7

Probability

Case Study The Monty Hall Problem

On the game show *Let's Make a Deal*, you are shown three doors, A, B, and C, and behind one of them is the Big Prize. After you select one of them—say, door A—to make things more interesting the host (Monty Hall) opens one of the other doors—say, door B—revealing that the Big Prize is not there. He then offers you the opportunity to change your selection to the remaining door, door C. Should you switch or stick with your original guess? **Does it make any difference?**

Everett Collection

Web Site

www.FiniteMath.org

At the Web site you will find:

- Section by section tutorials, including game tutorials with randomized quizzes
- A detailed chapter summary
- A true/false quiz
- A Markov system simulation and matrix algebra tool
- Additional review exercises

Introduction

What is the probability of winning the lottery twice? What are the chances that a college athlete whose drug test is positive for steroid use is actually using steroids? You are playing poker and have been dealt two iacks. What is the likelihood that one of the next three cards you are dealt will also be a jack? These are all questions about probability.

Understanding probability is important in many fields, ranging from risk management in business through hypothesis testing in psychology to quantum mechanics in physics. Historically, the theory of probability arose in the sixteenth and seventeenth centuries from attempts by mathematicians such as Gerolamo Cardano, Pierre de Fermat, Blaise Pascal, and Christiaan Huygens to understand games of chance. Andrey Nikolaevich Kolmogorov set forth the foundations of modern probability theory in his 1933 book *Foundations of the Theory of Probability.*

The goal of this chapter is to familiarize you with the basic concepts of modern probability theory and to give you a working knowledge that you can apply in a variety of situations. In the first two sections, the emphasis is on translating real-life situations into the language of sample spaces, events, and probability. Once we have mastered the language of probability, we spend the rest of the chapter studying some of its theory and applications. The last section gives an interesting application of both probability and matrix arithmetic.

7.1 Sample Spaces and Events

Sample Spaces

At the beginning of a football game, to ensure fairness, the referee tosses a coin to decide who will get the ball first. When the ref tosses the coin and observes which side faces up, there are two possible results: heads (H) and tails (T). These are the *only* possible results, ignoring the (remote) possibility that the coin lands on its edge. The act of tossing the coin is an example of an **experiment**. The two possible results, H and T, are possible **outcomes** of the experiment, and the set $S = \{H, T\}$ of all possible outcomes is the **sample space** for the experiment.

Experiments, Outcomes, and Sample Spaces

An **experiment** is an occurrence with a result, or **outcome**, that is uncertain before the experiment takes place. The set of all possible outcomes is called the **sample space** for the experiment.

Quick Examples

1. *Experiment:* Flip a coin and observe the side facing up.
   Outcomes: H, T
   Sample Space: $S = \{H,\ T\}$
2. *Experiment:* Select a student in your class.
   Outcomes: The students in your class
   Sample Space: The set of students in your class

3. *Experiment:* Select a student in your class and observe the color of his or her hair.
Outcomes: red, black, brown, blond, green, . . .
Sample Space: {red, black, brown, blond, green, . . .}

4. *Experiment:* Cast a die and observe the number facing up.
Outcomes: 1, 2, 3, 4, 5, 6
Sample Space: $S = \{1, 2, 3, 4, 5, 6\}$

5. *Experiment:* Cast two distinguishable dice (see Example 1(a) of Section 6.1) and observe the numbers facing up.
Outcomes: (1, 1), (1, 2), . . . , (6, 6) (36 outcomes)

$$\text{Sample Space: } S = \left\{\begin{array}{llllll} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6), \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6), \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6), \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6), \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6), \\ (6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6) \end{array}\right\}$$

$$n(S) = \text{the number of outcomes in } S = 36$$

6. *Experiment:* Cast two indistinguishable dice (see Example 1(b) of Section 6.1) and observe the numbers facing up.
Outcomes: (1, 1), (1, 2), . . . , (6, 6) (21 outcomes)

$$\text{Sample Space: } S = \left\{\begin{array}{rrrrrr} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6), \\ & (2,2), & (2,3), & (2,4), & (2,5), & (2,6), \\ & & (3,3), & (3,4), & (3,5), & (3,6), \\ & & & (4,4), & (4,5), & (4,6), \\ & & & & (5,5), & (5,6), \\ & & & & & (6,6) \end{array}\right\}$$

$$n(S) = 21$$

7. *Experiment:* Cast two dice and observe the *sum* of the numbers facing up.
Outcomes: 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12
Sample Space: $S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

8. *Experiment:* Choose 2 cars (without regard to order) at random from a fleet of 10.
Outcomes: Collections of 2 cars chosen from 10
Sample Space: The set of all collections of 2 cars chosen from 10

$$n(S) = C(10, 2) = 45$$

The following example introduces a sample space that we'll use in several other examples.

Cindy Charles/PhotoEdit

EXAMPLE 1 **School and Work**

In a survey conducted by the Bureau of Labor Statistics,* the high school graduating class of 2007 was divided into those who went on to college and those who did not. Those who went on to college were further divided into those who went to two-year colleges and those who went to four-year colleges. All graduates were also asked whether they were working or not. Find the sample space for the experiment "Select a member of the high school graduating class of 2007 and classify his or her subsequent school and work activity."

Solution The tree in Figure 1 shows the various possibilities.

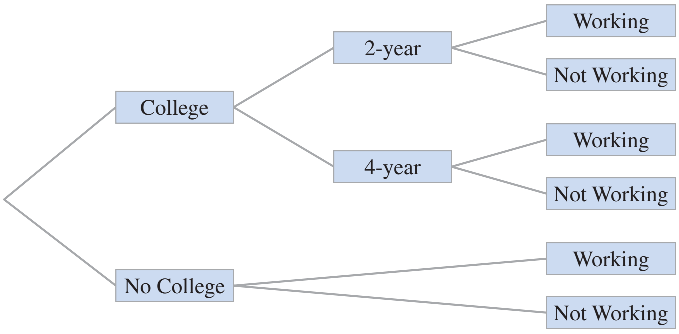


Figure 1

The sample space is

$$\begin{aligned} S = \{&\text{2-year college \& working, 2-year college \& not working,}\\ &\text{4-year college \& working, 4-year college \& not working,}\\ &\text{no college \& working, no college \& not working}\}. \end{aligned}$$

*"College Enrollment and Work Activity of High School Graduates News Release," U.S. Bureau of Labor Statistics, April 2008, available at www.bls.gov/news.release/hsgec.htm.

Events

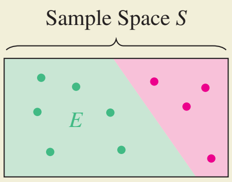
In Example 1, suppose we are interested in the event that a 2003 high school graduate was working. In mathematical language, we are interested in the *subset* of the sample space consisting of all outcomes in which the graduate was working.

Events

Given a sample space S, an **event** E is a subset of S. The outcomes in E are called the **favorable** outcomes. We say that E **occurs** in a particular experiment if the outcome of that experiment is one of the elements of E—that is, if the outcome of the experiment is favorable.

Visualizing an Event

In the following figure, the favorable outcomes (events in E) are shown in green.



Quick Examples

1. *Experiment:* Roll a die and observe the number facing up.

   $$S = \{1, 2, 3, 4, 5, 6\}$$

   Event: E: The number observed is odd.

   $$E = \{1, 3, 5\}$$

2. *Experiment:* Roll two distinguishable dice and observe the numbers facing up.

   $$S = \{(1, 1), (1, 2), \ldots, (6, 6)\}$$

   Event: F: The dice show the same number.

   $$F = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$$

3. *Experiment:* Roll two distinguishable dice and observe the numbers facing up.

   $$S = \{(1, 1), (1, 2), \ldots, (6, 6)\}$$

   Event: G: The sum of the numbers is 1.

   $G = \emptyset$ — There are no favorable outcomes.

4. *Experiment:* Select a city beginning with "J."
   Event: E: The city is Johannesburg.

   $E = \{\text{Johannesburg}\}$ — An event can consist of a single outcome.

5. *Experiment:* Roll a die and observe the number facing up.
   Event: E: The number observed is either even or odd.

   $E = S = \{1, 2, 3, 4, 5, 6\}$ — An event can consist of all possible outcomes.

6. *Experiment:* Select a student in your class.
   Event: E: The student has red hair.

   $$E = \{\text{red-haired students in your class}\}$$

7. *Experiment:* Draw a hand of two cards from a deck of 52.
   Event: H: Both cards are diamonds.

   H is the set of all hands of 2 cards chosen from 52 such that both cards are diamonds.

Here are some more examples of events.

EXAMPLE 2 Dice

We roll a red die and a green die and observe the numbers facing up. Describe the following events as subsets of the sample space.

a. E: The sum of the numbers showing is 6.

b. F: The sum of the numbers showing is 2.

