A Relatively Complete Axiomatisation of Projection onto State in the Duration Calculus

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Abstract

We present a complete axiomatisation of the operator of projection onto state in the Duration Calculus (DC) relative to validity in DC without extending constructs. Projection onto state was introduced and studied extensively in our earlier works. We first establish the completeness of a system of axioms and proof rules for the operator relative to validity in the extension of DC by neighbourhood formulas, which express the neighbourhood values of boolean DC state expressions. By establishing a relatively complete axiomatisation for the neighbourhood formulas in DC, we then achieve completeness of our system relative to basic DC.

Keywords: duration calculus, projection, relative completeness.

1 Introduction

The Duration Calculus (DC) was introduced in [ZHR91] as a first order temporal logic for reasoning about real-time systems. DC can be viewed as an extension of the real-time variant of Interval Temporal Logic (ITL, [HMM83, Mos85, CMZ]). DC has been extended by various operators both in order to increase its expressivity and to make specification more convenient and concise. For example, the state-variable-binding quantifier and the least-fixed-point operator, which were added to DC in [Pan95], enabled the straightforward specification of the behaviour of programs with local variables and recursive calls. Validity in DC is undecidable. Decision procedures are known only for subsets of DC. Validity in DC is not even recursively enumerable, and therefore no finitary complete proof system for DC exists. A finitary proof system for DC which is complete with respect to real time relative to the ITL theory of real time was first presented in [HZ92]. An ω-complete proof system for DC with respect to abstract time was first presented in [Gue98]. However, that system contains and infinitary rule.

Projection onto state is regarded as an additional operator in DC, relative to the basic system of DC as known from [ZHR91]. It can be viewed as a real-time counterpart of the discrete-time ITL operator Π, which was introduced in [HMM83]. A family of

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different operators in ITL and DC are known as projection operators too. They have been studied in [Mos86, Mos95, He99a, BT03, Gue04a]. In order to distinguish the operator studied in this paper from those other ones, we call it projection *onto state*.

One application of projection onto state in DC is to facilitate the specification of requirements on collections of interleaving real-time processes. Another one is to formalise the abstraction known as the *true synchrony hypothesis* about real-time systems with digital control. The true synchrony hypothesis is the assumption that digital computation does not take time in this kind of real-time systems. In reality computation does take time. Yet it is difficult to calculate accurately and of negligible size. Taking this time in account is still reasonable in order to keep the causal ordering of computation steps clear. By means of projection onto state requirements on concurrent real-time programs' behaviour which have been formulated without taking computation time into account, and specifications of this behaviour where computation time is explicitly accounted of can be put together in DC formulas [DV99, GDV02]. Projection onto state has also been used to formulate a special form of logical interpolation which describes the possibility of obtaining explicit descriptions of the interaction between the components of a real-time system in [Gue03]. The possibility to write a requirement on a component of a system in a form which accounts of the behaviour of the system only at the times when the component is active facilitates compositionality in specification of systems with *features* by DC [GRS03]. Projection can be used to write requirements in such form, and this way to avoid apparent interactions between features which do not account of really incorrect behaviour, but are detected just because of inflexibly formulated requirements. Details on some uses of projection can be found in our work [GDV02]. In this paper we focus on the axiomatisation of this modality.

In this paper we propose a finitary proof system for the extension of DC by the operator of projection onto state and *neighbourhood formulas*, which is complete relative to validity in DC without extending operators. Neighbourhood formulas have interesting uses in DC of their own (cf. e.g. [ZL94, HZ96, ZH98]). They appear in this paper just because they seem to facilitate our axiomatisation of projection. Projection onto state was introduced to DC in [DV99] and later studied extensively in [GDV02]. Yet none of those previous works dealt with the issue of completeness that we address here. The result presented in this paper subsumes the relative completeness result for a subset of DC with projection onto state from our earlier work [GDV03]. Furthermore, the system in this paper is much simpler and more streamlined than that in [GDV03].

Structure of the paper. We first give brief preliminaries on DC, neighbourhood formulas and projection onto state. Then we present our proof system and demonstrate its relative completeness. To do this, we first discuss the definition of projection, and present and motivate the axioms about projections of atomic formulas. Then we introduce a special form for formulas in DC with neighbourhood formulas and projection where only atomic formulas in a certain form can occur in the scope of projection, and present axioms which allow to demonstrate the equivalence of every formula to one in the special form. Next we show that our axiomatic system is sufficient to derive every valid formula in the special form using premises which are valid in DC with neighbourhood formulas only. Since neighbourhood formulas have an axiomatisation which is complete relative to validity in DC without extending constructs, this entails the completeness of our system relative to basic DC too. Finally we show that our completeness result holds for projection in DC with iteration and the general least-fixed-point operator relative to validity in these extensions of DC without projection.

2 Preliminaries on DC with projection and neighbourhood formulas

2.1 The definition of DC

DC is a classical first order modal logic with one normal binary modality called *chop*. We denote the chop modality by $(.;.)$. The possible worlds in the standard semantics of DC are closed and bounded intervals of real numbers. For this reason DC is also an *interval-based real-time* temporal logic. A comprehensive introduction to DC can be found in recent monograph [ZH04]. Here we only give a brief formal introduction for the sake of self-containedness.

2.1.1 Languages

Along with the customary first order logic symbols, DC vocabularies include *state variables* P, Q, $\ldots$. State variables are used to build *state expressions* S, which have the syntax:

$S ::= \mathbf{0} \mid P \mid S \Rightarrow S$

State expressions S occur in formulas as part of *duration terms* $\int S$. The syntax of DC *terms* t and *formulas* φ extends that of first order logic by duration terms and formulas built using the modality $(.;.)$, respectively:

$t ::= c \mid x \mid \int S \mid f(t, \ldots, t)$

$\varphi ::= \bot \mid R(t, \ldots, t) \mid \varphi \Rightarrow \varphi \mid (\varphi;\varphi) \mid \exists x\varphi$

Here and below x, y, $\ldots$ denote individual variables, c, d, $\ldots$ denote constants, f, g, $\ldots$ denote function symbols, and R, $\ldots$ denote relation symbols. Constant, function and relation symbols can be either *rigid* or *flexible* in DC. The interpretations of rigid symbols are required not to depend on the reference interval. Individual variables are rigid. State variables are flexible. We denote the *arity* of non-logical symbol s by $\#s$. Flexible relation symbols of arity 0 and flexible constant symbols are also called *temporal propositional letters* and *temporal variables*, respectively. The rigid constant 0, the temporal variable ℓ, the rigid binary function symbol $+$, the rigid binary relation symbols $=$ and $\leq$, and an infinite set of individual variables are mandatory in DC vocabularies.

We denote the set of state variables occurring in a DC state expression, term or formula E by $SV(E)$.

2.1.2 Semantics

The model of time in DC is the linearly ordered group of the reals. We denote the set $\{[\tau_1, \tau_2] : \tau_1, \tau_2 \in \mathbf{R}, \tau_1 \leq \tau_2\}$ by $\mathbf{I}$.

Definition 1 A function $f : \mathbf{R} \to \{0, 1\}$ has the *finite variability property* if, given $\tau_1, \tau_2 \in \mathbf{R}$, $\{\tau : f(\tau) = 0$ and $\tau_1 \leq \tau < \tau_2\}$ is either empty, or a finite union of intervals of the kind $[\tau', \tau'')$.

The finite variability property reflects the natural assumption that $\{0, 1\}$-valued signals, which appear in systems modelled by DC, change their values only finitely many times in any given bounded interval of time.

Definition 2 An *interpretation I of a DC language* $\mathbf{L}$ is a function on the vocabulary of $\mathbf{L}$. The types of the values of I for symbols of the various kinds are as follows:

$I(x), I(c) \in \mathbf{R}$	for individual variables x and rigid constants c
$I(c) : \mathbf{I} \to \mathbf{R}$	for flexible constants c
$I(f) : \mathbf{R}^{\#f} \to \mathbf{R}$, $I(R) : \mathbf{R}^{\#R} \to \{0, 1\}$	for rigid function symbols f and relation symbols R
$I(f) : \mathbf{I} \times \mathbf{R}^{\#f} \to \mathbf{R}$, $I(R) : \mathbf{I} \times \mathbf{R}^{\#R} \to \{0, 1\}$	for flexible f, R
$I(P) : \mathbf{R} \to \{0, 1\}$	for state variables P

$I(0)$, $I(+)$, $I(\leq)$, $I(=)$ and $I(\ell)$ should be the corresponding components of the linearly ordered group $\langle \mathbf{R}, 0, +, \leq \rangle$, equality on $\mathbf{R}$ and $\lambda\sigma. \max\sigma - \min\sigma$, respectively. Interpretations of state variables are required to have the finite variability property.

The impossibility to axiomatise DC completely by finitary means can be ascribed to the requirement on the interpretations of state variables to have the finite variability property. This can be seen by comparing the *abstract time* variant of ITL [Dut95], where finite variability is not present, and the abstract time variant of DC [Gue98], where it is. The former system admits complete finitary axiomatisation while the latter does not.

Definition 3 Given an interpretation I, the value $I_\tau(S)$ of state expression S at time $\tau \in \mathbf{R}$ is defined by the clauses:

$$\begin{array}{lcl} I_\tau(\mathbf{0}) & = & 0 \\ I_\tau(P) & = & I(P)(\tau) \\ I_\tau(S_1 \Rightarrow S_2) & = & \max\{1 - I_\tau(S_1), I_\tau(S_2)\} \end{array}$$

The value $I_\sigma(t)$ of a term t at interval $\sigma \in \mathbf{I}$ is defined by the clauses:

$$\begin{array}{lcl} I_\sigma(x) & = & I(x) \\ I_\sigma(c) & = & I(c) \text{ for rigid } c \\ I_\sigma(c) & = & I(c)(\sigma) \text{ for flexible } c \\ I_\sigma(\int S) & = & \int\limits_{\min\sigma}^{\max\sigma} I_\tau(S)d\tau \\ I_\sigma(f(t_1, \ldots, t_{\#f})) & = & I(f)(I_\sigma(t_1), \ldots, I_\sigma(t_{\#f})) \text{ for rigid } f \\ I_\sigma(f(t_1, \ldots, t_{\#f})) & = & I(f)(\sigma, I_\sigma(t_1), \ldots, I_\sigma(t_{\#f})) \text{ for flexible } f \end{array}$$

The modelling relation $\models$ is defined on interpretations I of $\mathbf{L}$, intervals $\sigma \in \mathbf{I}$ and formulas φ from $\mathbf{L}$ by the clauses:

$I, \sigma \not\models \bot$	
$I, \sigma \models R(t_1, \ldots, t_{\#R})$	iff $I(R)(I_\sigma(t_1), \ldots, I_\sigma(t_{\#R})) = 1$ for rigid R
$I, \sigma \models R(t_1, \ldots, t_{\#R})$	iff $I(R)(\sigma, I_\sigma(t_1), \ldots, I_\sigma(t_{\#R})) = 1$ for flexible R
$I, \sigma \models \varphi \Rightarrow \psi$	iff either $I, \sigma \models \psi$ or $I, \sigma \not\models \varphi$
$I, \sigma \models (\varphi; \psi)$	iff $I, \sigma_1 \models \varphi$ and $I, \sigma_2 \models \psi$ for some $\sigma_1, \sigma_2 \in \mathbf{I}$ such that $\sigma = \sigma_1 \cup \sigma_2$ and $\min \sigma_2 = \max \sigma_1$.
$I, \sigma \models \exists x \varphi$	iff $J, \sigma \models \varphi$ for some J which is a x-variant of I

2.1.3 Abbreviations and precedence of the operators

The symbols $\top$, $\neg$, $\vee$, $\wedge$, $\Leftrightarrow$, $\forall$, $\neq$, $\geq$, $<$ and $>$ are used as abbreviations in the usual way in formulas. Infix notation is used wherever $+$, $=$ and $\leq$ occur. $\forall\varphi$ denotes the *universal closure* of a formula φ, that is $\forall x_1 \ldots \forall x_n \varphi$, where $x_1, \ldots, x_n$ are all the free individual variables of φ. The connectives $\neg$, $\vee$, $\wedge$ and $\Leftrightarrow$ are used as abbreviations in state expressions too. The following abbreviations are specific to DC:

$$\Diamond\varphi \rightleftharpoons ((\top;\varphi);\top)\,, \qquad \Box\varphi \rightleftharpoons \neg\Diamond\neg\varphi\,,$$
$$(\varphi_1;\varphi_2;\ldots;\varphi_n) \rightleftharpoons (\varphi_1;\ldots;(\varphi_{n-1};\varphi_n)\ldots)\,,$$
$$\mathbf{1} \rightleftharpoons \mathbf{0} \Rightarrow \mathbf{0}\,, \qquad \lceil S \rceil \rightleftharpoons \textstyle\int S = \ell \wedge \ell \neq 0\,.$$

We assume the usual precedence conventions about the propositional connectives, $\exists$ and $\forall$. We always write parentheses when using the chop modality $(.;.)$. We assign $(.;.)$ the lowest precedence. For example, $(A \wedge B; C \Leftrightarrow D)$ is the same as $((A \wedge B); (C \Leftrightarrow D))$, and $A \wedge B; C \Leftrightarrow D$ is not well-formed in our setting, because the parentheses for ; are missing.

2.1.4 Completeness of finitary proof systems for DC can only be relative

A finitary proof system for DC and, consequently, for its extensions, can be no more than relatively complete. The impossibility to have a complete finitary axiomatisation for DC follows from the assumption that a finitary proof system is supposed to define a decidable notion of proof - whether a sequence of formulas is a valid proof can be checked mechanically. This entails that the corresponding notion of provability, that is, the existence of a valid proof for a given formula, can be no worse than semi-decidable. On the other hand, validity in DC is not semi-decidable. This is already so in the rather restricted subset of DC whose syntax is

$$\varphi ::= \bot \mid \lceil S \rceil \mid R \mid \varphi \Rightarrow \varphi \mid (\varphi;\varphi)$$

where R stands for a temporal propositional letter [Gue04b]. That is why relative completeness results like the one in this work are the best that can be obtained with respect to the scope of completeness in this setting.

