

REMARK 4.1. The theory presented in this paper can be adopted with some small modifications to the case of the infinite domain $\Omega = G \times (0, \infty),\ G \subset I\!\!R^n$, too, following the lines of the classical case of the linear parabolic PDEs.

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