Solution Here (again) is the sample space for the experiment of throwing two dice.

$$S = \left\{\begin{matrix} (1, 1), & (1, 2), & (1, 3), & (1, 4), & (1, 5), & (1, 6), \\ (2, 1), & (2, 2), & (2, 3), & (2, 4), & (2, 5), & (2, 6), \\ (3, 1), & (3, 2), & (3, 3), & (3, 4), & (3, 5), & (3, 6), \\ (4, 1), & (4, 2), & (4, 3), & (4, 4), & (4, 5), & (4, 6), \\ (5, 1), & (5, 2), & (5, 3), & (5, 4), & (5, 5), & (5, 6), \\ (6, 1), & (6, 2), & (6, 3), & (6, 4), & (6, 5), & (6, 6) \end{matrix}\right\}$$

a. In mathematical language, E is the subset of S that consists of all those outcomes in which the sum of the numbers showing is 6. Here is the sample space once again, with the outcomes in question shown in color:

$$S = \left\{\begin{matrix} (1, 1), & (1, 2), & (1, 3), & (1, 4), & \color{magenta}{(1, 5),} & (1, 6), \\ (2, 1), & (2, 2), & (2, 3), & \color{magenta}{(2, 4),} & (2, 5), & (2, 6), \\ (3, 1), & (3, 2), & \color{magenta}{(3, 3),} & (3, 4), & (3, 5), & (3, 6), \\ (4, 1), & \color{magenta}{(4, 2),} & (4, 3), & (4, 4), & (4, 5), & (4, 6), \\ \color{magenta}{(5, 1),} & (5, 2), & (5, 3), & (5, 4), & (5, 5), & (5, 6), \\ (6, 1), & (6, 2), & (6, 3), & (6, 4), & (6, 5), & (6, 6) \end{matrix}\right\}$$

Thus, $E = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}$.

b. The only outcome in which the numbers showing add to 2 is (1, 1). Thus,

$$F = \{(1, 1)\}.$$

EXAMPLE 3 School and Work

Let S be the sample space of Example 1. List the elements in the following events:

a. The event E that a 2007 high school graduate was working.

b. The event F that a 2007 high school graduate was not going to a two-year college.

Solution

a. We had this sample space:

$S = \{$2-year college & working, two-year college & not working,
4-year college & working, four-year college & not working,
no college & working, no college & not working$\}$.

We are asked for the event that a graduate was working. Whenever we encounter a phrase involving "the event that . . . ," we mentally translate this into mathematical language by changing the wording.

Replace the phrase "the event that . . . " by the phrase "the subset of the sample space consisting of all outcomes in which . . . "

Thus we are interested in the subset of the sample space consisting of all outcomes in which the graduate was working. This gives

$E = \{$two-year college & working, four-year college & working, no college & working$\}$.

The outcomes in E are illustrated by the shaded cells in Figure 2.

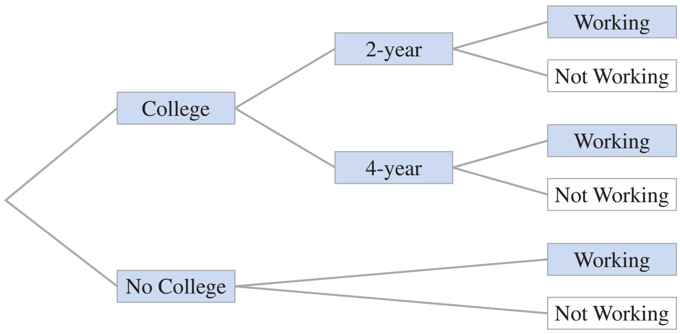


Figure 2

b. We are looking for the event that a graduate was not going to a two-year college, that is, the subset of the sample space consisting of all outcomes in which the graduate was not going to a two-year college. Thus,

$F = \{$four-year college & working, four-year college & not working, no college & working, no college & not working$\}$.

The outcomes in F are illustrated by the shaded cells in Figure 3.

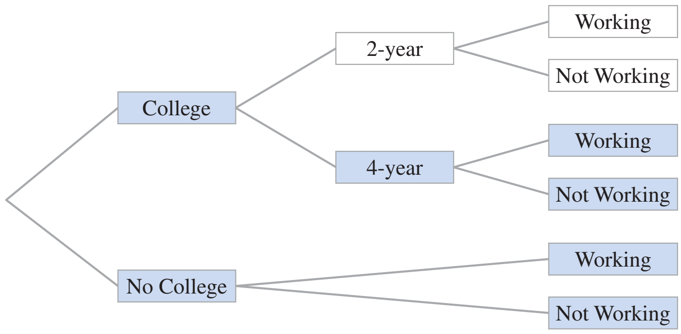


Figure 3

Complement, Union, and Intersection of Events

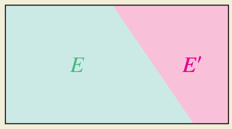
Events may often be described in terms of other events, using set operations such as complement, union, and intersection.

Complement of an Event

The **complement** of an event E is the set of outcomes not in E. Thus, the complement of E represents the event that E *does not occur*.

Visualizing the Complement

Sample Space S



Quick Examples

1. You take four shots at the goal during a soccer game and record the number of times you score. Describe the event that you score at least twice, and also its complement.

$S = \{0, 1, 2, 3, 4\}$ Set of outcomes

$E = \{2, 3, 4\}$ Event that you score at least twice

$E' = \{0, 1\}$ Event that you do not score at least twice

2. You roll a red die and a green die and observe the two numbers facing up. Describe the event that the sum of the numbers is not 6.

$S = \{(1, 1), (1, 2), \ldots, (6, 6)\}$

$F = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}$ Sum of numbers is 6.

$$F' = \left\{\begin{array}{cccccc} (1,1), & (1,2), & (1,3), & (1,4), & & (1,6), \\ (2,1), & (2,2), & (2,3), & & (2,5), & (2,6), \\ (3,1), & (3,2), & & (3,4), & (3,5), & (3,6), \\ (4,1), & & (4,3), & (4,4), & (4,5), & (4,6), \\ & (5,2), & (5,3), & (5,4), & (5,5), & (5,6), \\ (6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6) \end{array}\right\}$$

Sum of numbers is not 6.

Union of Events

The **union** of the events E and F is the set of all outcomes in E or F (or both). Thus, $E \cup F$ represents the event that E occurs *or* F occurs (or both).*

* **NOTE** As in the preceding chapter, when we use the word *or*, we agree to mean one or the other *or both*. This is called the **inclusive or** and mathematicians have agreed to take this as the meaning of *or* to avoid confusion.

Quick Example

Roll a die.

E: The outcome is a 5; $E = \{5\}$.

F: The outcome is an even number; $F = \{2, 4, 6\}$.

$E \cup F$: The outcome is either a 5 *or* an even number;
$E \cup F = \{2, 4, 5, 6\}$.

Intersection of Events

The **intersection** of the events E and F is the set of all outcomes common to E and F. Thus, $E \cap F$ represents the event that both E *and* F occur.

Quick Example

Roll two dice; one red and one green.

E: The red die is 2.

F: The green die is odd.

$E \cap F$: The red die is 2 and the green die is odd;

$E \cap F = \{(2, 1), (2, 3), (2, 5)\}$.

EXAMPLE 4 Weather

Let R be the event that it will rain tomorrow, let P be the event that it will be pleasant, let C be the event that it will be cold, and let H be the event that it will be hot.

a. Express in words: $R \cap P'$, $R \cup (P \cap C)$.

b. Express in symbols: Tomorrow will be either a pleasant day or a cold and rainy day; it will not, however, be hot.

Solution The key here is to remember that intersection corresponds to *and* and union to *or*.

a. $R \cap P'$ is the event that it will rain *and* it will not be pleasant.
$R \cup (P \cap C)$ is the event that either it will rain, or it will be pleasant and cold.

b. If we rephrase the given statement using *and* and *or* we get "Tomorrow will be either a pleasant day or a cold and rainy day, and it will not be hot."

$$[P \cup (C \cap R)] \cap H' \qquad \text{Pleasant, or cold and rainy, and not hot.}$$

The nuances of the English language play an important role in this formulation. For instance, the effect of the pause (comma) after "rainy day" suggests placing the preceding clause $P \cup (C \cap R)$ in parentheses. In addition, the phrase "cold and rainy" suggests that C and R should be grouped together in their own parentheses.

EXAMPLE 5 iPods, iPhones, and Macs

(Compare Example 3 in Section 6.2.) The following table shows sales, in millions of units, of iPods, iPhones, and Macintosh computers in the last three quarters of 2008.*

	iPods	iPhones	Macs	Total
2008 Q2	10.6	1.7	2.3	14.6
2008 Q3	11.0	0.7	2.5	14.2
2008 Q4	11.1	6.9	2.6	20.6
Total	32.7	9.3	7.4	49.4

*Figures are rounded. Source: Company reports (www.apple.com/investor/).