2.2 Projection onto state

Given a state expression H and a formula φ, the *projection of* φ *onto* H is a new formula denoted by (φ/H). Roughly speaking, (φ/H) holds at interval σ under interpretation I, if φ holds at the interval obtained from σ by cutting off its subintervals where H evaluates to 0 under an interpretation which preserves the (truth) values of non-logical symbols in this remaining (shortened) interval as best as possible. The auxiliary notation below is to make this precise.

Let $h : \mathbf{R} \to \{0, 1\}$ have the finite variability property. Let $\delta_h(\tau) = \int\limits_0^\tau h(\tau')d\tau'$. Let $\Sigma_h = \{\delta_h(\tau) : \tau \in \mathbf{R}\}$. Clearly Σ_h is either a closed interval, or a semi-closed unbounded interval, or the entire $\mathbf{R}$, and $0 \in \Sigma_h$. The function δ_h "glues" the collection of intervals $\{\tau \in \mathbf{R} : h(\tau) = 1\}$ into the single interval Σ_h. To transfer arbitrary interpretations from $\mathbf{R}$ to Σ_h as embedded in $\mathbf{R}$, we need to invert δ_h. The *multiple-valued* inverse of δ_h is defined by the equality

$$\delta_h^{-1}(\tau') = \{\tau \in \mathbf{R} : \delta_h(\tau) = \tau'\}.$$

We need a monotonic extension to $\mathbf{R}$ of a single-valued branch of δ_h^{-1}, that is, a monotonic function γ_h of type $\mathbf{R} \to \mathbf{R}$ such that if $\delta_h^{-1}(\tau') \neq \emptyset$, then $\gamma_h(\tau') \in \delta_h^{-1}(\tau')$. The extension with this property that we choose to employ can be defined as follows:

$$\gamma_h(\tau') = \begin{cases} \tau' - \inf\Sigma_h + \max\delta_h^{-1}(\inf\Sigma_h) & \text{if } \tau' < \inf\Sigma_h \leq \sup\Sigma_h; \\ \max\delta_h^{-1}(\tau') & \text{if } \inf\Sigma \leq \tau' < \sup\Sigma_h; \\ \tau' - \sup\Sigma_h + \min\delta_h^{-1}(\sup\Sigma_h) & \text{if } \inf\Sigma_h \leq \sup\Sigma_h \leq \tau'. \end{cases}$$

Note that the cases in the definition of γ_h depend on the kind of interval Σ_h is and not just on τ'. The idea is that $\gamma_h(\tau')$ is the maximal value of $\delta_h^{-1}(\tau')$ when $\delta_h^{-1}(\tau')$ is not empty, except for the case $\tau' = \sup\Sigma_h$. Otherwise, $\gamma_h(\tau')$ is defined to preserve the distance from τ' to Σ_h, i.e $\inf\Sigma_h - \tau' = \gamma_h(\inf\Sigma_h) - \gamma_h(\tau')$ in case $\tau' < \inf\Sigma_h$, and $\tau' - \sup\Sigma_h = \gamma_h(\tau') - \gamma_h(\sup\Sigma_h)$ in case $\tau' \geq \sup\Sigma_h$.

Definition 4 Given an interpretation I of some DC language $\mathbf{L}$, the *projection of I onto (the support of) h* is the DC interpretation I^h of $\mathbf{L}$ which is defined by the equalities:

$I^h(s)$	$=$	$I(s)$	for rigid s, including individual variables;
$I^h(c)(\sigma)$	$=$	$I(c)([\gamma_h(\min\sigma), \gamma_h(\max\sigma)])$	for flexible constants $c \not\equiv \ell$;
$I^h(s)(\sigma, d_1, \ldots, d_{\#s})$	$=$	$I(s)([\gamma_h(\min\sigma), \gamma_h(\max\sigma)], d_1, \ldots, d_{\#s})$	for flexible function and relation symbols s;
$I^h(P)(\tau)$	$=$	$I(P)(\gamma_h(\tau))$	for state variables P.

Given $\sigma \in \mathbf{I}$, the *projection σ^h of σ onto (the support of) h* is $[\delta_h(\min\sigma), \delta_h(\max\sigma)]$.

With γ_h defined and used as above, I^h is obtained from I by clipping off parts of $\mathbf{R}$ which are surrounded by parts where h evaluates to 1 only. In words, I^h interprets a symbol s at interval σ' in the way in which I interprets s at the corresponding interval $[\gamma_h(\min\sigma'), \gamma_h(\max\sigma')]$. In case Σ_h is (semi)bounded, that is, if $\inf\Sigma_h > -\infty$, or $\sup\Sigma_h < \infty$, or both, the values of I on $(-\infty, \gamma_h(\inf\Sigma_h))$ and $[\gamma_h(\sup\Sigma_h), \infty)$ are transferred to I^h with no loss.

Definition 5 Let φ be a formula and H be a state expression in $\mathbf{L}$, respectively. Let $h = \lambda\tau.I_\tau(H)$. Then

$$I, \sigma \models (\varphi/H) \text{ iff } I^h, \sigma^h \models \varphi$$

Just like $(.;.)$, we always write projection with parentheses.

2.3 Neighbourhood formulas

Given a state expression S, the formulas $\overleftarrow{S}$ and $\overrightarrow{S}$ are called *left neighbourhood* and *right neighbourhood* of S, respectively. Neighbourhood formulas and neighbourhood terms have been studied in numerous works on DC [ZL94, HZ96, ZH98, He99b, ZGZ00, Zha00]. The relation $\models$ is defined on neighbourhood formulas by the clauses:

$$I, \sigma \models \overleftarrow{S} \text{ iff } I, [\tau, \min \sigma] \models \lceil S \rceil \text{ for some } \tau < \min \sigma$$

$$I, \sigma \models \overrightarrow{S} \text{ iff } I, [\max \sigma, \tau] \models \lceil S \rceil \text{ for some } \tau > \max \sigma$$

Consider the axioms

$$\overrightarrow{\mathbf{0}} \Leftrightarrow \bot \tag{1}$$

$$\overrightarrow{S_1 \Rightarrow S_2} \Leftrightarrow (\overrightarrow{S_1} \Rightarrow \overrightarrow{S_2}) \tag{2}$$

$$\overrightarrow{S} \Leftrightarrow (\top; \overrightarrow{S}) \tag{3}$$

$$\neg(\overrightarrow{S}; \lceil \neg S \rceil) \tag{4}$$

Theorem 6 *The axioms (1)-(4) are complete for right-neighbourhood formulas in DC.*

We formulate this result here and prove it in Appendix A for the sake of self-containedness. The axiomatisations of neighbourhood formulas and terms which are available from the literature apply to slightly different settings. Together with their left-neighbourhood mirror images, the axioms (1)-(4) are relatively complete for both left- and right-neighbourhood formulas in DC.

3 Relative completeness of DC with projection and neighbourhood formulas

In this section we obtain the relative completeness of a proof system for DC with projection onto state with respect to real time, which is the main result of this paper. The proof system we present is complete relative to validity in basic DC. We obtain completeness relative to DC with neighbourhood formulas first. We explain how the dependency on neighbourhood formulas can be eliminated in Section 4.

Originally, $(./.)$ was defined on the subset of DC where the only flexible symbols are state variables and ℓ, which can be regarded as an abbreviation for $\int \mathbf{1}$. Projection was extended to the entire DC, because of the convenience of using other flexible symbols to write abstract specifications. For instance, propositional temporal letters appear in the extension of DC by a least-fixed-point operator [Pan95]. As we mention in the introduction, a subset of DC with both $(./.)$ and least fixed points was studied in [GDV02], but no completeness result was given there. The relative completeness

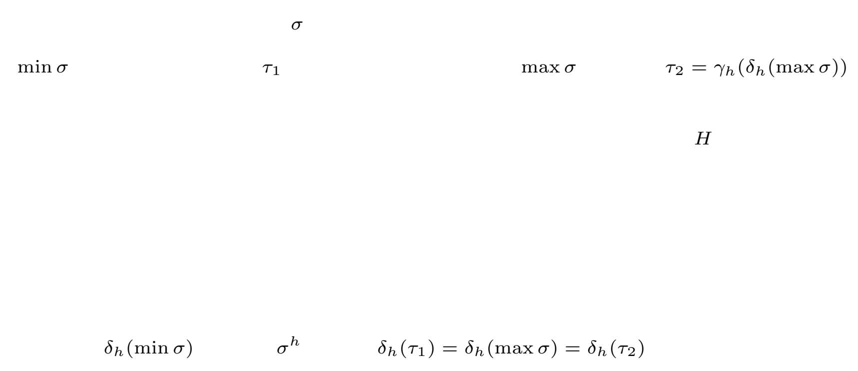

Figure 1: Projection onto H at σ depends on the interpretation of symbols outside σ.

result from [GDV03] applies only to a subset of the language, and the system involved is much more complex than the one in this paper.

We first discuss the definition of projection for DC with arbitrary non-logical symbols and motivate some of the new axioms which are needed in order to cope with it for the case of projections of atomic formulas. Then we present the rest of the proof system and prove its relative completeness.

The relative completeness proof goes through establishing that the axioms allow to derive the equivalence between an arbitrary DC formula and a corresponding formula in an appropriate special form. Valid formulas in this special form contain only certain atomic formulas in the scope of $(./.)$. Using the new axioms, all valid formulas of the special form can be derived in DC without $(./.)$.

3.1 On the definition of projection onto state

The role of the function γ_h in the definition of $(./.)$ for DC interpretations of languages with arbitrary flexible constant, function and relation symbols is particularly important. The main property of γ_h is that it is an inverse to the function δ_h, which maps $\mathbf{R}$ onto the time domain where the projected interpretation is defined. Intervals in $\mathbf{R}$ where the value of h is 0 are mapped by δ_h to single time points. There is a variety of possibilities for inverting δ_h at such time points. The exact choice is irrelevant in the case of state variables, because the finite variability of h implies that a projected interval σ^h can contain at most finitely many such time points, and therefore the values of state expressions at these points do not affect the values of duration terms under projected interpretations. Yet this is no longer so as soon as other types of flexible non-logical symbols get involved. One inevitable consequence of any possible choice of an inverse to δ_h is that the evaluation of projection at some intervals σ can depend on the interpretations of symbols outside σ. Let I be a DC interpretation, R be a propositional temporal letter, H be a state expression and $h = \lambda\tau.I_\tau(H)$. Let $\sigma \in \mathbf{I}$, $\tau_1, \tau_2 \in \mathbf{R}$, $\min \sigma < \tau_1 < \max \sigma < \tau_2$ and $h(\tau) = 1$ iff $\tau \in [\min \sigma, \tau_1) \cup [\tau_2, \infty)$. Then $\gamma_h(\delta_h(\min \sigma)) = \min \sigma$ and $\gamma_h(\delta_h(\max \sigma)) = \sup \delta_h^{-1}(\delta_h(\max \sigma)) = \sup[\tau_1, \tau_2] =$

τ_2 (see Figure 3.1). Hence $I^h, \sigma^h \models (R/H)$ if and only if $I, [\min\sigma, \tau_2] \models R$. In this case the restriction of $I(R)$ to the subintervals of σ is not sufficient to determine whether $I, \sigma \models (R/H)$ holds, because $\gamma_h(\delta_h(\max\sigma)) > \max\sigma$. Changing the definition of γ_h to map $\delta_h(\max\sigma)$ to $\min\delta_h^{-1}(\delta_h(\max\sigma)) = \tau_1$ would bring a similar inadequacy for intervals *starting* at points $\tau \in (\tau_1, \tau_2)$.

3.2 Bringing arbitrary formulas to special forms in DC with projection onto state

The axioms involving $(./.)$ that we present next make it possible to derive the equivalence between an arbitrary formula in the extension of DC by $(./.)$ and neighbourhood formulas and a corresponding formula in which only atomic formulas appear in the scope of $(./.)$:

$$\varphi \Leftrightarrow (\varphi/H) \text{ for rigid } \varphi \tag{5}$$

$$(\varphi \Rightarrow \psi/H) \Leftrightarrow (\varphi/H) \Rightarrow (\psi/H) \tag{6}$$

$$((\varphi;\psi)/H) \Leftrightarrow ((\varphi/H);(\psi/H)) \tag{7}$$

$$(\exists x\varphi/H) \Leftrightarrow \exists x(\varphi/H) \tag{8}$$

$$((\varphi/S)/H) \Leftrightarrow (\varphi/S \wedge H) \tag{9}$$

The correctness of the axioms (5)-(9) can be established by a direct check.

Proposition 7 *Let φ be a formula in some language for DC with $(./.)$ and neighbourhood formulas. Then there exists a formula ψ in the same language which contains only atomic formulas in the scope of $(./.)$ and is such that the equivalence $\varphi \Leftrightarrow \psi$ can be derived in DC using the axioms (5)-(9). Furthermore, using the axioms (1)-(3), (5) and (6), it can be achieved that only state variables occur as the state expressions in the neighbourhood subformulas of ψ.*

Proof: Induction on the construction of φ. $\dashv$

The axioms (5)-(9) were first introduced in [DV99]. Proposition 7 entails the expressibility of $(./.)$ in DC languages where state variables are the only flexible non-logical symbols:

Corollary 8 ([DV99]) *Let φ be a formula in some language for DC with $(./.)$. Then there exists a projection-free formula ψ in the same language such that the equivalence $\varphi \Leftrightarrow \psi$ can be derived in DC using the axioms (5)-(9) and (13).*

Proof: Induction on the construction of φ, using the equivalences (14) and (15). $\dashv$

The special form established in Proposition 7 is basically sufficient for us to carry out our relative completeness argument. However some further specialisations are possible with respect to the occurrences of neighbourhood formulas and their projections:

Proposition 9 ([GDV03]) *Let φ be a formula in some language for DC with $(./.)$ and neighbourhood formulas. Let only atomic subformulas occur in the scope of $(./.)$ in φ. Then there exists a boolean combination ψ of neighbourhood formulas, formulas of the forms $(\overleftarrow{S}/H)$ and $(\overrightarrow{S}/H)$, and formulas which contain no neighbourhood subformulas in the same language, such that $\models \varphi \Leftrightarrow \psi$.*

The proof of this proposition can be found in [GDV03], where there are also axioms which make the considered equivalence derivable.

