SURVEY PAPER

THE MELLIN INTEGRAL TRANSFORM IN FRACTIONAL CALCULUS

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Dedicated to Professor Francesco Mainardi
on the occasion of his 70th anniversary

Abstract

In Fractional Calculus (FC), the Laplace and the Fourier integral transforms are traditionally employed for solving different problems. In this paper, we demonstrate the role of the Mellin integral transform in FC. We note that the Laplace integral transform, the sin- and cos-Fourier transforms, and the FC operators can all be represented as Mellin convolution type integral transforms. Moreover, the special functions of FC are all particular cases of the Fox $H$-function that is defined as an inverse Mellin transform of a quotient of some products of the Gamma functions.

In this paper, several known and some new applications of the Mellin integral transform to different problems in FC are exemplarily presented. The Mellin integral transform is employed to derive the inversion formulas for the FC operators and to evaluate some FC related integrals and in particular, the Laplace transforms and the sin- and cos-Fourier transforms of some special functions of FC. We show how to use the Mellin integral transform to prove the Post-Widder formula (and to obtain its new modification), to derive some new Leibniz type rules for the FC operators, and to get new completely monotone functions from the known ones.

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Key Words and Phrases: fractional calculus, Mellin transform, Laplace transform, Fourier transform, $H$-function, Mittag-Leffler function, Wright function, Erdélyi-Kober derivatives and integrals, Post-Widder formula, Leibniz rules for FC operators, completely monotone functions

1. Introduction

Like in the classical Calculus and in (integer order) differential equations, the Laplace and the Fourier integral transforms are routinely employed in Fractional Calculus (FC) in general, and especially in fractional order differential equations. In some cases, this is inevitable. Thus the Riesz and the Riesz-Feller fractional derivatives are defined as pseudo-differential operators in terms of the Fourier and the inverse Fourier transforms (see e.g. [6], [9] or [25]). Another prominent example is given by the Laplace transform formulas for the Caputo and the Riemann-Liouville fractional derivatives that are routinely used in solving of fractional differential equations ([24], [25], [31]).

The aim of this paper is to demonstrate the role of the Mellin integral transform in FC and to show that applying the Mellin transform can essentially simplify some of the FC operations and derivations. Until now, the Mellin integral transform was only sporadically employed in the FC publications. We mention here e.g. the papers [11] and [25], where the Mellin integral transform was used to get a representation of the Green function for the space-time fractional diffusion equation in terms of the Mellin-Barnes integrals (Fox $H$-function) and to analyze its properties. In [19], [20], the mixed operators of the Erdélyi-Kober type were shown to be generating operators for the integral transforms of Mellin convolution type. In particular, the hyper-Bessel differential operator is a generating operator for the Obrechkoff transform and for the related generalized Hankel transform, see details in [1], [4], [15, Ch.3], [20], [23], etc. Leibniz type rules for several FC operators were deduced in [34], [36] by applying the technique of the Mellin integral transform. Of course, the Mellin integral transform was employed in connection with the FC special functions, like the Mittag-Leffler and the Wright functions and their generalizations. These functions are particular cases of the Fox $H$-function that can be interpreted as an inverse Mellin transform (see e.g. [15], [16], [18], [20], [22], [25], to refer to only few of many publications). Finally, we mention the book [36], where a theory of the integral transforms of the Mellin convolution type and some of its applications, also in connection with the FC operators, was presented. Some of the examples we deal with in this paper are motivated by [36].

In this paper, we first emphasize that many of the FC operators like the Riemann-Liouville and the Erdélyi-Kober derivatives and integrals can be interpreted as Mellin convolution type integral transforms. Using this