Consider the experiment in which a device is selected at random from those represented in the table. Let D be the event that it was an iPod, let M be the event that it was a Mac, and let A be the event that it was sold in the second quarter of 2008. Describe the following events and compute their cardinality.

a. A' **b.** $D \cap A$ **c.** $M \cup A'$

Solution Before we answer the questions, note that S is the set of all items represented in the table, so S has a total of 49.4 million outcomes.

a. A' is the event that the item was not sold in the second quarter of 2008. Its cardinality is

$$n(A') = n(S) - n(A) = 49.4 - 14.6 = 34.8 \text{ million items.}$$

b. $D \cap A$ is the event that it was an iPod sold in the second quarter of 2008. Referring to the table, we find

$$n(D \cap A) = 10.6 \text{ million items.}$$

c. $M \cup A'$ is the event that either it was a Mac or it was not sold in the second quarter of 2008:

	iPods	iPhones	Macs	Total
2008 Q2	10.6	1.7	2.3	14.6
2008 Q3	11.0	0.7	2.5	14.2
2008 Q4	11.1	6.9	2.6	20.6
Total	32.7	9.3	7.4	49.4

To compute its cardinality, we use the formula for the cardinality of a union:

$$\begin{aligned} n(M \cup A') &= n(M) + n(A') - n(M \cap A') \\ &= 7.4 + 34.8 - (2.5 + 2.6) = 37.1 \text{ million items.} \end{aligned}$$

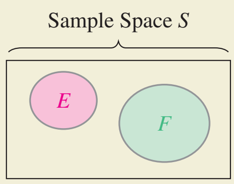
Before we go on... We could shorten the calculation in part (c) of Example 5 even further using De Morgan's law to write $n(M \cup A') = n((M' \cap A)') = 49.4 - (10.6 + 1.7) = 37.1$ million items, a calculation suggested by looking at the table. ■

The case where $E \cap F$ is empty is interesting, and we give it a name.

Mutually Exclusive Events

If E and F are events, then E and F are said to be **disjoint** or **mutually exclusive** if $E \cap F$ is empty. (Hence, they have no outcomes in common.)

Visualizing Mutually Exclusive Events



Interpretation

It is impossible for mutually exclusive events to occur simultaneously.

Quick Examples

In each of the following examples, E and F are mutually exclusive events.

1. Roll a die and observe the number facing up. E: The outcome is even; F: The outcome is odd.

$$E = \{2, 4, 6\}, F = \{1, 3, 5\}$$

2. Toss a coin three times and record the sequence of heads and tails. E: All three tosses land the same way up, F: One toss shows heads and the other two show tails.

$$E = \{\text{HHH, TTT}\}, F = \{\text{HTT, THT, TTH}\}$$

3. Observe tomorrow's weather. E: It is raining; F: There is not a cloud in the sky.

FAQs

Specifying the Sample Space

Q: *How do I determine the sample space in a given application?*

A: Strictly speaking, an experiment should include a description of what kinds of objects are in the sample space, as in:

Cast a die and observe the number facing up.
Sample space: the possible numbers facing up, {1, 2, 3, 4, 5, 6}.

Choose a person at random and record her social security number and whether she is blonde.
Sample space: pairs (9-digit number, Y/N).

However, in many of the scenarios discussed in this chapter and the next, an experiment is specified more vaguely, as in "Select a student in your class." In cases like this, the nature of the sample space should be determined from the context. For example, if the discussion is about grade-point averages and gender, the sample space can be taken to consist of pairs (grade-point average, M/F).

7.1 EXERCISES

▼ more advanced ◆ challenging

T indicates exercises that should be solved using technology

In Exercises 1–18, describe the sample space S of the experiment and list the elements of the given event. (Assume that the coins are distinguishable and that what is observed are the faces or numbers that face up.) HINT [See Examples 1–3.]

1. Two coins are tossed; the result is at most one tail.
2. Two coins are tossed; the result is one or more heads.
3. Three coins are tossed; the result is at most one head.
4. Three coins are tossed; the result is more tails than heads.
5. Two distinguishable dice are rolled; the numbers add to 5.
6. Two distinguishable dice are rolled; the numbers add to 9.
7. Two indistinguishable dice are rolled; the numbers add to 4.
8. Two indistinguishable dice are rolled; one of the numbers is even and the other is odd.

9. Two indistinguishable dice are rolled; both numbers are prime.[1]

10. Two indistinguishable dice are rolled; neither number is prime.

11. A letter is chosen at random from those in the word *Mozart*; the letter is a vowel.

12. A letter is chosen at random from those in the word *Mozart*; the letter is neither *a* nor *m*.

13. A sequence of two different letters is randomly chosen from those of the word *sore*; the first letter is a vowel.

14. A sequence of two different letters is randomly chosen from those of the word *hear*; the second letter is not a vowel.

15. A sequence of two different digits is randomly chosen from the digits 0–4; the first digit is larger than the second.

16. A sequence of two different digits is randomly chosen from the digits 0–4; the first digit is twice the second.

17. You are considering purchasing either a domestic car, an imported car, a van, an antique car, or an antique truck; you do not buy a car.

18. You are deciding whether to enroll for Psychology 1, Psychology 2, Economics 1, General Economics, or Math for Poets; you decide to avoid economics.

19. A packet of gummy candy contains four strawberry gums, four lime gums, two black currant gums, and two orange gums. April May sticks her hand in and selects four at random. Complete the following sentences:

    a. The sample space is the set of . . .

    b. April is particularly fond of combinations of two strawberry and two black currant gums. The event that April will get the combination she desires is the set of . . .

20. A bag contains three red marbles, two blue ones, and four yellow ones, and Alexandra pulls out three of them at random. Complete the following sentences:

    a. The sample space is the set of . . .

    b. The event that Alexandra gets one of each color is the set of . . .

21. ▼ President Barack H. Obama's cabinet consisted of the Secretaries of Agriculture, Commerce, Defense, Education, Energy, Health and Human Services, Homeland Security, Housing and Urban Development, Interior, Labor, State, Transportation, Treasury, Veterans Affairs, and the Attorney General.[2] Assuming that President Obama had 20 candidates, including Hillary Clinton, to fill these posts (and wished to assign no one to more than one post), complete the following sentences:

    a. The sample space is the set of . . .

    b. The event that Hillary Clinton is the Secretary of State is the set of . . .

[1] A positive integer is **prime** if it is neither 1 nor a product of smaller integers.

[2] Source: The White House Web site (www.whitehouse.gov).

22. ▼ A poker hand consists of a set of 5 cards chosen from a standard deck of 52 playing cards. You are dealt a poker hand. Complete the following sentences:

    a. The sample space is the set of . . .

    b. The event "a full house" is the set of . . . (Recall that a full house is three cards of one denomination and two of another.)

Suppose two dice (one red, one green) are rolled. Consider the following events. A: the red die shows 1; B: the numbers add to 4; C: at least one of the numbers is 1; and D: the numbers do not add to 11. In Exercises 23–30, express the given event in symbols and say how many elements it contains. HINT [See Example 5.]

23. The red die shows 1 and the numbers add to 4.

24. The red die shows 1 but the numbers do not add to 11.

25. The numbers do not add to 4.

26. The numbers add to 11.

27. The numbers do not add to 4, but they do add to 11.

28. Either the numbers add to 11 or the red die shows a 1.

29. At least one of the numbers is 1 or the numbers add to 4.

30. Either the numbers add to 4, or they add to 11, or at least one of them is 1.

Let W be the event that you will use the Web site tonight, let I be the event that your math grade will improve, and let E be the event that you will use the Web site every night. In Exercises 31–38, express the given event in symbols.

31. You will use the Web site tonight and your math grade will improve.

32. You will use the Web site tonight or your math grade will not improve.

33. Either you will use the Web site every night, or your math grade will not improve.

34. Your math grade will not improve even though you use the Web site every night.

35. ▼ Either your math grade will improve, or you will use the Web site tonight but not every night.