3.3 Axioms about projection onto state of atomic formulas

Let I, H and h be like above. Let there exist at least one $\tau \in \mathbf{R}$ such that $h(\tau) = 1$. Then the time points which participate in the definition of projected interpretations I^h at projected intervals σ^h are the ones at which h evaluates to 1, and the time point $\sup\{\tau \in \mathbf{R} : h(\tau) = 1\}$, in case $\sup \Sigma_h$ is finite. In the sequel, we call these time points *definitive*. Other time points are involved in the definition of I^h too, but never affect the values of I^h *within projections* σ^h of intervals $\sigma \in \mathbf{I}$. We call $\sigma \in \mathbf{I}$ definitive, if both $\min \sigma$ and $\max \sigma$ are definitive.

$I_{\max \sigma}(H) = 1$ if and only if $I, \sigma \models \overrightarrow{H}$, because of the form of the finite variability property we have adopted (see Definition 1). A direct check shows that $\max \sigma = \sup\{\tau \in \mathbf{R} : h(\tau) = 1\}$ is equivalent to $I, \sigma \models (\overrightarrow{\neg H}/H) \wedge (\top; \ell = 0 \wedge \overleftarrow{H})$. Let

$$H! \rightleftharpoons \overrightarrow{H} \vee ((\overrightarrow{\neg H}/H) \wedge (\top; \ell = 0 \wedge \overleftarrow{H}))$$

Then $\max \sigma$ is definitive for projection onto H if and only if $I, \sigma \models H!$. To determine whether the beginning point of an interval is definitive for projection onto H, we can use formulas like $(\ell = 0 \wedge H!; \top)$. A direct check shows that $h(\tau) = 0$ for all $\tau \in \mathbf{R}$ if and only if

$$I, \sigma \models \textstyle\int H = 0 \wedge (\overleftarrow{\neg H}/H) \wedge (\overrightarrow{\neg H}/H)$$

at some, and, consequently, at all $\sigma \in \mathbf{I}$. In this case σ^h is the interval $[0, 0]$ for all $\sigma \in \mathbf{I}$ and I^h is defined on σ^h using only the restriction of I to $[0, 0]$.

Here follow some axioms which, according to the above observations, can be used to determine the truth values of projection formulas:

$$(\varphi/H) \Leftrightarrow (\textstyle\int H = 0; (\varphi/H); \int H = 0) \tag{10}$$

Axiom (10) states that changing the end points of a reference interval does not affect the truth value of projection formulas as long as there are no definitive points between the pairs of corresponding end points. Let ε denote either $\neg$ or nothing. Then the axioms

$$(\ell = 0 \wedge H!; (\varepsilon R(x_1, \ldots, x_n)/H) \wedge H!) \Rightarrow \varepsilon R(x_1, \ldots, x_n) \tag{11}$$

and

$$(\ell = 0 \wedge H!; (y = f(x_1, \ldots, x_n)/H) \wedge H!) \Rightarrow y = f(x_1, \ldots, x_n) \tag{12}$$

state that if both the beginning point and the end point of an interval are definitive, then the projection of a flexible atomic formula at this interval is equivalent to that atomic formula itself. The corresponding axiom about atomic formulas with duration terms, on the other hand, does not involve definitiveness of endpoints:

$$(y = \int S/H) \Leftrightarrow y = \int(S \wedge H) \tag{13}$$

Note the special form of the atomic formulas which appear in (11) and (12). Every formula has an equivalent one where all the atomic formulas built using relation symbols have this form, because of the predicate logic equivalences

$$R(t_1, \ldots, t_n) \Leftrightarrow \exists x_1 \ldots \exists x_n \left(R(x_1, \ldots, x_n) \wedge \bigwedge_{i=1}^{n} x_i = t_i \right) \tag{14}$$

and

$$y = f(t_1, \ldots, t_n) \Leftrightarrow \exists x_1 \ldots \exists x_n \left(y = f(x_1, \ldots, x_n) \wedge \bigwedge_{i=1}^{n} x_i = t_i \right), \tag{15}$$

where $x_1, \ldots, x_n \notin FV(t_1), \ldots, FV(t_n)$. The corresponding axioms about projections of neighbourhood formulas are as follows:

$$\overleftarrow{S \wedge H} \Rightarrow (\overleftarrow{S}/H) \tag{16}$$

$$H! \Rightarrow (\overrightarrow{S} \Leftrightarrow (\overrightarrow{S}/H)) \tag{17}$$

Note that the axiom (16) about projections of left neighbourhood formulas does not explicitly refer to the definitiveness of the beginning point of the reference interval. Projections of neighbourhood formulas satisfy the following axioms and their mirror images too:

$$\vdash H_1 \Rightarrow H_2 \text{ implies } (\overrightarrow{H_1}/H_2) \Rightarrow (\overrightarrow{H_1}/H_1) \tag{18}$$

$$\vdash H_1 \Rightarrow H_2 \text{ implies } (\overrightarrow{H_1}/H_1) \Rightarrow (\overrightarrow{H_2}/H_2) \tag{19}$$

$$(\overrightarrow{S}/H_1 \vee H_2) \Rightarrow (\overrightarrow{S}/H_1) \vee (\overrightarrow{S}/H_2) \tag{20}$$

$$(\overrightarrow{H_1}/H_1 \vee H_2) \wedge (\overrightarrow{H_2}/H_2 \vee H_3) \Rightarrow (\overrightarrow{H_1}/H_1 \vee H_3) \tag{21}$$

$$\ell = 0 \wedge (\overleftarrow{H_1 \wedge \neg H_2} \wedge \overrightarrow{H_2}/H_2) \Rightarrow (\overrightarrow{H_1 \wedge \neg H_2}/H_1 \vee H_2) \vee \overrightarrow{H_2} \tag{22}$$

Now let us return to the case in which the beginning and the end points of a reference interval σ are not known to be definitive. If $\min \sigma$ is not definitive, but $\max \sigma$ is, then there exists a subinterval σ' of σ such that $\max \sigma' = \max \sigma$ and $\min \sigma'$ is the closest definitive time point on the right of $\min \sigma$. In this case the truth value of a projection formula can be determined using σ' and axiom (10). The case when $\max \sigma$ is not definitive is more subtle. Then the truth value of projection formulas (α/H) at σ depends on the truth value of α at some interval σ' whose endpoint is on the right of $\max \sigma$. All that can be said here is that the endpoint $\max \sigma'$ of this interval σ' is *the same* for all projection formulas (α/H') onto states H' which are related to H in a certain way. Here follows a detailed explanation.

Let the state variables occurring in all the considered formulas be $P_1, \ldots, P_n$. Then, given the interpretation I and the reference interval σ, because of the finite variability, we can define a (possibly infinite) ascending sequence of time points $\tau_0 = \max\sigma$, τ_1, $\tau_2, \ldots$ such that the interpretations of all the state variables $P_1, \ldots, P_n$ are constant in the intervals of the form $[\tau_i, \tau_{i+1})$ and at least one of these variables changes its value at τ_i for every $i = 1, 2, \ldots$. Let $\tau_0' < \min\sigma$, τ_1', τ_2', ..., be a descending sequence of time points on the left of σ with the same property. Note that H is an expression built from P_1, ..., P_n. Let there exist an i such that $I_{\tau_i}(H) = 1$. A direct check shows that this is equivalent to $I, \sigma \models (\overrightarrow{H}/H)$. In this case the truth value of the projection (α/H) of an atomic formula α onto state expression H at σ depends on the truth value of α at an interval which ends at some of the time points τ_i, $i = 0, 1, \ldots$. In case $\max\sigma$ is definitive for projections onto H, this interval ends at τ_0, which is $\max\sigma$ itself. In general, the interval in question ends at τ_i where i is the smallest number such that $I, [\tau_i, \tau_{i+1}] \models \lceil H\rceil$. In case $I_\tau(H) = 0$ for all $\tau \geq \max\sigma$, we have $I, \sigma \models \neg(\overrightarrow{H}/H)$ and either $I_\tau(H) = 0$ for all $\tau \in \mathbf{R}$, or the truth value of (α/H) at σ is equal to the truth value of α at an interval which ends (and possibly also begins) at $\tau' = \gamma_h(\sup\Sigma_h)$. Furthermore, in this case either $\tau' \in [\min\sigma, \max\sigma)$ or $\sigma = \tau_i'$ for some i.

Now consider the projections (α/H_1) and (α/H_2) of the same atomic formula α onto two state expressions H_1 and H_2. Let $h_j = \lambda\tau.I_\tau(H_j)$, $j = 1, 2$. In case the definitive intervals

$$[\gamma_{h_1}(\delta_{h_1}(\min\sigma)), \gamma_{h_1}(\delta_{h_1}(\max\sigma))]$$

and

$$[\gamma_{h_2}(\delta_{h_2}(\min\sigma)), \gamma_{h_2}(\delta_{h_2}(\max\sigma))] \tag{23}$$

for these projections are the same, we need to be able to establish that

$$I, \sigma \models (\alpha/H_1) \Leftrightarrow (\alpha/H_2). \tag{24}$$

For this purpose we introduce a formula which holds at σ if and only if either $h_1, h_2 \neq \lambda\tau.0$ and $\gamma_{h_1}(\delta_{h_1}(\max\sigma)) = \gamma_{h_2}(\delta_{h_2}(\max\sigma))$, or $h_1 = h_2 = \lambda\tau.0$. We denote this formula by $H_1 \equiv H_2$. To define $\equiv$, we first introduce one more auxiliary binary connective, which we denote by $\leq$. $H_1 \leq H_2$ holds at σ if and only if either $h_1 = h_2 = \lambda\tau.0$, or $\gamma_{h_1}(\delta_{h_1}(\max\sigma)) = \gamma_{\max(h_1,h_2)}(\delta_{\max(h_1,h_2)}(\max\sigma))$ (where $\max(h_1, h_2)(t) = \max(h_1(t), h_2(t))$). $H_1 \leq H_2$ is defined as the formula

$$\left(\begin{array}{l} (\overrightarrow{H_1}/H_1 \vee H_2) \vee (\neg(\overrightarrow{H_1 \vee H_2}/H_1 \vee H_2) \wedge (\top; \ell = 0 \wedge (\overleftarrow{H_1}/H_1 \vee H_2))) \vee \\ (\neg(\overrightarrow{H_1 \vee H_2}/H_1 \vee H_2) \wedge \neg(\overleftarrow{H_1 \vee H_2}/H_1 \vee H_2) \wedge \int(H_1 \vee H_2) = 0)) \end{array}\right).$$

The definition of $\equiv$ is

$$H_1 \equiv H_2 \rightleftharpoons H_1 \leq H_2 \wedge H_2 \leq H_1.$$

A direct check shows that $H_1 \equiv H_2$ and $H_1 \leq H_2$ really express the conditions on h_1 and h_2 formulated above. Defining $\equiv$ in terms of $\leq$ is technically convenient for the proof of Lemma 14. Two more derived operators are involved in the proof of that lemma. They are defined by the clauses

$$H_1 \preceq_l H_2 \rightleftharpoons (\overleftarrow{H_1}/H_1 \vee H_2) \text{ and } H_1 \preceq_r H_2 \rightleftharpoons (\overrightarrow{H_1}/H_1 \vee H_2).$$

These operators can be defined semantically too:

$$I, \sigma \models H_1 \preceq_l H_2 \text{ iff } I, [\tau', \tau''] \models \lceil H_1 \rceil \text{ and } I, [\tau'', \min \sigma] \models \textstyle\int (H_1 \vee H_2) = 0$$

for some τ', τ'' such that $\tau' < \tau'' \leq \min \sigma$, and

$$I, \sigma \models H_1 \preceq_r H_2 \text{ iff } I, [\tau', \tau''] \models \lceil H_1 \rceil \text{ and } I, [\max \sigma, \tau'] \models \textstyle\int (H_1 \vee H_2) = 0$$

for some τ', τ'' such that $\max \sigma \leq \tau' < \tau'' \leq \min \sigma$.

Using $\preceq_l$ and $\preceq_r$, $H_1 \leq H_2$ can be written as

$$\begin{array}{c} H_1 \preceq_r H_2 \vee (\neg(\overrightarrow{H_1 \vee H_2}/H_1 \vee H_2) \wedge (\top; \ell = 0 \wedge H_1 \preceq_l H_2)) \vee \\ (\neg(\overrightarrow{H_1 \vee H_2}/H_1 \vee H_2) \wedge \neg(\overleftarrow{H_1 \vee H_2}/H_1 \vee H_2) \wedge \textstyle\int (H_1 \vee H_2) = 0)) \end{array}$$

The properties of $\preceq_l$, $\preceq_r$ and $\leq$ to be used in the proof of Lemma 14 are listed in the two lemmata below. Their proofs are given in Appendix B.