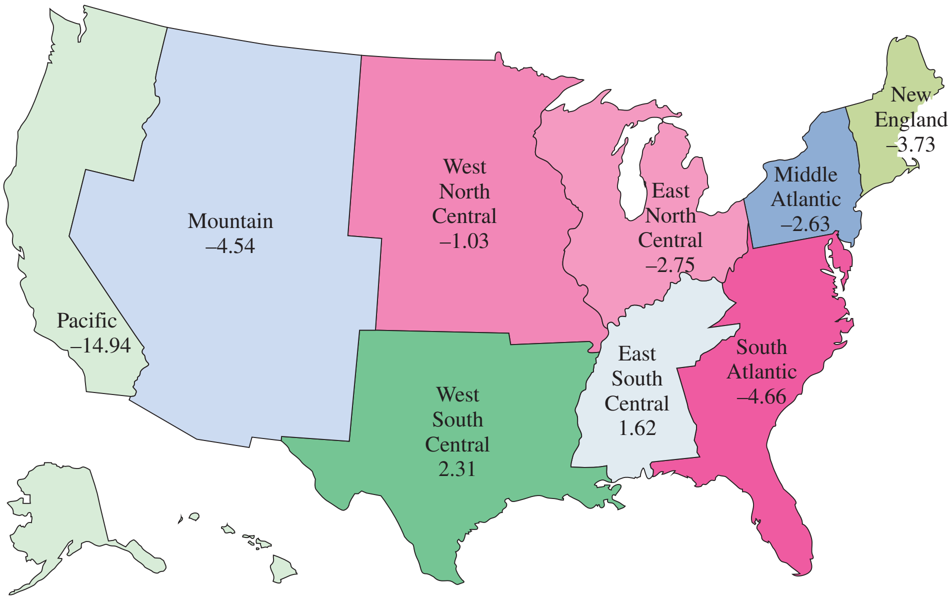
36. ▼ You will use the Web site either tonight or every night, and your grade will improve.

37. ▼ (Compare Exercise 35.) Either your math grade will improve or you will use the Web site tonight, but you will not use it every night.

38. ▼ (Compare Exercise 36.) You will either use the Web site tonight, or you will use it every night and your grade will improve.

APPLICATIONS

Housing Prices *Exercises 39–44 are based on the following map, which shows the percentage increase in housing prices*



from September 30, 2007 to September 30, 2008 in each of nine regions (U.S. Census divisions)[3].

39. You are choosing a region of the country to move to. Describe the event E that the region you choose saw a decrease in housing prices of 4% or more.

40. You are choosing a region of the country to move to. Describe the event E that the region you choose saw an increase in housing prices.

41. You are choosing a region of the country to move to. Let E be the event that the region you choose saw a decrease in housing prices of 4% or more, and let F be the event that the region you choose is on the east coast. Describe the events $E \cup F$ and $E \cap F$ both in words and by listing the outcomes of each.

42. You are choosing a region of the country to move to. Let E be the event that the region you choose saw an increase in housing prices, and let F be the event that the region you choose is not on the east coast. Describe the events $E \cup F$ and $E \cap F$ both in words and by listing the outcomes of each.

43. ▼ You are choosing a region of the country to move to. Which of the following pairs of events are mutually exclusive?

a. E: You choose a region from among the two with the highest percentage decrease in housing prices.
F: You choose a region that is not on the east or west coast.

b. E: You choose a region from among the two with the highest percentage decrease in housing prices.
F: You choose a region that is not on the west coast.

[3] Source: Third Quarter 2008 House Price Index, Office of Federal Housing Enterprise Oversight; available online at www.ofheo.gov/media/pdf/3q08hpi.pdf.

44. ▼ You are choosing a region of the country to move to. Which of the following pairs of events are mutually exclusive?

a. E: You choose a region from among the three best performing in terms of housing prices.
F: You choose a region from among the central divisions.

b. E: You choose a region from among the three best performing in terms of housing prices.
F: You choose a region from among the noncentral divisions.

Publishing *Exercises 45–52 are based on the following table, which shows the results of a survey of authors by a (fictitious) publishing company.* HINT [See Example 5.]

	New Authors	Established Authors	Total
Successful	5	25	30
Unsuccessful	15	55	70
Total	20	80	100

Consider the following events: S: an author is successful; U: an author is unsuccessful; N: an author is new; and E: an author is established.

45. Describe the events $S \cap N$ and $S \cup N$ in words. Use the table to compute $n(S \cap N)$ and $n(S \cup N)$.

46. Describe the events $N \cap U$ and $N \cup U$ in words. Use the table to compute $n(N \cap U)$ and $n(N \cup U)$.

47. Which of the following pairs of events are mutually exclusive: N and E; N and S; S and E?

48. Which of the following pairs of events are mutually exclusive: U and E; U and S; S and N?

49. Describe the event $S \cap N'$ in words and find the number of elements it contains.

50. Describe the event $U \cup E'$ in words and find the number of elements it contains.

51. ▼ What percentage of established authors are successful? What percentage of successful authors are established?

52. ▼ What percentage of new authors are unsuccessful? What percentage of unsuccessful authors are new?

Exercises 53–60 are based on the following table, which shows the performance of a selection of 100 stocks after one year. (Take S to be the set of all stocks represented in the table.)

	Companies			
	Pharmaceutical ***P***	**Electronic** ***E***	**Internet** ***I***	**Total**
Increased ***V***	10	5	15	30
Unchanged* ***N***	30	0	10	40
Decreased ***D***	10	5	15	30
Total	50	10	40	100

* If a stock stayed within 20% of its original value, it is classified as "unchanged."

53. Use symbols to describe the event that a stock's value increased but it was not an Internet stock. How many elements are in this event?

54. Use symbols to describe the event that an Internet stock did not increase. How many elements are in this event?

55. Compute $n(P' \cup N)$. What does this number represent?

56. Compute $n(P \cup N')$. What does this number represent?

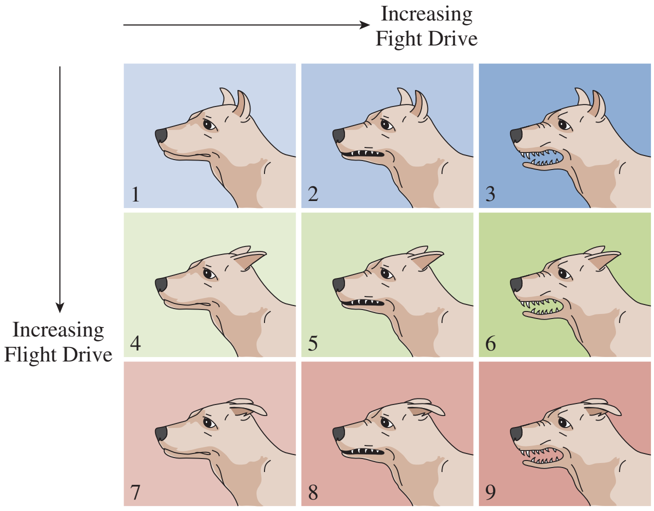
57. Find all pairs of mutually exclusive events among the events P, E, I, V, N, and D.

58. Find all pairs of events that are not mutually exclusive among the events P, E, I, V, N, and D.

59. ▼ Calculate $\dfrac{n(V \cap I)}{n(I)}$. What does the answer represent?

60. ▼ Calculate $\dfrac{n(D \cap I)}{n(D)}$. What does the answer represent?

Animal Psychology *Exercises 61–66 concern the following chart, which shows the way in which a dog moves its facial muscles when torn between the drives of fight and flight.*[4] *The "fight" drive increases from left to right; the "flight" drive increases from top to bottom. (Notice that an increase in the "fight" drive causes its upper lip to lift, while an increase in the "flight" drive draws its ears downward.)*



61. ▼ Let E be the event that the dog's flight drive is the strongest, let F be the event that the dog's flight drive is weakest, let G be the event that the dog's fight drive is the strongest, and let H be the event that the dog's fight drive is weakest. Describe the following events in terms of E, F, G, and H using the symbols $\cap$, $\cup$, and $'$.

a. The dog's flight drive is not strongest and its fight drive is weakest.
b. The dog's flight drive is strongest or its fight drive is weakest.
c. Neither the dog's flight drive nor its fight drive is strongest.

62. ▼ Let E be the event that the dog's flight drive is the strongest, let F be the event that the dog's flight drive is weakest, let G be the event that the dog's fight drive is the strongest, and let H be the event that the dog's fight drive is weakest. Describe the following events in terms of E, F, G, and H using the symbols $\cap$, $\cup$, and $'$.

a. The dog's flight drive is weakest and its fight drive is not weakest.
b. The dog's flight drive is not strongest or its fight drive is weakest.
c. Either the dog's flight drive or its fight drive fails to be strongest.

63. ▼ Describe the following events explicitly (as subsets of the sample space):

a. The dog's fight and flight drives are both strongest.
b. The dog's fight drive is strongest, but its flight drive is neither weakest nor strongest.

64. ▼ Describe the following events explicitly (as subsets of the sample space):

a. Neither the dog's fight drive nor its flight drive is strongest.
b. The dog's fight drive is weakest, but its flight drive is neither weakest nor strongest.

[4] Source: *On Aggression* by Konrad Lorenz (Fakenham, Norfolk: University Paperback Edition, Cox & Wyman Limited, 1967).