Lemma 10 *The following formulas are provable using our axioms and rules about* $(./.)$*:*

$$(\overleftarrow{H}/H) \Rightarrow H \preceq_l H, \qquad (\overrightarrow{H}/H) \Rightarrow H \preceq_r H \tag{25}$$

$$H_1 \preceq_l H_2 \wedge H_2 \preceq_l H_3 \Rightarrow H_1 \preceq_l H_3 \;\; \textit{for} \;\; \preceq \in \{\preceq_l, \preceq_r\} \tag{26}$$

$$\vdash \neg(H_1 \wedge H_2) \;\textit{implies}\; \neg(H_1 \preceq H_2 \wedge H_2 \preceq H_1) \;\; \textit{for} \;\; \preceq \in \{\preceq_l, \preceq_r\} \tag{27}$$

$$(\overleftarrow{H_1}/H_1) \Rightarrow H_1 \preceq_l H_2 \vee H_2 \preceq_l H_1, \tag{28}$$

$$(\overrightarrow{H_1}/H_1) \Rightarrow H_1 \preceq_r H_2 \vee H_2 \preceq_r H_1 \tag{29}$$

$$H_1 \preceq_r H_2 \Rightarrow H_1 \leq H_2 \tag{30}$$

$$\textstyle\int (H_1 \vee H_2) = 0 \wedge \neg(\overrightarrow{H_1 \vee H_2}/H_1 \vee H_2) \Rightarrow (H_1 \preceq_l H_2 \Rightarrow H_1 \leq H_2) \tag{31}$$

Lemma 11 *The following formulas are provable using our axioms and rules about* $(./.)$*:*

$$\vdash H_2 \Rightarrow H_1 \;\textit{implies}\; H_1 \leq H_2 \tag{32}$$

$$H \equiv H \tag{33}$$

$$H_1 \leq H_2 \wedge H_2 \leq H_3 \Rightarrow H_1 \leq H_3 \tag{34}$$

$$S \equiv (H_1 \vee H_2) \Rightarrow S \equiv H_1 \vee S \equiv H_2 \tag{35}$$

$$H_1 \leq H_2 \Rightarrow (H_1 \vee H_2) \equiv H_1 \tag{36}$$

$$H_1 \leq H_2 \vee H_2 \leq H_1 \tag{37}$$

$$H_1 \leq H_2 \wedge \neg(\overrightarrow{H_1}/H_1) \Rightarrow \neg(\overrightarrow{H_2}/H_2) \tag{38}$$

$$H_1 \leq H_2 \wedge \neg(\overrightarrow{H_1}/H_1) \Rightarrow \neg(\top; \lceil H_2 \wedge \neg H_1 \rceil; \textstyle\int(H_1 \vee H_2) = 0) \tag{39}$$

$$H_1 \leq H_2 \wedge \neg(\overrightarrow{H_1}/H_1) \Rightarrow (\textstyle\int H_1 = 0 \Rightarrow \int H_2 = 0) \tag{40}$$

$$H_1 \leq H_2 \wedge \neg(\overrightarrow{H_1}/H_1) \wedge \textstyle\int H_1 = 0 \Rightarrow ((\overleftarrow{H_2}/H_2) \Rightarrow (\overleftarrow{H_1}/H_1)) \tag{41}$$

$H_1 \equiv H_2$ holds at σ iff the *end* points of the intervals in (23) are the same. The equality of the *beginning* points of these intervals is equivalent to the satisfaction of $H_1 \equiv H_2$ at $[\min \sigma, \min \sigma]$. This means that we can formulate the following axiom about equivalences between projections of formulas α of the forms $R(x_1, \ldots, x_{\#R})$, $y = f(x_1, \ldots, x_{\#f})$ and $y = c$, where c is not ℓ:

$$(H_1 \equiv H_2 \wedge \ell = 0; H_1 \equiv H_2) \Rightarrow ((\alpha/H_1) \Leftrightarrow (\alpha/H_2)) \tag{42}$$

The axioms below and their mirror images apply to projections of neighbourhood formulas:

$$(\overrightarrow{S}/H) \Leftrightarrow (\top; (\overrightarrow{S}/H)) \tag{43}$$

$$H_1 \equiv H_2 \Rightarrow ((\overrightarrow{S}/H_1) \Leftrightarrow (\overrightarrow{S}/H_2)) \tag{44}$$

To enable the replacement of equivalents in the scope of $(./.)$, we introduce the rule

$$\vdash \varphi \Rightarrow \psi \text{ and } \vdash H_1 \Leftrightarrow H_2 \text{ imply } \vdash (\varphi/H_1) \Rightarrow (\psi/H_2) \tag{45}$$

Using (42)-(44) and (45), one can easily derive

$$\textstyle\int H = 0 \wedge (\overleftarrow{\neg H}/H) \wedge (\overrightarrow{\neg H}/H) \Rightarrow ((\alpha/H) \Leftrightarrow (\alpha/\mathbf{0})) \tag{46}$$

for all atomic α.

$\Sigma_{\lambda\tau.I_\tau(\mathbf{0})}$ is the singleton interval $[0,0]$ and $\gamma_{\lambda\tau.I_\tau(\mathbf{0})}$ maps it onto $[0,0]$. $I^{\lambda\tau.I_\tau(\mathbf{0})}$ is equal to I, the only difference is that only a restriction of $I^{\lambda\tau.I_\tau(\mathbf{0})}$ to a neighbourhood of $[0,0]$ is ever referred to in determining the semantics of a formula, and it is involved only in determining the truth values of projections onto $\mathbf{0}$. Hence $I, \sigma \models (\varphi/\mathbf{0})$ iff $I, [0,0] \models \varphi$ for *all* σ. This makes formulas of the form $(\varphi/\mathbf{0})$ behave like rigid formulas. The following axioms reflect this:

$$((\varphi/\mathbf{0}); \psi) \Rightarrow (\varphi/\mathbf{0}),\ (\psi; (\varphi/\mathbf{0})) \Rightarrow (\varphi/\mathbf{0}) \tag{47}$$

The axiom below describes the effects of $[0,0]$ being located either on the left, or on the right, or inside the reference interval:

$$(\overleftarrow{S_l} \wedge \overrightarrow{S_r}/\mathbf{0}) \Rightarrow \left(\begin{array}{l} ((\overleftarrow{S_l}/S_l) \wedge (\overleftarrow{S_r}/S_r)) \vee ((\overrightarrow{S_l}/S_l) \wedge (\overrightarrow{S_r}/S_r)) \vee \\ \Diamond \left(\ell = 0 \wedge \bigwedge\limits_{\varphi \in \Phi} \forall (\varphi \Leftrightarrow (\varphi/\mathbf{0})) \right) \end{array} \right) \tag{48}$$

where Φ stands for a finite set of arbitrary formulas. Note that axioms (10), (45) and (47) apply to arbitrary formulas φ and ψ, and not only to atomic ones too.

3.4 The proof of relative completeness

Now we are ready to prove that the axioms (10)-(13), (16)-(22), (42)-(44), (47)-(9), their mirror images and the rule (45) are complete for the extension of DC by projection onto state and neighbourhood formulas relative to validity in the extension of DC by neighbourhood formulas only.

In order to use validity in DC with neighbourhood formulas, relative to which the completeness of our set of axioms is being established, we extend the considered DC vocabularies by flexible constant, function and relation symbols to denote the flexible constants, functions and relations which are defined by the projections of atomic formulas. Then we translate the axiomatic system for DC with projection into a theory in the language for DC with neighbourhood formulas only based on the extended vocabulary. We demonstrate that the consistency of a formula with this theory is equivalent to the satisfiability of the result of substituting the extending non-logical symbols in the formula by their corresponding projection formulas. We do this by showing that the appropriate instances of our axioms imply that the interpretations of the extending non-logical symbols are the same as the interpretations of the corresponding projection formulas themselves. Hence, the validity of a formula in this theory is equivalent to the derivability of its counterpart formula using our axiomatic system and formulas which are valid in DC with neighbourhood formulas only.

For the rest of the section **L** is some language for DC with $(./.)$ and neighbourhood formulas.

Definition 12 Let H be a state expression in **L**. Let c_H, f_H, R_H, $P_H^{\rightarrow}$, $P_H^{\leftarrow}$ be fresh flexible relation symbols for every flexible constant c other than ℓ, every flexible function symbol f, flexible relation symbol R and state variable P in **L**, respectively. Let $\#P_H^{\rightarrow} = \#P_H^{\leftarrow} = 0$, $\#c_H = 1$, $\#f_H = \#f + 1$ and $\#R_H = \#R$. Let **L**′ be the language for DC with neighbourhood formulas (without the operator $(./.)$) based on the extension of the vocabulary of **L** by these fresh symbols for all H.

Definition 13 We define the translation t of the formulas from **L**′ into formulas from **L** as follows. If the terms t_0, t_1, …, are rigid, then

$$\begin{array}{l}
\mathsf{t}(P_H^{\leftarrow}) \rightleftharpoons (\overleftarrow{P}/H)\\
\mathsf{t}(P_H^{\rightarrow}) \rightleftharpoons (\overrightarrow{P}/H)\\
\mathsf{t}(c_H(t_0)) \rightleftharpoons (c = t_0/H)\\
\mathsf{t}(f_H(t_0, t_1, \ldots, t_{\#f})) \rightleftharpoons (t_0 = f(t_1, \ldots, t_{\#f})/H)\\
\mathsf{t}(R_H(t_1, \ldots, t_{\#R})) \rightleftharpoons (R(t_1, \ldots, t_{\#R})/H)
\end{array}$$

If α is an atomic formula built using a relation symbol from the vocabulary of **L**, then

$$\mathsf{t}(\alpha) \rightleftharpoons \alpha$$

The clauses for compound formulas are

$$\begin{array}{l}
\mathsf{t}(\varphi \Rightarrow \psi) \rightleftharpoons \mathsf{t}(\varphi) \Rightarrow \mathsf{t}(\psi)\\
\mathsf{t}((\varphi; \psi)) \rightleftharpoons (\mathsf{t}(\varphi); \mathsf{t}(\psi))\\
\mathsf{t}(\exists x \varphi) \rightleftharpoons \exists x(\mathsf{t}(\varphi))
\end{array}$$

The translation $\mathtt{t}$ is defined on atomic formulas built using the relation symbols introduced in Definition 12 and flexible terms as the translation of their flexible-term-free equivalents which can be obtained using (14) and (15).

Proposition 7 implies that every formula in $\mathbf{L}$ is equivalent to the $\mathtt{t}$-translation of some formula in $\mathbf{L}'$. The translation $\mathtt{t}$ is invertible for formulas of the form mentioned in Proposition 7. We extend the subset of $\mathbf{L}$ in which $\mathtt{t}$ is invertible as follows:

If S is a state expression, then $\mathtt{t}^{-1}(\overrightarrow{S}/H)$ denotes the boolean combination built of propositional temporal letters of the form $P_{\overrightarrow{H}}$ in the way the corresponding state variables P are used to build S. We extend $\mathtt{t}^{-1}$ in the same way to projections of left neighbourhood formulas. Similarly, if φ is a formula built using atomic formulas and their projections, then $\mathtt{t}^{-1}(\varphi)$ stands for the result of eliminating the occurrence of compound terms in φ by means of (14) and (15), distributing the projections over the newly introduced connectives and quantifier prefixes and then replacing the projections of atomic formulas in the obtained formula by atomic formulas built using the corresponding symbols from the vocabulary of $\mathbf{L}'$. For example, our convention about extended $\mathtt{t}^{-1}$ means that

$$\mathtt{t}^{-1}((\overrightarrow{P \wedge \neg Q}/H) \vee (R(x) \Rightarrow (R(f(x))/\neg H)))$$

is

$$(P_{\overrightarrow{H}} \wedge \neg Q_{\overrightarrow{H}}) \vee (R(x) \Rightarrow \exists y(f_{\neg H}(y,x) \wedge R_{\neg H}(y))).$$

Note that parts of the formula which are not in the scope of $(./.)$ are not affected by $\mathtt{t}^{-1}$.

Let $DC_{\mathbf{L}}^{(./.)}$ be the set of the formulas of $\mathbf{L}$ which can be derived using valid DC formulas and the $(./.)$-specific axioms enumerated in the beginning of this section. Let $DC_{\mathbf{L}}^{\mathtt{t}}$ be the set of those formulas from $\mathbf{L}'$ whose $\mathtt{t}$-translations are in $DC_{\mathbf{L}}^{(./.)}$.

The key step in our proof is Lemma 14 which provides the possibility of using the extending non-logical symbols and the translation $\mathtt{t}$ from Definitions 12 and 13 in the intended way by showing that these symbols have the same meaning as the corresponding projections of atomic formulas under interpretations which satisfy the formulas from $DC_{\mathbf{L}}^{\mathtt{t}}$. To prove Lemma 14, we use some properties of the derived operators $\leq$, $\preceq_l$ and $\preceq_r$ defined in Subsection 3.3 These properties are listed in Lemmata 10 and 11 from Subsection 3.3.

Lemma 14 (Truth Lemma) *Let the vocabulary of* $\mathbf{L}$ *contain finitely many flexible symbols. Let* I_0 *be an interpretation of* $\mathbf{L}'$ *and* $\sigma_0 \in \mathbf{I}$*. Let* $I_0, \sigma_0 \models \varphi$ *for all* $\varphi \in DC_{\mathbf{L}}^{\mathtt{t}}$*. Then there exist two interpretations* I *and* I' *of* $\mathbf{L}'$ *and an interval* $\sigma \in \mathbf{I}$ *such that* $I, \sigma \models \varphi$ *iff* $I_0, \sigma_0 \models \varphi$ *for all* φ *from* $\mathbf{L}'$*, and* I' *has the same restriction to subintervals of* σ *as* I *and satisfies*

$I', \sigma' \models P_{\overleftarrow{H}} \Leftrightarrow (\overleftarrow{P}/H)$
$I', \sigma' \models P_{\overrightarrow{H}} \Leftrightarrow (\overrightarrow{P}/H)$
$I', \sigma' \models \forall x(c_H(x) \Leftrightarrow (c = x/H))$
$I', \sigma' \models \forall y \forall x_1 \ldots \forall x_{\#f}(f_H(y, x_1, \ldots, x_{\#f}) \Leftrightarrow (y = f(x_1, \ldots, x_{\#f})/H))$
$I', \sigma' \models \forall x_1 \ldots \forall x_{\#R}(R_H(x_1, \ldots, x_{\#R}) \Leftrightarrow (R(x_1, \ldots, x_{\#R})/H))$

for all subintervals σ' of σ and all the non-logical symbols P, c, f and R of their respective types from $\mathbf{L}$.

We need to use σ and I in the lemma instead of σ_0 and I_0 themselves, because projection onto $\mathbf{0}$ makes the location of reference intervals relative to 0 relevant, and the location of σ_0 may happen to be different from the one described by the t-translations of the $\mathbf{L}'$ formulas which it satisfies under I_0.

Proof: Throughout this proof we refer to our axioms directly, despite the fact that they are written using $(./.)$ and we are actually working with formulas from $\mathbf{L}'$. When referring to an axiom, we mean a formula which t maps to an instance of this axiom, or to some formula which is straightforwardly derivable from such an instance, in order to achieve brevity.

Let $P_1, \dots, P_n$ be all the state variables from $\mathbf{L}$. Let E be the set of the conjunctions $\bigwedge\limits_{i=1}^{n} \varepsilon_i P_i$ where ε_i is either $\neg$ or nothing, $i = 1, \dots, n$. There are 2^n such conjunctions. Every state expression from $\mathbf{L}$ has a propositionally equivalent one of the form $\bigvee E'$ where $E' \subseteq E$. Rule (45) implies that we can assume that all the involved state expressions H are of this form.