65. ▼ Describe the following events in words:

a. $\{1, 4, 7\}$
b. $\{1, 9\}$
c. $\{3, 6, 7, 8, 9\}$

66. ▼ Describe the following events in words:

a. $\{7, 8, 9\}$
b. $\{3, 7\}$
c. $\{1, 2, 3, 4, 7\}$

Exercises 67–74 use counting arguments from the preceding chapter.

67. ▼ ***Gummy Bears*** A bag contains six gummy bears. Noel picks four at random. How many possible outcomes are there? If one of the gummy bears is raspberry, how many of these outcomes include the raspberry gummy bear?

68. ▼ ***Chocolates*** My couch potato friend enjoys sitting in front of the TV and grabbing handfuls of 5 chocolates at random from his snack jar. Unbeknownst to him, I have replaced one of the 20 chocolates in his jar with a cashew. (He hates cashews with a passion.) How many possible outcomes are there the first time he grabs 5 chocolates? How many of these include the cashew?

69. ▼ ***Horse Races*** The seven contenders in the fifth horse race at Aqueduct on February 18, 2002, were: Pipe Bomb, Expect a Ship, All That Magic, Electoral College, Celera, Cliff Glider, and Inca Halo.[5] You are interested in the first three places (winner, second place, and third place) for the race.

a. Find the cardinality $n(S)$ of the sample space S of all possible finishes of the race. (A finish for the race consists of a first, second, and third place winner.)

b. Let E be the event that Electoral College is in second or third place, and let F be the event that Celera is the winner. Express the event $E \cap F$ in words, and find its cardinality.

70. ▼ ***Intramurals*** The following five teams will be participating in Urban University's hockey intramural tournament: the Independent Wildcats, the Phi Chi Bulldogs, the Gate Crashers, the Slide Rule Nerds, and the City Slickers. Prizes will be awarded for the winner and runner-up.

a. Find the cardinality $n(S)$ of the sample space S of all possible outcomes of the tournament. (An outcome of the tournament consists of a winner and a runner-up.)

b. Let E be the event that the City Slickers are runners-up, and let F be the event that the Independent Wildcats are neither the winners nor runners-up. Express the event $E \cup F$ in words, and find its cardinality.

[5] Source: *Newsday*, Feb. 18, 2002, p. A36.

In Exercises 71–74, Pablo randomly picks three marbles from a bag of eight marbles (four red ones, two green ones, and two yellow ones).

71. ▼ How many outcomes are there in the sample space?

72. ▼ How many outcomes are there in the event that Pablo grabs three red marbles?

73. ▼ How many outcomes are there in the event that Pablo grabs one marble of each color?

74. ▼ How many outcomes are there in the event that Pablo's marbles are not all the same color?

COMMUNICATION AND REASONING EXERCISES

75. Complete the following sentence. An event is a ____.

76. Complete the following sentence. Two events E and F are mutually exclusive if their intersection is _____.

77. If E and F are events, then $(E \cap F)'$ is the event that ____.

78. If E and F are events, then $(E' \cap F')$ is the event that ____.

79. Let E be the event that you meet a tall dark stranger. Which of the following could reasonably represent the experiment and sample space in question?

(A) You go on vacation and lie in the sun; S is the set of cloudy days.
(B) You go on vacation and spend an evening at the local dance club; S is the set of people you meet.
(C) You go on vacation and spend an evening at the local dance club; S is the set of people you do not meet.

80. Let E be the event that you buy a Porsche. Which of the following could reasonably represent the experiment and sample space in question?

(A) You go to an auto dealership and select a Mustang; S is the set of colors available.
(B) You go to an auto dealership and select a red car; S is the set of cars you decide not to buy.
(C) You go to an auto dealership and select a red car; S is the set of car models available.

81. ▼ True or false? Every set S is the sample space for some experiment. Explain.

82. ▼ True or false? Every sample space S is a finite set. Explain.

83. ▼ Describe an experiment in which a die is cast and the set of outcomes is $\{0, 1\}$.

84. ▼ Describe an experiment in which two coins are flipped and the set of outcomes is $\{0, 1, 2\}$.

85. ▼ Two distinguishable dice are rolled. Could there be two mutually exclusive events that both contain outcomes in which the numbers facing up add to 7?

86. ▼ Describe an experiment in which two dice are rolled and describe two mutually exclusive events that both contain outcomes in which both dice show a 1.

7.2 Relative Frequency

Suppose you have a coin that you think is not fair and you would like to determine the likelihood that heads will come up when it is tossed. You could estimate this likelihood by tossing the coin a large number of times and counting the number of times heads comes up. Suppose, for instance, that in 100 tosses of the coin, heads comes up 58 times. The fraction of times that heads comes up, $58/100 = .58$, is the **relative frequency**, or **estimated probability** of heads coming up when the coin is tossed. In other words, saying that the relative frequency of heads coming up is .58 is the same as saying that heads came up 58% of the time in your series of experiments.

Now let's think about this example in terms of sample spaces and events. First of all, there is an experiment that has been repeated $N = 100$ times: Toss the coin and observe the side facing up. The sample space for this experiment is $S = \{\mathrm{H}, \mathrm{T}\}$. Also, there is an event E in which we are interested: the event that heads comes up, which is $E = \{\mathrm{H}\}$. The number of times E has occurred, or the **frequency** of E, is $fr(E) = 58$. The relative frequency of the event E is then

$$P(E) = \frac{fr(E)}{N} \qquad \frac{\text{Frequency of event } E}{\text{Number of repetitions } N}$$

$$= \frac{58}{100} = .58.$$

Notes

1. The relative frequency gives us an *estimate* of the likelihood that heads will come up when that particular coin is tossed. This is why statisticians often use the alternative term *estimated probability* to describe it.
2. The larger the number of times the experiment is performed, the more accurate an estimate we expect this estimated probability to be. ■

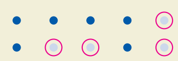
Relative Frequency

When an experiment is performed a number of times, the **relative frequency** or **estimated probability** of an event E is the fraction of times that the event E occurs. If the experiment is performed N times and the event E occurs $fr(E)$ times, then the relative frequency is given by

$$P(E) = \frac{fr(E)}{N}. \qquad \text{Fraction of times } E \text{ occurs}$$

The number $fr(E)$ is called the **frequency** of E. N, the number of times that the experiment is performed, is called the number of **trials** or the **sample size**. If E consists of a single outcome s, then we refer to $P(E)$ as the relative frequency or estimated probability of the outcome s, and we write $P(s)$.

Visualizing Relative Frequency



$$P(E) = \frac{fr(E)}{N} = \frac{4}{10} = .4$$

The collection of the estimated probabilities of *all* the outcomes is the **relative frequency distribution** or **estimated probability distribution**.

Quick Examples

1. **Experiment:** Roll a pair of dice and add the numbers that face up.

   Event: E: The sum is 5.

   If the experiment is repeated 100 times and E occurs on 10 of the rolls, then the relative frequency of E is

   $$P(E) = \frac{fr(E)}{N} = \frac{10}{100} = .10.$$

2. If 10 rolls of a single die resulted in the outcomes 2, 1, 4, 4, 5, 6, 1, 2, 2, 1, then the associated relative frequency distribution is shown in the following table:

Outcome	1	2	3	4	5	6
Rel. Frequency	.3	.3	0	.2	.1	.1

3. **Experiment:** Note the cloud conditions on a particular day in April.

   If the experiment is repeated a number of times, and it is clear 20% of those times, partly cloudy 30% of those times, and overcast the rest of those times, then the relative frequency distribution is:

Outcome	Clear	Partly Cloudy	Overcast
Rel. Frequency	.20	.30	.50

EXAMPLE 1 Sales of Hybrid SUVs

In a survey of 120 hybrid SUVs sold in the United States, 48 were Toyota Highlanders, 28 were Lexus RX 400s, 38 were Ford Escapes, and 6 were other brands.* What is the relative frequency that a hybrid SUV sold in the United States is not a Toyota Highlander?

Solution The experiment consists of choosing a hybrid SUV sold in the United States and determining its type. The sample space suggested by the information given is

$$S = \{\text{Toyota Highlander, Lexus RX 400, Ford Escape, Other}\}$$

and we are interested in the event

$$E = \{\text{Lexus RX 400, Ford Escape, Other}\}.$$

The sample size is $N = 120$ and the frequency of E is $fr(E) = 28 + 38 + 6 = 72$. Thus, the relative frequency of E is

$$P(E) = \frac{fr(E)}{N} = \frac{72}{120} = .6.$$

* The proportions are based on actual U.S. sales in February 2008. Source: www.hybridsuv.com/hybrid-resources/hybrid-suv-sales.

Before we go on... In Example 1, you might ask how accurate the estimate of .6 is or how well it reflects *all* of the hybrid SUVs sold in the United States absent any information about national sales figures. The field of statistics provides the tools needed to say to what extent this estimated probability can be trusted. ■

David Young-Wolff/PhotoEdit

EXAMPLE 2 Auctions on eBay

The following chart shows the results of a survey of 50 paintings on eBay whose listings were close to expiration, examining the number of bids each had received.*

Bids	0	1	2–10	>10
Frequency	31	9	7	3

Consider the experiment in which a painting whose listing is close to expiration is chosen and the number of bids is observed.

a. Find the relative frequency distribution.

b. Find the relative frequency that a painting whose listing was close to expiration had received no more than 10 bids.

Solution

a. The following table shows the relative frequency of each outcome, which we find by dividing each frequency by the sum $N = 50$:

Relative Frequency Distribution

Bids	0	1	2–10	>10
Rel. Frequency	$\frac{31}{50} = .62$	$\frac{9}{50} = .18$	$\frac{7}{50} = .14$	$\frac{3}{50} = .06$

b. Method 1: Computing Directly
$E = \{0 \text{ bids}, 1 \text{ bid}, 2\text{–}10 \text{ bids}\}$; thus,

$$P(E) = \frac{fr(E)}{N} = \frac{31 + 9 + 7}{50} = \frac{47}{50} = .94.$$

Method 2: Using the Relative Frequency Distribution
Notice that we can obtain the same answer from the distribution in part (a) by simply adding the relative frequencies of the outcomes in E:

$$P(E) = .62 + .18 + .14 = .94.$$

*The 50 paintings whose listings were closest to expiration in the category "Art-Direct from Artist" on January 13, 2009 (www.eBay.com).

Before we go on...

Q: *Why did we get the same result in part (b) of Example 2 by simply adding the relative frequencies of the outcomes in E?*

A: The reason can be seen by doing the calculation in the first method a slightly different way:

$$P(E) = \frac{fr(E)}{N} = \frac{31+9+7}{50}$$
$$= \frac{31}{50} + \frac{9}{50} + \frac{7}{50} \qquad \text{Sum of rel. frequencies of the individual outcomes}$$
$$= .62 + .18 + .14 = .94.$$

This property of relative frequency distributions is discussed below.

Following are some important properties of estimated probability that we can observe in Example 2.

Some Properties of Relative Frequency Distributions

Let $S = \{s_1, s_2, \ldots, s_n\}$ be a sample space and let $P(s_i)$ be the relative frequency of the event $\{s_i\}$. Then

1. $0 \le P(s_i) \le 1$
2. $P(s_1) + P(s_2) + \cdots + P(s_n) = 1$
3. If $E = \{e_1, e_2, \ldots, e_r\}$, then $P(E) = P(e_1) + P(e_2) + \cdots + P(e_r)$.

In words:

1. The relative frequency of each outcome is a number between 0 and 1 (inclusive).
2. The relative frequencies of all the outcomes add up to 1.
3. The relative frequency of an event E is the sum of the relative frequencies of the individual outcomes in E.

Relative Frequency and Increasing Sample Size

using Technology

See the Technology Guides at the end of the chapter to see how to use a TI-83/84 Plus or Excel to simulate experiments.

A "fair" coin is one that is as likely to come up heads as it is to come up tails. In other words, we expect heads to come up 50% of the time if we toss such a coin many times. Put more precisely, we expect the relative frequency to approach .5 as the number of trials gets larger. Figure 4 shows how the relative frequency behaved for one sequence of coin tosses. For each N we have plotted what fraction of times the coin came up heads in the first N tosses.

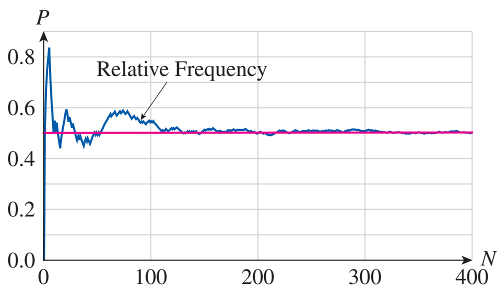


Figure 4

Notice that the relative frequency graph meanders as N increases, sometimes getting closer to .5, and sometimes drifting away again. However, the graph tends to meander within smaller and smaller distances of .5 as N increases.*

In general, this is how relative frequency seems to behave; as N gets large, the relative frequency appears to approach some fixed value. Some refer to this value as the "actual" probability, whereas others point out that there are difficulties with this this notion. For instance, how can we actually determine this limit to any accuracy by experiment? How exactly is the experiment conducted? Technical and philosophical issues aside, the relative frequencies do approach a fixed value and, in the next section, we will talk about how we use probability models to predict this limiting value.

* **NOTE** This can be made more precise by the concept of "limit" used in calculus.

7.2 EXERCISES

▼ more advanced ◆ challenging

T indicates exercises that should be solved using technology

In Exercises 1–6, calculate the relative frequency $P(E)$ using the given information.

1. $N = 100,\ fr(E) = 40$ **2.** $N = 500,\ fr(E) = 300$

3. Eight hundred adults are polled and 640 of them support universal health-care coverage. E is the event that an adult supports universal health coverage. HINT [See Example 1.]

4. Eight Hundred adults are polled and 640 of them support universal health-care coverage. E is the event that an adult does not support universal health coverage. HINT [See Example 1.]

5. A die is rolled 60 times with the following result: 1, 2, and 3 each come up 8 times, and 4, 5, and 6 each come up 12 times. E is the event that the number that comes up is at most 4.

6. A die is rolled 90 times with the following result: 1 and 2 never come up, 3 and 4 each come up 30 times, and 5 and 6 each come up 15 times. E is the event that the number that comes up is at least 4.

Exercises 7–12 are based on the following table, which shows the frequency of outcomes when two distinguishable coins were tossed 4,000 times and the uppermost faces were observed. HINT [See Example 2.]

Outcome	HH	HT	TH	TT
Frequency	1,100	950	1,200	750

7. Determine the relative frequency distribution.

8. What is the relative frequency that heads comes up at least once?

9. What is the relative frequency that the second coin lands with heads up?

10. What is the relative frequency that the first coin lands with heads up?

11. Would you judge the second coin to be fair? Give a reason for your answer.

12. Would you judge the first coin to be fair? Give a reason for your answer.

In Exercises 13–18, say whether the given distribution can be a relative frequency distribution. If your answer is no, indicate why not. HINT [See the properties of relative frequency distributions on page 463.]

13.

Outcome	1	2	3	5
Rel. Frequency	.4	.6	0	0

14.

Outcome	A	B	C	D
Rel. Frequency	.2	.1	.2	.1

15.

Outcome	HH	HT	TH	TT
Rel. Frequency	.5	.4	.5	−.4

16.

Outcome	2	4	6	8
Rel. Frequency	25	25	25	25

17.

Outcome	−3	−2	−1	0
Rel. Frequency	.2	.3	.2	.3

18.

Outcome	HH	HT	TH	TT
Rel. Frequency	0	0	0	1

In Exercises 19 and 20, complete the given relative frequency distribution and compute the stated relative frequencies. HINT [See the properties of relative frequency distributions on page 463.]

19.

Outcome	1	2	3	4	5
Rel. Frequency	.2	.3	.1	.1	

a. $P(\{1, 3, 5\})$ **b.** $P(E')$ where $E = \{1, 2, 3\}$

20.

Outcome	1	2	3	4	5
Rel. Frequency	.4		.3	.1	.1

a. $P(\{2, 3, 4\})$ **b.** $P(E')$ where $E = \{3, 4\}$

T *Exercises 21–24 require the use of a calculator or computer with a random number generator.*

21. Simulate 100 tosses of a fair coin, and compute the estimated probability that heads comes up.

22. Simulate 100 throws of a fair die, and calculate the estimated probability that the result is a 6.

23. Simulate 50 tosses of two coins, and compute the estimated probability that the outcome is one head and one tail (in any order).

24. Simulate 100 throws of two fair dice, and calculate the estimated probability that the result is a double 6.

APPLICATIONS

25. ***Music Sales*** In a survey of 500 music downloads, 160 were rock music, 55 were hip-hop, and 20 were classical, while the remainder were other genres.[6] Calculate the following relative frequencies:

a. That a music download was rock music.

b. That a music download was either classical music or hip-hop.

c. That a music download was not rock music. HINT [See Example 1.]

26. ***Music Sales*** In a survey of 400 music downloads, 45 were country music, 16 were religious music, and 16 were jazz, while the remainder were other genres.[7] Calculate the following relative frequencies:

a. That a music download was jazz.

b. That a music download was either religious music or jazz.

c. That a music download was not country music. HINT [See Example 1.]