We first define σ and I. Let $S_l, S_r \in E$ satisfy $I_0, \sigma_0 \models \mathsf{t}^{-1}((\overleftarrow{S_l} \wedge \overrightarrow{S_r}/\mathbf{0}))$. The existence of a unique pair of such state expressions follows from the facts $\vdash \bigvee E$ and $\vdash \neg(S \wedge S')$ for different $S, S' \in E$, and axioms (1)-(3), (5) and (6). Let Φ consist of the formulas $\overleftarrow{S_l}$, $\overrightarrow{S_r}$ and all the formulas of the forms $R(x_1, \dots, x_{\#R})$, $y = f(x_1, \dots, x_{\#f})$ and $y = c$, where R, f and c are some of the finitely many flexible symbols in $\mathbf{L}$ and y, x_1, x_2, $\dots$ is some fixed sequence of distinct individual variables. Then axiom (48) implies that at least one of the three formulas

$$\mathsf{t}^{-1}((\overleftarrow{S_l}/S_l) \wedge (\overleftarrow{S_r}/S_r)),\ \mathsf{t}^{-1}((\overrightarrow{S_l}/S_l) \wedge (\overrightarrow{S_r}/S_r))$$

and

$$\mathsf{t}^{-1}\left(\Diamond\left(\ell = 0 \wedge \bigwedge_{\varphi \in \Phi} \forall(\varphi \Leftrightarrow (\varphi/\mathbf{0}))\right)\right)$$

holds at σ_0 under I_0. Let $\delta = \min \sigma_0 - 1$ or $\delta = \max \sigma_0 + 1$ in case it is the first one or the second one, respectively. In case it is the third formula, let $\delta \in \sigma_0$ be such that

$$I_0, [\delta, \delta] \models \mathsf{t}^{-1}\left(\bigwedge_{\varphi \in \Phi} \forall(\varphi \Leftrightarrow (\varphi/\mathbf{0}))\right). \tag{49}$$

We choose σ to be $[\min \sigma_0 - \delta, \max \sigma_0 - \delta]$. We define I by the equalities

$I(s) = I_0(s)$ for all rigid s;

$I(P)(\tau) = I_0(P)(\tau + \delta)$ for state variables P;

$I(c)(\sigma') = I_0(c)([\min \sigma' + \delta, \max \sigma' + \delta])$ for flexible $c \not\equiv \ell$;

$I(s)(\sigma', d_1, \dots, d_{\#s}) = I_0(s)([\min \sigma' + \delta, \max \sigma' + \delta], d_1, \dots, d_{\#s})$ for all other flexible s, except ℓ.

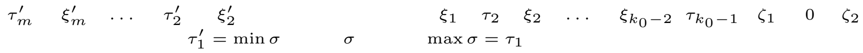

Figure 2: Definitive time points for projections under I', assuming that $\max\sigma < 0$.

The correspondence between I, σ, I_0 and σ_0 described in the lemma can be established by a direct check. Furthermore, in case $\delta \in \sigma_0$, we have $0 \in \sigma$ and (49) implies

$$I, [0,0] \models \begin{array}{l} \overleftarrow{P} \Leftrightarrow P_{\mathbf{0}}^{\leftarrow}, \overrightarrow{P} \Leftrightarrow P_{\mathbf{0}}^{\rightarrow}, c_{\mathbf{0}}(c), \\ \forall x_1 \ldots \forall x_{\#f} f_{\mathbf{0}}(f(x_1, \ldots, x_{\#f}), x_1, \ldots, x_{\#f}), \\ \forall x_1 \ldots \forall x_{\#R} R(x_1, \ldots, x_{\#R}) \Leftrightarrow R_{\mathbf{0}}(x_1, \ldots, x_{\#R}) \end{array} \tag{50}$$

for all the flexible symbols P, c, f and R of their respective types from $\mathbf{L}$.

We define I' only as much as necessary to prove the lemma. The values of I' on symbols and at intervals not mentioned here are irrelevant to the required properties of I' and can be arbitrary. As required by the lemma, I' is the same as I at subintervals of σ.

Let $\zeta_1, \zeta_2 \in \mathbf{R}$ be such that $0 \in (\zeta_1, \zeta_2)$ and $[\zeta_1, \zeta_2] \cap \sigma = \emptyset$, in case $0 \notin \sigma$.

Let $k = 2^n$ and the sequence $S_1, \ldots, S_k$ contain all the conjunctions from E and satisfy $I, \sigma \models \mathsf{t}^{-1}(S_i \leq S_{i+1})$, $i = 1, \ldots, k-1$. The existence of such a sequence follows from (34) and (37) of Lemma 11. Let k_0 be the smallest number such that $I, \sigma \models \mathsf{t}^{-1}(\neg(\overrightarrow{S_{k_0}}/S_{k_0}))$. Let $k_0 = k + 1$, in case $I, \sigma \models \mathsf{t}^{-1}((\overrightarrow{S_i}/S_i))$ for all $i = 1, \ldots, k$. Then

$$I, \sigma \models \mathsf{t}^{-1}(S_i \preceq_r S_{i+1}), i = 1, \ldots, k_0 - 1, \text{ and } I, \sigma \models \mathsf{t}^{-1}(\neg(\overrightarrow{S_i}/S_i))$$

for $i = k_0, \ldots, k$, because of (25)-(30) from Lemma 10 and (38) from Lemma 11, respectively. Obviously $I, \sigma \models \overrightarrow{S_{i_0}}$ for some $i_0 \in \{1, \ldots, k\}$. Then $I, \sigma \models (S_{i_0} \vee S_j)!$ for all $j = 1, \ldots, k$, whence axiom (17) implies that $I, \sigma \models \mathsf{t}^{-1}((\overrightarrow{S_{i_0}}/S_{i_0} \vee S_j))$. Hence $I, \sigma \models \mathsf{t}^{-1}(S_{i_0} \leq S_j)$, $j = 1, \ldots, k$ by the definition of $\leq$, which means that $i_0 = 1$. Furthermore, axiom (18) implies that $I, \sigma \models \mathsf{t}^{-1}((\overrightarrow{S_1}/S_1))$, which means that $k_0 > 1$. Let $\tau_1 = \max\sigma$ and $\tau_2, \ldots, \tau_{k_0-1}, \xi_1, \ldots, \xi_{k_0-2} \in \mathbf{R}$ be such that $\tau_1 < \xi_1 < \tau_2 < \ldots < \xi_{k_0-2} < \tau_{k_0-1}$ (see Figure 3.4). Let there be no such ξs, in case $k_0 = 2$. Let $\tau_{k_0-1} < \zeta_1$, in case $\max\sigma < 0$. We put $I'(P)(\tau) = 1$ if and only if P occurs positively in S_i for $\tau \in [\tau_i, \xi_i)$, $i = 1, \ldots, k_0 - 2$, or P occurs positively in S_{k_0-1} for $\tau \geq \tau_{k_0-1}$, in case $0 \leq \max\sigma$, and for $\tau \in [\tau_{k_0-1}, \zeta_1)$, in case $\max\sigma < 0$. If $I, [\max\sigma, \max\sigma] \models \mathsf{t}^{-1}((\overleftarrow{\neg S_i}/S_i))$, then we put $I'(P)(\tau) = 1$ if and only if $I, [\max\sigma, \max\sigma] \models P_{S_i}^{\leftarrow}$ for $\tau \in [\xi_{i-1}, \tau_i)$; otherwise we put $I'(P)(\tau) = I'(P)(\tau_{i-1})$ for $\tau \in [\xi_{i-1}, \tau_i)$, $i = 2, \ldots, k_0 - 1$. According to this definition, $I'(P)$ agrees with $I(P)$ at $\max\sigma$, and indeed at $[\max\sigma, \tau)$ for some $\tau > \max\sigma$.

Let k_1 be the smallest number such that $k_0 \leq k_1$ and $I, \sigma \models \int S_{k_1} = 0$. Let $k_1 = k + 1$, if $I, \sigma \models \int S_i > 0$ for all $i = k_0, \ldots, k$. Then

$$I, \sigma \models \textstyle\int S_i = 0,\ i = k_1, \ldots, k,$$

because of (40) from Lemma 11. Let k_2 be the smallest number such that $k_1 \leq k_2$ and $I, \sigma \models \mathsf{t}^{-1}(\neg(\overleftarrow{S_{k_2}}/S_{k_2}))$. Let $k_2 = k+1$, in case $I, \sigma \models \mathsf{t}^{-1}((\overleftarrow{S_i}/S_i))$ for all $i = k_1, \ldots, k$. Then

$$I, \sigma \models \mathsf{t}^{-1}(\neg(\overleftarrow{S_i}/S_i)),\ i = k_2, \ldots, k, \tag{51}$$

because of (41) from Lemma 11.

Let $E^{\leftarrow} = \{S \in E : I, \sigma \models \mathsf{t}^{-1}((\overleftarrow{S_i}/S_i))\}$. (51) implies $E^{\leftarrow} \subseteq \{S_1, \ldots, S_{k_2-1}\}$. There exists a unique $S \in E$ such that $I, \sigma \models \overleftarrow{S}$. Then axiom (16) implies that $I, \sigma \models \mathsf{t}^{-1}((\overleftarrow{S}/S))$ and $I, \sigma \models \mathsf{t}^{-1}(S \preceq_l S')$ for all other $S' \in E$. Hence $E^{\leftarrow} \neq \emptyset$. Let $E^{\leftarrow}$ contain m conjunctions and the sequence $S'_1, \ldots, S'_m$ contain all the conjunctions from $E^{\leftarrow}$ and satisfy $I, \sigma \models S'_i \preceq_l S'_{i+1}$, $i = 1, \ldots, m-1$. Lemma 10 implies that there is a unique such sequence and we have just shown that $I, \sigma \models \overleftarrow{S'_1}$. Furthermore, together with axiom (10), (31) from Lemma 10 implies that $S_{k_1}, \ldots, S_{k_2-1}$ is a subsequence of $S'_1, \ldots, S'_m$.

Let $\tau'_1 = \min \sigma$ and $\tau'_2, \ldots, \tau'_m, \xi'_2, \ldots, \xi'_m \in \mathbf{R}$ be such that $\tau'_m < \xi'_m < \tau'_{m-1} \ldots < \tau'_2 < \xi'_2 < \tau'_1$. Let $\tau'_m > \zeta_2$, in case $0 < \min \sigma$. We put $I'(P)(\tau) = 1$ if and only if P occurs positively in S'_i for $\tau \in [\xi'_{i+1}, \tau'_i)$, $i = 1, \ldots, m-1$, or P occurs positively in S'_m for $\tau \in [\zeta_2, \tau'_m)$, in case $0 < \min \sigma$, and for all $\tau < \tau'_m$, in case $\min \sigma \leq 0$. If $I, [\min \sigma, \min \sigma] \models \mathsf{t}^{-1}((\neg S'_i/S'_i))$, then we put $I'(P)(\tau) = 1$ if and only if $I, [\min \sigma, \min \sigma] \models P_{\overrightarrow{S'_i}}$ for $\tau \in [\tau_i, \xi_i)$; otherwise we put $I'(P)(\tau) = I'(P)(\xi_i)$ for $\tau \in [\tau_i, \xi_i)$, $i = 2, \ldots, m$. Axiom (16) implies that $I', \sigma \models \overleftarrow{S'_1}$. This implies that I and I' agree at $[\tau, \min \sigma)$ for some $\tau < \min \sigma$.

In case $0 \notin \sigma$, we put $I'(P)(\tau) = 1$ if and only if P occurs positively in S_l for $\tau \in [\zeta_1, 0)$, and we put $I'(P)(\tau) = 1$ if and only if P occurs positively in S_r for $\tau \in [0, \zeta_2)$. If $0 < \min \sigma$, we put $I'(P)(\tau) = I'(P)(\zeta_1)$ for all $\tau < \zeta_1$. If $\max \sigma < 0$, we put $I'(P)(\tau) = I'(P)(0)$ for all $\tau \geq \zeta_2$.

Axioms (5), (6) and the choice of k_2 guarantee that $I'(P)(\tau) = 0$ for all τ and all P which occur positively in some of the conjunctions $S_{k_2}, \ldots, S_k$.

Now let us prove that I' satisfies the equivalence about $P_{\overrightarrow{H}}$ from the lemma. Let $\sigma' \in \mathbf{I}$ and $\sigma' \subseteq \sigma$.

Let $I, [\max \sigma', \max \sigma] \models \int H > 0$. Let τ be the least time point in $[\max \sigma', \max \sigma)$ such that $I_\tau(H) = 1$. Then $I, \sigma' \models P_{\overrightarrow{H}}$ iff $I, [\min \sigma', \tau] \models P_{\overrightarrow{H}}$ by axiom (10) and $I, [\min \sigma', \tau] \models P_{\overrightarrow{H}}$ iff $I, [\min \sigma', \tau] \models \overrightarrow{P}$ by axiom (17). Hence in this case $I', \sigma' \models P_{\overrightarrow{H}} \Leftrightarrow (\overrightarrow{P}/H)$.

If $I, [\max \sigma', \max \sigma] \models \int H = 0$, then $I, \sigma' \models P_{\overrightarrow{H}}$ iff $I, \sigma \models P_{\overrightarrow{H}}$ and $I', \sigma' \models (\overrightarrow{P}/H)$ iff $I', \sigma \models (\overrightarrow{P}/H)$ by axioms (10) and (43). Let H be $\bigvee E'$ where $E' \subseteq E$.

Let $E' \neq \emptyset$ and i be the least number such that $S_i \in E'$. Then we have $I, \sigma \models \mathsf{t}^{-1}(H \equiv S_i)$ by (32), (35) and (36) from Lemma 11. A lengthy but simple check shows that $I', \sigma \models H \equiv S_i$. Now axiom (44) implies that $I', \sigma \models P_{\overrightarrow{H}} \Leftrightarrow P_{\overrightarrow{S_i}}$ and $I, \sigma \models (\overrightarrow{P}/H) \Leftrightarrow (\overrightarrow{P}/S_i)$. Hence it is sufficient to prove that $I', \sigma \models P_{\overrightarrow{S_i}} \Leftrightarrow (\overrightarrow{P}/S_i)$. We have the following four cases:

Case 1. $1 \leq i < k_0$. Then $I, \sigma \models \mathsf{t}^{-1}((\overrightarrow{S_i}/S_i))$, which means that $I, \sigma \models P_{\overrightarrow{S_i}}$ iff P occurs in S_i positively. By the definition of I', this is equivalent to $I', [\tau_i, \xi_i) \models \lceil S_i \wedge P \rceil$, in case $i < k_0 - 1$, and $I', [\tau_i, \tau) \models \lceil S_i \wedge P \rceil$ for some $\tau > \tau_i$, in case

$i = k_0 - 1$. Axiom (22) and the position of S_i in the sequence $S_1, \ldots, S_k$ imply that $I', [\max\sigma, \tau_i] \models \int S_i = 0$. Hence $I', \sigma \models (\overrightarrow{P}/S_i)$ iff P occurs in S_i positively too.

Case 2. $k_0 \leq i < k_1$. Then $I, \sigma \models \mathsf{t}^{-1}(\neg(\overrightarrow{S_i}/S_i)) \wedge \int S_i > 0$. Let τ be the least time point such that $I, [\tau, \max\sigma] \models \int S_i = 0$. Then axiom (10) implies that $I', \sigma \models (\overrightarrow{P}/S_i)$ iff $I', [\min\sigma, \tau] \models (\overrightarrow{P}/S_i)$, and $I', \sigma \models P_{\overrightarrow{S_i}}$ iff $I', [\min\sigma, \tau] \models P_{\overrightarrow{S_i}}$. It can be shown that $I'_{\tau'}(S_i) = 0$ for all $\tau' \geq \max\sigma$, and, consequently, for all $\tau' \geq \tau$. This implies $I, [\min\sigma, \tau] \models \neg(\overrightarrow{S_i}/S_i)$. $I', \sigma \models \int S_i > 0$ and the choice of τ imply $I', [\min\sigma, \tau] \models (\top; \ell = 0 \wedge \overleftarrow{S_i})$. Hence $I', [\min\sigma, \tau] \models S_i!$. Furthermore, $I, [\min\sigma, \tau] \models \mathsf{t}^{-1}(\neg(\overrightarrow{S_i}/S_i))$ by axiom (10), because $I', [\tau, \max\sigma] \models \int S_i = 0$. This implies $I', [\min\sigma, \tau] \models \mathsf{t}^{-1}(S_i!)$. Hence both $I', [\min\sigma, \tau] \models P_{\overrightarrow{S_i}}$ and $I', [\min\sigma, \tau] \models (\overrightarrow{P}/S_i)$ are equivalent to $I', [\min\sigma, \tau] \models \overrightarrow{P}$ by axiom (17).

Case 3. $k_1 \leq i < k_2$. Then $I, \sigma \models \mathsf{t}^{-1}(\neg(\overrightarrow{S_i}/S_i) \wedge \int S_i = 0 \wedge (\overleftarrow{S_i}/S_i))$. In this case $I', \sigma \models (\overrightarrow{P}/S_i)$ and $I', \sigma \models P_{\overrightarrow{S_i}}$ are equivalent to $I', [\min\sigma, \min\sigma] \models (\overrightarrow{P}/S_i)$ and $I', [\min\sigma, \min\sigma] \models P_{\overrightarrow{S_i}}$ by axiom (10), respectively. Another lengthy simple check shows that $I'_\tau(S_i) = 0$ for all $\tau \geq \min\sigma$. $I, \sigma \models \mathsf{t}^{-1}((\overleftarrow{S_i}/S_i))$ implies that $S_i \in E^{\leftarrow}$. Now recall the sequence $S'_1, \ldots, S'_m$ of the conjunctions from $E^{\leftarrow}$. Let S_i be S'_1 from this sequence. Then $I', [\min\sigma, \min\sigma] \models \overleftarrow{S_i}$. Since $I'_\tau(S_i) = 0$ for all $\tau \geq \min\sigma$, $I', [\min\sigma, \min\sigma] \models \neg(\overrightarrow{S_i}/S_i)$. $I, \sigma \models \int S_i = 0$ and $I, \sigma \models \mathsf{t}^{-1}(\neg(\overrightarrow{S_i}/S_i))$ imply $I, [\min\sigma, \min\sigma] \models \mathsf{t}^{-1}(\neg(\overrightarrow{S_i}/S_i))$ by axiom (10). Hence $I', [\min\sigma, \min\sigma] \models S_i!$ and $I', [\min\sigma, \min\sigma] \models \mathsf{t}^{-1}(S_i!)$. Then axiom (17) implies that both $I', [\min\sigma, \min\sigma] \models (\overrightarrow{P}/S_i)$ and $I', [\min\sigma, \min\sigma] \models P_{\overrightarrow{S_i}}$ are equivalent to $I', [\min\sigma, \min\sigma] \models \overrightarrow{P}$. If S_i is S'_j for some $j > 1$, then the definition of I' and axiom (22) imply that $I'_\tau(S_i) = 0$ for all $\tau \geq \tau'_j$, there exists a $\tau' < \tau'_j$ such that $I'_\tau(S_i) = 0$ for all $\tau \in [\tau', \tau'_j)$ and $I'_\tau(P) = 1$ for $\tau \in [\tau'_j, \xi'_j)$ iff $I', [\min\sigma, \min\sigma] \models P_{\overrightarrow{S'_j}}$. Furthermore, then $I', [\min\sigma, \min\sigma] \models (\overrightarrow{P}/S_i)$ is equivalent to the existence of $\tau' > \tau'_j$ such that $I'_\tau(P) = 1$ for $\tau \in [\tau'_j, \tau')$ too. Hence $I', [\min\sigma, \min\sigma] \models P_{\overrightarrow{S'_j}}$ and $I', [\min\sigma, \min\sigma] \models (\overrightarrow{P}/S_j)$ are equivalent again.

Case 4. $k_2 \leq i \leq k$. Then $I, \sigma \models \mathsf{t}^{-1}(\neg(\overrightarrow{S_i}/S_i) \wedge \int S_i = 0 \wedge \neg(\overleftarrow{S_i}/S_i))$ and $I'_\tau(S_i) = 0$ for all $\tau \in \mathbf{R}$. Axiom (46) implies that $I', \sigma \models P_{\overrightarrow{S_i}} \Leftrightarrow P_{\overrightarrow{\mathbf{0}}}$ and $I', \sigma \models (\overrightarrow{P}/S_i) \Leftrightarrow (\overrightarrow{P}/\mathbf{0})$. If $0 \notin \sigma$, then the definition of $I'(P)(\tau)$ for $\tau \in [0, \zeta_2)$ implies $I', \sigma \models P_{\overrightarrow{\mathbf{0}}} \Leftrightarrow (\overrightarrow{P}/\mathbf{0})$. Otherwise we have $I', \sigma \models P_{\overrightarrow{\mathbf{0}}}$ iff $I', [0,0] \models P_{\overrightarrow{\mathbf{0}}}$ and $I', \sigma \models (\overrightarrow{P}/\mathbf{0})$ iff $I', [0,0] \models (\overrightarrow{P}/\mathbf{0})$ by axiom (10). Then $I', \sigma \models P_{\overrightarrow{\mathbf{0}}} \Leftrightarrow (\overrightarrow{P}/\mathbf{0})$ follows from (50), because I' and I agree on state variables at a proper neighbourhood of σ.

If $E' = \emptyset$, which is equivalent to $\vdash H \Leftrightarrow \mathbf{0}$, then the rule (45) implies that $I', \sigma' \models P_{\overrightarrow{H}} \Leftrightarrow P_{\overrightarrow{\mathbf{0}}}$, and $I', \sigma \models P_{\overrightarrow{\mathbf{0}}} \Leftrightarrow (\overrightarrow{P}/\mathbf{0})$ is established as in Case 4 above.

The equivalence about $P_{\overleftarrow{H}}$ is established similarly. To establish the equivalences about formulas built using relation symbols, we use the now established equivalences about the projections of neighbourhood formulas involved in the definition of the formulas abbreviated using $(.)!$, which appear in the axioms (11), (12) and (42). The truth values of projections of atomic formulas built using relation symbols at subintervals σ' of σ can be defined using the interpretations of the involved flexible constant, function and relation symbols at intervals whose end points are in the set $\{\tau'_m, \ldots, \tau'_1\} \cup \sigma \cup \{\tau_1, \ldots, \tau_{k_0-1}\} \cup \{0\}$. We only need to define I' on such intervals.

It can be easily shown that $DC^{\mathfrak{t}}_{\mathbf{L}}$ contains the formulas

$$\exists y f_H(y, x_1, \ldots, x_{\#f}),$$

$$f_H(y_1, x_1, \ldots, x_{\#f}) \wedge f_H(y_2, x_1, \ldots, x_{\#f}) \Rightarrow y_1 = y_2$$

and

$$\exists y c_H(y),\ c_H(y_1) \wedge c_H(y_2) \Rightarrow y_1 = y_2$$

for all flexible function symbols f and constants c from $\mathbf{L}$. This means that for every interval $\sigma' \subseteq \sigma$ and all $d_1, \ldots, d_{\#f} \in \mathbf{R}$ there exists a unique $e \in \mathbf{R}$ such that $I(f_H)(\sigma', e, d_1, \ldots, d_{\#f})$ and $I(c_H)(\sigma', e)$.

Let H be a state expression. If $\not\vdash S_i \Rightarrow H$ for all $i = 1, \ldots, k$, then $\vdash H \Leftrightarrow \mathbf{0}$. Then, to establish the equivalences about c_H, f_H and R_H from the lemma, we use that (50) holds together with the equalities

$$I'(c_H)(\sigma', e) = I'(c_H)([0, 0], e)$$

$$I'(f_H)(\sigma', e, d_1, \ldots, d_{\#f}) = I'(f_H)([0, 0], e, d_1, \ldots, d_{\#f})$$

and

$$I'(R_H)(\sigma', d_1, \ldots, d_{\#R}) = I'(R_H)([0, 0], d_1, \ldots, d_{\#R}),$$

respectively, which follow from axiom (10), for the case $0 \in \sigma$. We use (50) to define I' at $[0, 0]$, in case $0 \notin \sigma$. For the rest of the proof we assume that i is the least number such that $\vdash S_i \Rightarrow H$ and $h = \lambda\tau.I'_\tau(H)$.

Let $i < k_0$. If $\tau \in \sigma$, then $\gamma_h(\delta_h(\tau)) \in \{\tau \in \sigma : h(\tau) = 1\} \cup \{\tau_i\}$, where γ_h and δ_h are as in the definition of $(./.)$. We need to define I' on flexible non-logical symbols at intervals whose endpoints are in this set. I' and I coincide at intervals whose both endpoints are in σ. Given $\tau \in \sigma$ such that $h(\tau) = 1$, we put $I'(c)([\tau, \tau_i]) = e$ where e is the unique element of $\mathbf{R}$ such that $I(c_H)([\tau, \max\sigma], e)$ for all flexible constants c, and we put $I'(f)([\tau, \tau_i], d_1, \ldots, d_{\#f}) = e$ where e is the unique element of $\mathbf{R}$ such that $I(f_H)([\tau, \max\sigma], e, d_1, \ldots, d_{\#f})$ for all flexible function symbols f and $d_1, \ldots, d_{\#f} \in \mathbf{R}$. We put

$$I'(R)([\tau, \tau_i], d_1, \ldots, d_{\#R}) = I(R_H)([\tau, \max\sigma], d_1, \ldots, d_{\#R})$$

for all flexible relation symbols R and $d_1, \ldots, d_{\#R} \in \mathbf{R}$. We put

$$I'(c)([\tau_i, \tau_i]) = e \text{ iff } I(c_H)([\max\sigma, \max\sigma], e),$$

$$I'(f)([\tau_i, \tau_i], d_1, \ldots, d_{\#f}) = e \text{ iff } I(f_H)([\max\sigma, \max\sigma], e, d_1, \ldots, d_{\#f}),$$

and

$$I'(R)([\tau_i, \tau_i], d_1, \ldots, d_{\#R}) = I(R_H)([\max\sigma, \max\sigma], d_1, \ldots, d_{\#R}).$$

In case $k_0 \le i < k_1$, projections onto H at subintervals of σ depend only on the interpretation I of flexible non-logical symbols at subintervals of σ and cause no need to provide values for I', because I' is the same as I at such intervals.

Let $k_1 \le i < k_2$. Then there exists a unique $j \in \{1, \dots, m\}$ such that $S_i = S'_j$. Since $S_{k_1}, \dots, S_{k_2} - 1$ is a subsequence of $S'_1, \dots, S'_m$, $\gamma_h(\delta_h(\tau)) = \tau_j$ for all $\tau \in \sigma$. We put

$$I'(c)([\tau_j, \tau_j]) = e \text{ iff } I(c_H)(\sigma, e),$$

$$I'(f)([\tau_j, \tau_j], d_1, \dots, d_{\#f}) = e \text{ iff } I(f_H)(\sigma, e, d_1, \dots, d_{\#f})$$

and

$$I'(R)([\tau_j, \tau_j], d_1, \dots, d_{\#R}) = I(R_H)(\sigma, d_1, \dots, d_{\#R})$$

for all flexible constants c, function symbols f and relation symbols R, respectively.