27. ***Subprime Mortgages*** The following chart shows the results of a survey of the status of subprime home mortgages in Texas in November 2008:[8]

Mortgage Status	Current	Past Due	In Foreclosure	Repossessed
Frequency	134	52	9	5

(The four categories are mutually exclusive; for instance, "Past Due" refers to a mortgage whose payment status is past due but is not in foreclosure, and "In Foreclosure" refers to a mortgage that is in the process of being foreclosed but not yet repossessed.)

a. Find the relative frequency distribution for the experiment of randomly selecting a subprime mortgage in Texas and determining its status.

b. What is the relative frequency that a randomly selected subprime mortgage in Texas was not current? HINT [See Example 2.]

28. ***Subprime Mortgages*** The following chart shows the results of a survey of the status of subprime home mortgages in Florida in November 2008:[9]

Mortgage Status	Current	Past Due	In Foreclosure	Repossessed
Frequency	110	65	60	15

(The four categories are mutually exclusive; for instance, "Past Due" refers to a mortgage whose payment status is past due but is not in foreclosure, and "In Foreclosure" refers to a mortgage that is in the process of being foreclosed but not yet repossessed.)

a. Find the relative frequency distribution for the experiment of randomly selecting a subprime mortgage in Florida and determining its status.

b. What is the relative frequency that a randomly selected subprime mortgage in Florida was neither in foreclosure nor repossessed? HINT [See Example 2.]

29. ***Employment in Mexico by Age*** The following table shows the results of a survey of employed adult (age 18 and higher) residents of Mexico:[10]

Age	18–25	26–35	36–45	46–55	>55
Percentage	29	47	20	3	1

a. Find the associated relative frequency distribution. HINT [See Quick Example 3 page 461.]

b. Find the relative frequency that an employed adult resident of Mexico is *not* between 26 and 45.

30. ***Unemployment in Mexico by Age*** The following table shows the results of a survey of unemployed adult (age 18 and higher) residents of Mexico:[11]

Age	18–25	26–35	36–45	46–55	>55
Percentage	40	37	13	7	3

a. Find the associated relative frequency distribution. HINT [See Quick Example 3 page 461.]

b. Find the relative frequency that an employed adult resident of Mexico is either younger than 26 or older than 55.

[6] Based on actual sales in 2007. Source: The Recording Industry Association of America (www.riaa.com).

[7] Ibid.

[8] Based on actual data in 2008. Source: Federal Reserve Bank of New York (www.newyorkfed.org/regional/subprime.html).

[9] Ibid.

[10] Source: Profeco/*Empresas y Empresarios*, April 23, 2007, p. 2.

[11] Ibid.

31. ***Motor Vehicle Safety*** The following table shows crashworthiness ratings for 10 small SUVs.[12] (3 = Good, 2 = Acceptable, 1 = Marginal, 0 = Poor)

Frontal Crash Test Rating	3	2	1	0
Frequency	1	4	4	1

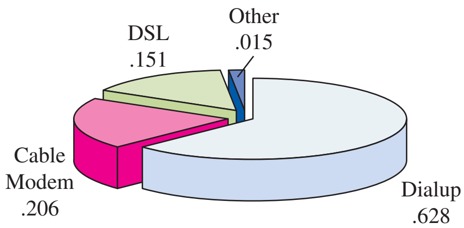
a. Find the relative frequency distribution for the experiment of choosing a small SUV at random and determining its frontal crash rating.
b. What is the relative frequency that a randomly selected small SUV will have a crash test rating of "Acceptable" or better?

32. ***Motor Vehicle Safety*** The following table shows crashworthiness ratings for 16 small cars.[13] (3=Good, 2 = Acceptable, 1 = Marginal, 0 = Poor)

Frontal Crash Test Rating	3	2	1	0
Frequency	1	11	2	2

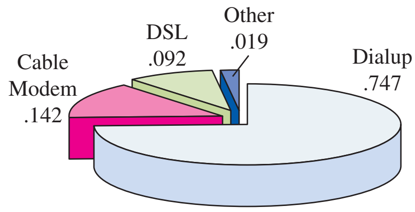
a. Find the relative frequency distribution for the experiment of choosing a small car at random and determining its frontal crash rating.
b. What is the relative frequency that a randomly selected small car will have a crash test rating of "Marginal" or worse?

33. ***Internet Connections*** The following pie chart shows the relative frequency distribution resulting from a survey of 2000 U.S. households with Internet connections back in 2003.[14]



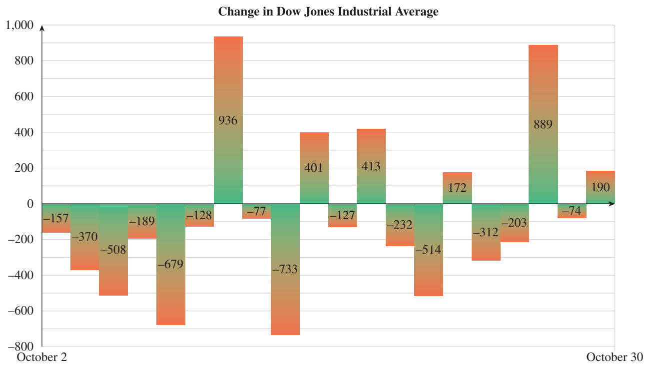
Determine the **frequency distribution**; that is, the total number of households with each type of Internet connection in the survey.

34. ***Internet Connections*** The following pie chart shows the relative frequency distribution resulting from a survey of 3,000 U.S. rural households with Internet connections back in 2003.[15]



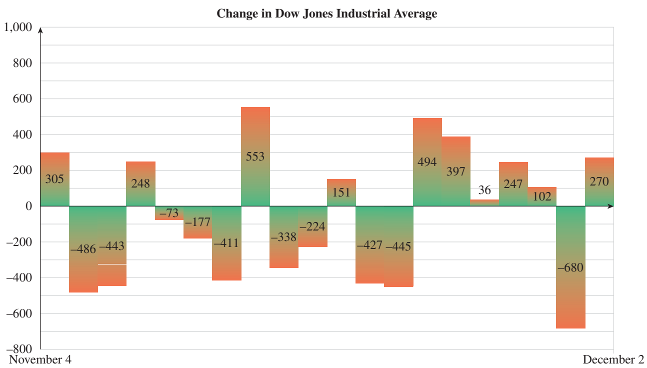
Determine the **frequency distribution**; that is, the total number of households with each type of Internet connection in the survey.

35. ▼ ***Stock Market Gyrations*** The following chart shows the day-by-day change in the Dow Jones Industrial Average during 20 successive business days in October 2008:[16]



Use the chart to construct the relative frequency distribution using the following three outcomes. Surge: The Dow was up by more than 300 points; Plunge: The Dow was down by more than 300 points; Steady: The Dow changed by 300 points or less.

36. ▼ ***Stock Market Gyrations*** Repeat Exercise 35 using the following chart for November–December 2008:[17]



[12] Ratings by the Insurance Institute for Highway Safety. Sources: Oak Ridge National Laboratory: "An Analysis of the Impact of Sport Utility Vehicles in the United States" Stacy C. Davis, Lorena F. Truett (August 2000) Insurance Institute for Highway Safety (www-cta.ornl.gov/Publications/Final SUV report.pdf).

[13] Ratings by the Insurance Institute for Highway Safety. Sources: Oak Ridge National Laboratory: "An Analysis of the Impact of Sport Utility Vehicles in the United States" Stacy C. Davis, Lorena F. Truett (August 2000) Insurance Institute for Highway Safety (www-cta.ornl.gov/Publications/Final SUV report.pdf www.highwaysafety.org/vehicle_ratings/).

[14] Based on a 2003 survey. Source: "A Nation Online: Entering the Broadband Age," U.S. Department of Commerce, September 2004 (www.ntia.doc.gov/reports/anol/index.html).

[15] Ibid.

[16] Source: http://finance.google.com.

[17] Ibid.

Publishing *Exercises 37–46 are based on the following table, which shows the results of a survey of 100 authors by a publishing company.*

	New Authors	Established Authors	Total
Successful	5	25	30
Unsuccessful	15	55	70
Total	20	80	100

Compute the relative frequencies of the given events if an author as specified is chosen at random.

37. ▼ An author is established and successful.

38. ▼ An author is unsuccessful and new.

39. ▼ An author is a new author.

40. ▼ An author is successful.

41. ▼ An author is unsuccessful.

42. ▼ An author is established.

43. ▼ A successful author is established.

44. ▼ An unsuccessful author is established.

45. ▼ An established author is successful.

46. ▼ A new author is unsuccessful.

47. ▼ ***Public Health*** A random sampling of chicken in supermarkets revealed that approximately 80% was contaminated with the organism *Campylobacter*.[18] Of the contaminated chicken, 20% had the strain resistant to antibiotics. Construct a relative frequency distribution showing the following outcomes when chicken is purchased at a supermarket: *U:* the chicken is not infected with *Campylobacter*; *C:* the chicken is infected with nonresistant *Campylobacter*; *R:* the chicken is infected with resistant *Campylobacter*.