The case $i \ge k_2$ is similar to the case $\vdash H \Leftrightarrow \mathbf{0}$

The correctness of the above clauses follows from axiom (42) and (35) and (36) from Lemma 11. A direct check using these theorems and axiom together with axiom (10) shows that I' now satisfies the equivalences about the symbols c_H, f_H and R_H from the lemma. $\dashv$

Theorem 15 (Relative completeness of DC with $(./.)$ and neighbourhood formulas) *Let φ be a valid formula in $\mathbf{L}$. Then φ can be derived from formulas valid in DC with neighbourhood formulas by means of the axioms (10)-(13), (16)-(22), (42)-(44), (47)-(48) and the rule (45).*

Proof: We may assume that $\mathbf{L}$ has only the flexible symbols occurring in φ in its vocabulary. Then Proposition 7 implies that there exists a formula ψ such that $\varphi \Leftrightarrow \psi$ is derivable using the axioms and rule mentioned above and $(./.)$ occurs in ψ only in subformulas of the forms occurring on the right sides of the equivalences from Lemma 14. Assume that φ is not derivable in the above way for the sake of contradiction. Then neither is ψ. Hence $\neg\psi$ is consistent with the set of formulas $DC_{\mathbf{L}}^{(./.)}$, and therefore $\mathtt{t}^{-1}(\neg\psi)$ is consistent with the set $DC_{\mathbf{L}}^{\mathtt{t}}$. Since $DC_{\mathbf{L}}^{\mathtt{t}}$ contains all the valid formulas in $\mathbf{L}'$, then there exists an interpretation I_0 of $\mathbf{L}'$ and an interval $\sigma_0 \in \mathbf{I}$ such that $I_0, \sigma_0 \models \mathtt{t}^{-1}(\neg\psi)$ and $I_0, \sigma_0 \models \chi$ for all $\chi \in DC_{\mathbf{L}}^{\mathtt{t}}$. Now Lemma 14 implies that there exist interpretations I and I' of $\mathbf{L}'$ and an interval $\sigma \in \mathbf{I}$ such that $I, \sigma \models \chi$ iff $I_0, \sigma_0 \models \chi$ for all $\chi \in \mathbf{L}'$, I' coincides with I at subintervals of σ and the I'-interpretations of the extending non-logical symbols of $\mathbf{L}'$ are the same as those of their corresponding projection formulas at subintervals of σ. In particular, $I', \sigma \models \mathtt{t}^{-1}(\neg\psi)$. An induction on the construction of ψ shows that $I', \sigma \models \mathtt{t}^{-1}(\psi) \Leftrightarrow \psi$. Hence, $I', \sigma \models \neg\psi$ and, consequently, $I', \sigma \models \neg\varphi$, which is a contradiction. This means that φ can be derived from formulas valid in the extension of DC by neighbourhood formulas using only our axioms and rule. $\dashv$

4 The scope of relative completeness

In this section we discuss the part neighbourhood formulas have in our relative completeness result and how it applies to extensions of DC by other operators.

As pointed out in Subsection 2.3 and proved in Appendix A, the axioms (1)-(4) and their left-neighbourhood mirror images are complete for neighbourhood formulas relative to DC with no extending construct whatsoever. It is important to note that, given a finite set of neighbourhood formulas, it takes finitely many instances of (1)-(4) to achieve this axiomatisation. The detailed proof in Appendix A shows how the relevant instances can be determined from the given neighbourhood formulas. Let $\mathcal{N}_\varphi$ be the conjunction of the instances of (1)-(4) which are relevant to neighbourhood subformulas of some formula φ. Then the deduction theorem for DC (cf. e.g. [HZ92, HZ97, ZH04]) implies that the validity of φ in DC with neighbourhood formulas is equivalent to the validity of $\Box\mathcal{N}_\varphi \Rightarrow \varphi$ in DC without extending constructs, if the neighbourhood formulas in this implication be regarded as propositional temporal letters. Hence our completeness result can be stated relative to DC with no extending constructs and we can assume that our axiomatisation applies to projection onto state as the only extending operator in DC as well.

Finally, let us note that the relative completeness propagates straightforwardly to the extension DC^* of DC by the unary modality known as *iteration* and denoted by $(.)^*$ [DVW96, DVG99]. Iteration is defined in DC and, more generally, in ITL as follows

$I, \sigma \models \varphi^*$ iff either $\min\sigma = \max\sigma$, or there exist an $n < \omega$ and $\sigma_1, \ldots, \sigma_n \in \mathbf{I}$ such that $\sigma_1; \ldots; \sigma_n = \sigma$ and $I, \sigma_i \models \varphi$, $i = 1, \ldots, n$.

To extend our relative completeness result to DC^*, it is sufficient to extend the list of axioms (5)-(9) used to drive $(./.)$ down to atomic formulas by one for formulas built using $(.)^*$:

$$(\varphi^*/H) \Leftrightarrow (\varphi/H)^* \vee \textstyle\int H = 0. \qquad (52)$$

Proposition 9 can be extended to DC^* too. An axiom which generalises (52) was shown in [GDV02] to apply to a subset of the extension of DC by least-fixed-point operator μ [Pan95]. In the general it is more straightforward to take projection formulas out of the scope of μ by using a definitional extension for projection formulas. This means to prove, e.g.,

$$\Box(\forall x_1 \ldots \forall x_{\#R}(R_H(x_1, \ldots, x_{\#R}) \Leftrightarrow (R(x_1, \ldots, x_{\#R})/H))) \Rightarrow \varphi,$$

where R_H is some fresh flexible relation symbol, if the original formula to prove is

$$[\lambda x_1, \ldots, x_n.(R(x_1, \ldots, x_n)/H)/R_H]\varphi.$$

5 Concluding remarks

Obviously the definition of projection can have different variants for DC with more than state expressions only. In this article we stick to the variant from our earlier work [GDV02]. We believe that some alternative variants could be handled with the use of neighbourhood formulas in similar ways. The particular variant of the operator needed should be determined by the needs of the considered applications. It is worth noting that our technique involves axioms which allow detailed treatment of projection at the level of atomic formulas. This means that one can axiomatise different variants for the different flexible non-logical symbols in the same vocabulary, if an application requires that. An important element of choice in the definition is to consider DC interpretations which are defined on the entire **R** only. Alternatively, one can define DC on time domains of the forms of all the possible kinds of intervals $\Sigma_{\lambda\tau.I_\tau(H)}$, which includes **R** itself, and semibounded and bounded intervals. Then one can refer to an appropriate model with $\Sigma_{\lambda\tau.I_\tau(H)}$ as the time domain when defining projection onto state H. This may cause the use of $(\overleftarrow{S}/H)$ and $(\overrightarrow{S}/H)$ in a possible axiomatisation to change or vanish.

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A Appendix: Proof of Theorem 6

Proof: Let S be a state expression. Let

$$\mathsf{a}(S) \rightleftharpoons [\perp/\mathbf{0}][\overrightarrow{P}/P : P \in SV(S)]S.$$

The formula $\mathsf{a}(S)$ is isomorphic to S. A simple induction on the construction of S shows that

$$\overrightarrow{S} \Leftrightarrow \mathsf{a}(S) \tag{53}$$

is derivable using axioms (1) and (2). Hence every formula φ in DC with right-neighbourhood formulas has an equivalent one ψ which contains only state variables in its right-neighbourhood subformulas. The equivalence can be derived using (53). Given a ψ of this form we put

$$\mathsf{t}(\psi) \rightleftharpoons [R_P/\overrightarrow{P} : P \in SV(\psi)]\psi,$$

where R_P is a fresh propositional temporal letter for each $P \in SV(\psi)$. Let $\mathcal{N}_\psi$ denote the conjunction

$$\bigwedge_{p \in SV(\psi)} (\overrightarrow{P} \Rightarrow (\top; \overrightarrow{P})) \wedge (\overrightarrow{\neg P} \Rightarrow (\top; \overrightarrow{\neg P})) \wedge \neg(\overrightarrow{P}; \lceil \neg P \rceil) \wedge \neg(\overrightarrow{\neg P}; \lceil P \rceil)$$

of the instances of axioms (3) and (4) for $P \in SV(\psi)$. Let us prove that if ψ is valid, then $\mathsf{t}(\Box\mathcal{N}_\psi \Rightarrow \psi)$, which contains no right-neighboudhood formulas, is valid too. Assume it is not, for the sake of contradiction. Then there is an interpretation I and an interval $\sigma \in \mathbf{I}$ such that

$$I, \sigma \models \Box\mathsf{t}(\mathcal{N}_\psi) \wedge \neg\mathsf{t}(\psi).$$

Let the interpretation J coincide with I on all non-logical symbols in ψ, except possibly the state variables $P \in SV(P)$. Let

$$J(P)(\tau) = \begin{cases} I(P)(\tau), & \text{if } \tau < \max\sigma, \\ 1, & \text{if } \tau \geq \max\sigma \text{ and } I, \sigma \models R_P \\ 0, & \text{if } \tau \geq \max\sigma \text{ and } I, \sigma \not\models R_P \end{cases}$$

Then $J, \sigma \models \Box\mathsf{t}(\mathcal{N}_\psi) \wedge \neg\mathsf{t}(\psi)$ and $J, \sigma' \models R_P \Leftrightarrow \overrightarrow{P}$ for intervals $\sigma' \subseteq \sigma$. To realise that, consider the cases:

Case 1. $\max\sigma' = \max\sigma$. Then $J, \sigma' \models R_P$ iff $J, \sigma \models R_P$, because the t-translations of the instances $\overrightarrow{P} \Rightarrow (\top; \overrightarrow{P})$ and $\overrightarrow{\neg P} \Rightarrow (\top; \overrightarrow{\neg P})$ of axiom (4) for P and $\neg P$ are conjunctive members of $\mathcal{N}_\psi$, which I satisfies, and I coincides with J at σ. Hence, by the definition of $J(P)(\tau)$ for $\tau \geq \max\sigma$, we have $J, \sigma' \models R_P \Leftrightarrow \overrightarrow{P}$.

Case 2. $\max\sigma' < \max\sigma$. Then $J, \sigma' \models R_P$ iff $I, [\max\sigma', \max\sigma] \models \lceil P \rceil$, because the t-translations of the instances $\neg(\overrightarrow{P}; \lceil \neg P \rceil)$ and $\neg(\overrightarrow{\neg P}; \lceil P \rceil)$ of axiom (3) for P and $\neg P$ are conjunctive members of $\mathcal{N}_\psi$, which I satisfies, and I coincides with J at subintervals of σ. Hence $J, \sigma' \models R_P \Leftrightarrow \overrightarrow{P}$ again.

Now $J, \sigma \models \neg\psi$ follows from $J, \sigma \models \Box(R_P \Leftrightarrow \overrightarrow{P})$ for $P \in SV(P)$ and $J, \sigma \models \neg\mathsf{t}(\psi)$ by substitution of equivalents, which contradicts the validity of ψ. Hence, if ψ is valid, then $\Box\mathcal{N}_\psi \Rightarrow \psi$ is valid in DC, provided that right-neighbourhood formulas are treated as propositional temporal letters and to derive ψ itself from this implication only the instances of (3) and (4) which appear in $\mathcal{N}_\psi$ are needed. $\dashv$

B Appendix: Proofs of Lemmata 10 and 11

Proof:[Lemma 10](25) follows from the definitions of $\preceq_l$ and $\preceq_r$ by rule (45). (30) follows from the definitions of $\preceq_r$ and $\leq$ immediately. (26) is simply axiom (21). The first two deductions below and their mirror images prove (27) and (29). The last deduction proves (31). Let $H_{1,2} \rightleftharpoons H_1 \vee H_2$ for the sake of brevity.

(27):

1	$\neg(\overleftarrow{H_1 \wedge H_2}/H_{1,2})$	$\vdash \neg(H_1 \wedge H_2)$, rule (45)
2	$\neg(\overleftarrow{H_1 \wedge H_2}/H_{1,2}) \Rightarrow \neg(\overleftarrow{H_1}/H_{1,2}) \vee \neg(\overleftarrow{H_2}/H_{1,2})$	axioms (1)-(3), (5) and (6) and rule (45)
3	$\neg(H_1 \preceq_l H_2 \wedge H_2 \preceq_l H_1)$	1, 2, def. of $\preceq_l$

(29):

1	$(\overleftarrow{H_1}/H_1) \Rightarrow (\overleftarrow{H_{1,2}}/H_{1,2})$	axiom (19)
2	$(\overleftarrow{H_{1,2}}/H_{1,2}) \Leftrightarrow (\overleftarrow{H_1}/H_{1,2}) \vee (\overleftarrow{H_1}/H_{1,2})$	axioms (1)-(3), (5) and (6) and rule (45)
3	$(\overleftarrow{H_1}/H_1) \Rightarrow H_1 \preceq_l H_2 \vee H_2 \preceq_l H_1$	1, 2, def. of $\preceq_l$

(31):

1	$\int H_{1,2} = 0 \Rightarrow ((\overleftarrow{H_1}/H_{1,2}) \Rightarrow (\top; \ell = 0 \wedge (\overleftarrow{H_1}/H_{1,2})))$	axiom (10), DC
2	$\int H_{1,2} = 0 \wedge \neg(\overrightarrow{H_{1,2}}/H_{1,2}) \Rightarrow$ $((\top; \ell = 0 \wedge (\overleftarrow{H_1}/H_{1,2})) \Rightarrow H_1 \leq H_2$	def. of $\leq$
3	$\int H_{1,2} = 0 \wedge \neg(\overrightarrow{H_{1,2}}/H_{1,2}) \Rightarrow (H_1 \preceq_l H_2 \Rightarrow H_1 \leq H_2)$	1, 2, def. of $\preceq_l$

$\dashv$

Proof:[Lemma 11] Using axiom (10), one can easily derive (32). (32) implies (33) and (39) implies (40) in basic DC. Let $H_{i,j} \rightleftharpoons H_i \vee H_j$ for the sake of brevity in the deductions for (34)-(39) and (41) below. In these deductions we frequently use that

$$(\overrightarrow{H_{i,j}}/H_{i,j}) \Leftrightarrow (\overrightarrow{H_i}/H_{i,j}) \vee (\overrightarrow{H_j}/H_{i,j}) \qquad (54)$$

and its mirror image can be derived from axioms (1)-(3), (5) and (6) by rule (45).

(34):

1	$(\overrightarrow{H_1}/H_{1,2}) \wedge (\overrightarrow{H_2}/H_{2,3}) \Rightarrow (\overrightarrow{H_1}/H_{1,3})$	axiom (21)
2	$(\overrightarrow{H_1}/H_1) \Rightarrow (\overrightarrow{H_{1,3}}/H_{1,3})$	axiom (19)
3	$(\overrightarrow{H_{1,3}}/H_{1,3}) \Rightarrow (\overrightarrow{H_1}/H_{1,3}) \vee (\overrightarrow{H_3}/H_{1,3})$	(54)
4	$\neg(\overrightarrow{H_{2,3}}/H_{2,3}) \Rightarrow \neg(\overrightarrow{H_3}/H_3)$	axiom (19)
5	$\neg(\overrightarrow{H_3}/H_3) \Rightarrow \neg(\overrightarrow{H_3}/H_{1,3})$	axiom (18)
6	$(\overrightarrow{H_1}/H_{1,2}) \wedge \neg(\overrightarrow{H_{2,3}}/H_{2,3}) \Rightarrow (\overrightarrow{H_1}/H_{1,3})$	2-5
7	$H_1 \leq H_2 \wedge \neg(\overrightarrow{H_{1,2}}/H_{1,2}) \Rightarrow \neg(\overrightarrow{H_2}/H_{2,3})$	(38), axioms (18), (19)
8	$\neg(\overrightarrow{H_1}/H_{1,2}) \wedge \neg(\overrightarrow{H_2}/H_{1,2}) \Rightarrow \neg(\overrightarrow{H_{1,2}}/H_{1,2})$	(54)
9	$\neg(\overrightarrow{H_1}/H_{1,2}) \wedge H_1 \leq H_2 \Rightarrow \neg(\overrightarrow{H_2}/H_{1,2})$	(54) and def. of $\leq$
10	$\neg(\overrightarrow{H_1}/H_{1,2}) \wedge H_1 \leq H_2 \Rightarrow \neg(\overrightarrow{H_{1,2}}/H_{1,2})$	8, 9
11	$H_1 \leq H_2 \wedge \neg(\overrightarrow{H_1}/H_{1,2}) \Rightarrow \neg(\overrightarrow{H_2}/H_{2,3})$	7, 10
12	$(\overleftarrow{H_1}/H_{1,2}) \wedge (\overleftarrow{H_2}/H_{2,3}) \Rightarrow (\overleftarrow{H_1}/H_{1,3})$	axiom (21)
13	$(\top; \ell = 0 \wedge (\overleftarrow{H_1}/H_{1,2}))\wedge$	
	$(\top; \ell = 0 \wedge (\overleftarrow{H_2}/H_{2,3})) \Rightarrow$	
	$(\top; \ell = 0 \wedge (\overleftarrow{H_1}/H_{1,3}))$	12, DC
14	$(\overrightarrow{H_{i,j}}/H_{i,j}) \Leftrightarrow (\overrightarrow{H_i}/H_i) \vee (\overrightarrow{H_j}/H_j)$	$i, j = 1, 2, 3$, (54) and axioms (18) and (19)
15	$\int H_{i,j} = 0 \Leftrightarrow \int H_i = 0 \wedge \int H_j = 0$	$i, j = 1, 2, 3$, DC
16	$\neg(\overrightarrow{H_{1,2}}/H_{1,2}) \wedge \neg(\overleftarrow{H_{1,2}}/H_{1,2})\wedge$	
	$\int H_{1,2} = 0 \wedge \neg(\overrightarrow{H_{2,3}}/H_{2,3})\wedge$	
	$\neg(\overleftarrow{H_{2,3}}/H_{2,3}) \wedge \int H_{2,3} = 0 \Rightarrow$	
	$\neg(\overrightarrow{H_{1,3}}/H_{1,3}) \wedge \neg(\overleftarrow{H_{1,3}}/H_{1,3}) \wedge \int H_{1,3} = 0$	14, 15
17	$H_1 \leq H_2 \wedge H_2 \leq H_3 \Rightarrow H_1 \leq H_3$	1, 6, 11, 13, 16, def. of $\leq$

(35):

1	$(\overrightarrow{S}/S \vee H_{1,2}) \wedge (\overrightarrow{H_{1,2}}/S \vee H_{1,2}) \Rightarrow$	
	$\bigvee_{i=1}^{2} (\overrightarrow{S \wedge H_i}/S \vee H_{1,2})$	axioms (1)-(3), (5) and (6)
2	$(\overrightarrow{S \wedge H_i}/S \vee H_{1,2}) \Rightarrow (\overrightarrow{S \wedge H_i}/S \vee H_i)$	$i = 1, 2$, axiom (18)
3	$(\overrightarrow{S \wedge H_i}/S \vee H_i) \Rightarrow (\overrightarrow{S}/S \vee H_i) \wedge (\overrightarrow{H_i}/S \vee H_i)$	$i = 1, 2$, axioms (1)-(3), (5), (6)
4	$\left(\begin{array}{c} (\overrightarrow{S}/S \vee H_{1,2})\wedge \\ (\overrightarrow{H_{1,2}}/S \vee H_{1,2}) \end{array} \right) \Rightarrow$	
	$\bigvee_{i=1}^{2} (\overrightarrow{S}/S \vee H_i) \wedge (\overrightarrow{H_i}/S \vee H_i)$	1-3
5	$(\overrightarrow{S}/S \vee H_{1,2}) \vee (\overrightarrow{H_{1,2}}/S \vee H_{1,2}) \Rightarrow$	
	$(\overrightarrow{S \vee H_{1,2}}/S \vee H_{1,2})$	axioms (1)-(3),(5), (6)

6	$(\overleftarrow{S}/S \vee H_{1,2}) \wedge (\overleftarrow{H_{1,2}}/S \vee H_{1,2}) \Rightarrow \bigvee_{i=1}^{2} (\overleftarrow{S}/S \vee H_i) \wedge (\overleftarrow{H_i}/S \vee H_i)$	like 4
7	$\begin{pmatrix} (\top; \ell = 0 \wedge (\overleftarrow{S}/S \vee H_{1,2}))\wedge \\ (\top; \ell = 0 \wedge (\overleftarrow{H_{1,2}}/S \vee H_{1,2})) \end{pmatrix} \Rightarrow \bigvee_{i=1}^{2} \begin{pmatrix} (\top; \ell = 0 \wedge (\overleftarrow{S}/S \vee H_i))\wedge \\ (\top; \ell = 0 \wedge (\overleftarrow{H_i}/S \vee H_i)) \end{pmatrix}$	6, DC
8	$\begin{pmatrix} \neg(\overrightarrow{S \vee H_{1,2}}/S \vee H_{1,2})\wedge \\ \neg(\overleftarrow{S \vee H_{1,2}}/S \vee H_{1,2})\wedge \\ \int(S \vee H_{1,2}) = 0 \end{pmatrix} \Rightarrow \bigwedge_{i=1}^{2} \begin{pmatrix} \neg(\overrightarrow{S \vee H_i}/S \vee H_i)\wedge \\ \neg(\overleftarrow{S \vee H_i}/S \vee H_i)\wedge \\ \int(S \vee H_i) = 0 \end{pmatrix}$	axiom (19), DC
9	$S \equiv H_{1,2} \Rightarrow S \equiv H_1 \vee S \equiv H_2$	4-6, 8, def. of $\leq$

(36):

1	$(\overrightarrow{H_1}/H_{1,2}) \Rightarrow (\overrightarrow{H_{1,2}}/H_{1,2})$	rule (45)
2	$H_1 \leq H_2 \wedge \neg(\overrightarrow{H_1}/H_{1,2}) \Rightarrow \neg(\overrightarrow{H_{1,2}}/H_{1,2})$	def. of $\leq$
3	$(\overleftarrow{H_1}/H_{1,2}) \Rightarrow (\overleftarrow{H_{1,2}}/H_{1,2})$	rule (45)
4	$(\top; \ell = 0 \wedge (\overleftarrow{H_1}/H_{1,2})) \Rightarrow (\top; \ell = 0 \wedge (\overleftarrow{H_{1,2}}/H_{1,2}))$	3, DC
5	$H_1 \leq H_2 \Rightarrow H_{1,2} \equiv H_1$	1, 2, 4, def. of $\leq$

(37):

1	$\neg(H_i \leq H_j) \Rightarrow \begin{pmatrix} (\neg\overrightarrow{H_i}/H_{1,2})\wedge \\ ((\overrightarrow{H_{1,2}}/H_{1,2}) \vee (\top; \ell = 0 \wedge (\neg\overleftarrow{H_i}/H_{1,2})))\wedge \\ ((\overrightarrow{H_{1,2}}/H_{1,2}) \vee (\overleftarrow{H_{1,2}}/H_{1,2}) \vee \int H_{1,2} > 0)) \end{pmatrix}$	$i = 1, 2,\ j = 3 - i$, def. of $\leq$, rule (45), axioms (5) and (6), DC
2	$(\neg\overrightarrow{H_1}/H_{1,2}) \wedge (\neg\overrightarrow{H_2}/H_{1,2}) \Rightarrow \neg(\overrightarrow{H_{1,2}}/H_{1,2})$	(54) and rule (45)
3	$\bigwedge_{i=1}^{2} (\top; \ell = 0 \wedge (\neg\overleftarrow{H_i}/H_{1,2})) \Rightarrow (\top; \ell = 0 \wedge \neg(\overleftarrow{H_{1,2}}/H_{1,2}))$	(54) and rule (45), DC
4	$(\top; \lceil H_{1,2} \rceil; \int H_{1,2} = 0) \vee \int H_{1,2} = 0$	DC
5	$(\top; \lceil H_{1,2} \rceil; \int H_{1,2} = 0) \Rightarrow (\top; \lceil H_{1,2} \rceil; \int H_{1,2} = 0 \wedge \overleftarrow{H_{1,2}})$	axiom (4), DC
6	$(\top; \lceil H_{1,2} \rceil; \int H_{1,2} = 0 \wedge \overleftarrow{H_{1,2}}) \Rightarrow (\top; \int H_{1,2} = 0 \wedge (\overleftarrow{H_{1,2}}/H_{1,2}))$	axiom (16), DC
7	$(\top; \int H_{1,2} = 0 \wedge (\overleftarrow{H_{1,2}}/H_{1,2})) \Rightarrow (\top; \ell = 0 \wedge (\overleftarrow{H_{1,2}}/H_{1,2}))$	axiom (10), DC
8	$\bigwedge_{i=1}^{2} (\top; \ell = 0 \wedge (\neg\overleftarrow{H_i}/H_{1,2})) \Rightarrow \int H_{1,2} = 0$	3-7, DC
9	$\int H_{1,2} = 0 \wedge (\top; \ell = 0 \wedge (\neg\overleftarrow{H_i}/H_{1,2})) \Rightarrow \neg(\overleftarrow{H_i}/H_{1,2})$	axioms (5), (6), (10), DC
10	$H_1 \leq H_2 \vee H_2 \leq H_1$	1, 2, 8, 9 and (54)

(38):

1	$H_1 \leq H_2 \Rightarrow (\overrightarrow{H_1}/H_{1,2}) \vee \neg(\overrightarrow{H_{1,2}}/H_{1,2})$	def. of $\leq$
2	$\neg(\overrightarrow{H_1}/H_1) \Rightarrow \neg(\overrightarrow{H_1}/H_{1,2})$	axiom (18)
3	$\neg(\overrightarrow{H_{1,2}}/H_{1,2}) \Rightarrow \neg(\overrightarrow{H_2}/H_2)$	(54) and axiom (19)
4	$H_1 \leq H_2 \wedge \neg(\overrightarrow{H_1}/H_1) \Rightarrow \neg(\overrightarrow{H_2}/H_2)$	1-3

(39):

1	$\neg(\overrightarrow{H_1}/H_1) \Rightarrow \neg(\overrightarrow{H_1}/H_{1,2})$	axiom (18)
2	$H_1 \leq H_2 \wedge \neg(\overrightarrow{H_1}/H_1) \Rightarrow$ $((\top; \ell = 0 \wedge (\overleftarrow{H_1}/H_{1,2})) \vee \int H_{1,2} = 0$	1, def. of $\leq$
3	$\int H_{1,2} = 0 \Rightarrow \neg(\top; \lceil H_2 \wedge \neg H_1 \rceil; \int H_1 = 0)$	DC
4	$(\top; \lceil H_2 \wedge \neg H_1 \rceil; \int H_{1,2} = 0) \Rightarrow$ $(\top; \overleftarrow{H_2 \wedge \neg H_1} \wedge \int H_{1,2} = 0)$	axiom (4), DC
5	$(\top; \overleftarrow{\neg H_1 \wedge H_{1,2}} \wedge \int H_{1,2} = 0) \Rightarrow$ $(\top; (\overleftarrow{\neg H_1}/H_{1,2}) \wedge \int H_{1,2} = 0)$	axiom (16), DC
6	$(\top; (\overleftarrow{\neg H_1}/H_{1,2}) \wedge \int H_{1,2} = 0) \Rightarrow$ $(\top; (\overleftarrow{\neg H_1}/H_{1,2}) \wedge \ell = 0)$	axiom (10), DC
7	$H_1 \leq H_2 \wedge \neg(\overrightarrow{H_1}/H_1) \Rightarrow$ $\neg(\top; \lceil H_2 \wedge \neg H_1 \rceil; \int H_{1,2} = 0)$	2, 3, 6 and (54)

(41):

1	$H_1 \leq H_2 \wedge \neg(\overrightarrow{H_1}/H_1) \wedge \int H_1 = 0 \Rightarrow \int H_{1,2} = 0$	(40), DC
2	$\int H_{1,2} = 0 \wedge (\top; \ell = 0 \wedge (\overleftarrow{H_i}/H_{1,2})) \Rightarrow (\overleftarrow{H_1}/H_{1,2})$	axiom (10), DC
3	$(\overleftarrow{H_1}/H_{1,2}) \Rightarrow (\overleftarrow{H_1}/H_1)$	axiom (18)
4	$\neg(\overrightarrow{H_1}/H_1) \Rightarrow \neg(\overrightarrow{H_1}/H_{1,2})$	axiom (18)
5	$\neg(\overleftarrow{H_{1,2}}/H_{1,2}) \Rightarrow \neg(\overleftarrow{H_2}/H_2)$	axiom (19)
6	$H_1 \leq H_2 \wedge \neg(\overrightarrow{H_1}/H_1) \wedge \int H_1 = 0 \Rightarrow$ $((\overleftarrow{H_2}/H_2) \Rightarrow (\overleftarrow{H_1}/H_1))$	1-5, def. of $\leq$

$\dashv$