48. ▼ ***Public Health*** A random sampling of turkey in supermarkets found 58% to be contaminated with *Campylobacter*, and 84% of those to be resistant to antibiotics.[19] Construct a relative frequency distribution showing the following outcomes when turkey is purchased at a supermarket: *U:* the turkey is not infected with *Campylobacter*; *C:* the turkey is infected with nonresistant *Campylobacter*; and *R:* the turkey is infected with resistant *Campylobacter*.

49. ▼ ***Organic Produce*** A 2001 Agriculture Department study of more than 94,000 samples from more than 20 crops showed that 73% of conventionally grown foods had residues from at least one pesticide. Moreover, conventionally grown foods were six times as likely to contain multiple pesticides as organic foods. Of the organic foods tested, 23% had pesticide residues, which includes 10% with multiple pesticide residues.[20] Compute two estimated probability distributions: one for conventional produce and one for organic produce, showing the relative frequencies that a randomly selected product has no pesticide residues, has residues from a single pesticide, and has residues from multiple pesticides.

50. ▼ ***Organic Produce*** Repeat Exercise 49 using the following information for produce from California: 31% of conventional food and 6.5% of organic food had residues from at least one pesticide. Assume that, as in Exercise 49, conventionally grown foods were six times as likely to contain multiple pesticides as organic foods. Also assume that 3% of the organic food has residues from multiple pesticides.

51. ▼ ***Steroids Testing*** A pharmaceutical company is running trials on a new test for anabolic steroids. The company uses the test on 400 athletes known to be using steroids and 200 athletes known not to be using steroids. Of those using steroids, the new test is positive for 390 and negative for 10. Of those not using steroids, the test is positive for 10 and negative for 190. What is the relative frequency of a **false negative** result (the probability that an athlete using steroids will test negative)? What is the relative frequency of a **false positive** result (the probability that an athlete not using steroids will test positive)?

52. ▼ ***Lie Detectors*** A manufacturer of lie detectors is testing its newest design. It asks 300 subjects to lie deliberately and another 500 to tell the truth. Of those who lied, the lie detector caught 200. Of those who told the truth, the lie detector accused 200 of lying. What is the relative frequency of the machine wrongly letting a liar go, and what is the probability that it will falsely accuse someone who is telling the truth?

53. T ◆ ***Public Health*** Refer back to Exercise 47. Simulate the experiment of selecting chicken at a supermarket and determining the following outcomes: *U:* The chicken is not infected with *campylobacter; C:* The chicken is infected with nonresistant *campylobacter; R:* The chicken is infected with resistant *campylobacter.* [*Hint*: Generate integers in the range 1–100. The outcome is determined by the range. For instance if the number is in the range 1–20, regard the outcome as *U*, etc.]

54. T ◆ ***Public Health*** Repeat the preceding exercise, but use turkeys and the data given in Exercise 48.

COMMUNICATION AND REASONING EXERCISES

55. Complete the following. The relative frequency of an event *E* is defined to be _____.

56. If two people each flip a coin 100 times and compute the relative frequency that heads comes up, they will both obtain the same result, right?

[18] *Campylobacter* is one of the leading causes of food poisoning in humans. Thoroughly cooking the meat kills the bacteria. Source: *The New York Times*, October 20, 1997, p. A1. Publication of this article first brought *Campylobacter* to the attention of a wide audience.

[19] Ibid.

[20] The 10% figure is an estimate. Source: *New York Times*, May 8, 2002, p. A29.

57. How many different answers are possible if you flip a coin 100 times and compute the relative frequency that heads comes up? What are the possible answers?

58. Interpret the popularity rating of the student council president as a relative frequency by specifying an appropriate experiment and also what is observed.

59. ▼ Ruth tells you that when you roll a pair of fair dice, the probability of obtaining a pair of matching numbers is $1/6$. To test this claim, you roll a pair of fair dice 20 times, and never once get a pair of matching numbers. This proves that either Ruth is wrong or the dice are not fair, right?

60. ▼ Juan tells you that when you roll a pair of fair dice, the probability that the numbers add up to 7 is $1/6$. To test this claim, you roll a pair of fair dice 24 times, and the numbers add up to 7 exactly four times. This proves that Juan is right, right?

61. ▼ How would you measure the relative frequency that the weather service accurately predicts the next day's high temperature?

62. ▼ Suppose that you toss a coin 100 times and get 70 heads. If you continue tossing the coin, the estimated probability of heads overall should approach 50% if the coin is fair. Will you have to get more tails than heads in subsequent tosses to "correct" for the 70 heads you got in the first 100 tosses?

7.3 Probability and Probability Models

It is understandable if you are a little uncomfortable with using relative frequency as the estimated probability because it does not always agree with what you intuitively feel to be true. For instance, if you toss a fair coin (one as likely to come up heads as tails) 100 times and heads happen to come up 62 times, the experiment seems to suggest that the probability of heads is .62, even though you *know* that the "actual" probability is .50 (because the coin is fair).

Q: *So what do we mean by "actual" probability?*

A: There are various philosophical views as to exactly what we should mean by "actual" probability. For example, (finite) *frequentists* say that there is no such thing as "actual probability"—all we should really talk about is what we can actually measure, the relative frequency. *Propensitists* say that the actual probability p of an event is a (often physical) property of the event that makes its relative frequency tend to p in the long run; that is, p will be the limiting value of the relative frequency as the number of trials in a repeated experiment gets larger and larger. (See Figure 4 in the preceding section.) *Bayesians*, on the other hand, argue that the actual probability of an event is the degree to which we *expect* it to occur, given our knowledge about the nature of the experiment. These and other viewpoints have been debated in considerable depth in the literature.[21]

Mathematicians tend to avoid the whole debate, and talk instead about *abstract* probability, or **probability distributions**, based purely on the properties of relative frequency listed in Section 7.2. Specific probability distributions can then be used as models of probability in real life situations, such as flipping a coin or tossing a die, to predict (or model) relative frequency.

[21] The interested reader should consult references in the philosophy of probability. For an online summary, see, for example, the Stanford Encyclopedia of Philosophy (http://plato.stanford.edu/contents.html).

Probability Distribution; Probability

(Compare page 463.) A (finite) **probability distribution** is an assignment of a number $P(s_i)$, the **probability of s_i**, to each outcome of a finite sample space $S = \{s_1, s_2, \ldots, s_n\}$. The probabilities must satisfy

1. $0 \le P(s_i) \le 1$

and

2. $P(s_1) + P(s_2) + \cdots + P(s_n) = 1$.

We find the **probability of an event E**, written $P(E)$, by adding up the probabilities of the outcomes in E.

If $P(E) = 0$, we call E an **impossible event**. The empty event Ø is always impossible, since *something* must happen.

Quick Examples

1. All the examples of estimated probability distributions in Section 7.2 are examples of probability distributions. (See page 463.)
2. Let us take $S = \{\text{H, T}\}$ and make the assignments $P(\text{H}) = .5$, $P(\text{T}) = .5$. Because these numbers are between 0 and 1 and add to 1, they specify a probability distribution.
3. In Quick Example 2, we can instead make the assignments $P(\text{H}) = .2$, $P(\text{T}) = .8$. Because these numbers are between 0 and 1 and add to 1, they, too, specify a probability distribution.
4. With $S = \{\text{H, T}\}$ again, we could also take $P(\text{H}) = 1$, $P(\text{T}) = 0$, so that $\{\text{T}\}$ is an impossible event.
5. The following table gives a probability distribution for the sample space

   $S = \{1, 2, 3, 4, 5, 6\}$.

Outcome	1	2	3	4	5	6
Probability	.3	.3	0	.1	.2	.1

It follows that

$$P(\{1, 6\}) = .3 + .1 = .4$$
$$P(\{2, 3\}) = .3 + 0 = .3$$
$$P(3) = 0. \qquad \{3\} \text{ is an impossible event.}$$

The above Quick Examples included models for the experiments of flipping fair and unfair coins. In general:

Probability Models

A **probability model** for a particular experiment is a probability distribution that predicts the relative frequency of each outcome if the experiment is performed a large number of times (see Figure 4 at the end of the preceding section).* Just as we think of relative frequency as *estimated probability*, we can think of modeled probability as *theoretical probability*.

* **NOTE** Just how large is a "large number of times"? That depends on the nature of the

experiment. For example, if you toss a fair coin 100 times, then the relative frequency of heads will be between .45 and .55 about 73% of the time. If an outcome is extremely unlikely (such as winning the lotto) one might need to repeat the experiment billions or trillions of times before the relative frequencies approaches any specific number.

Quick Examples

1. **Fair Coin Model:** (See Quick Example 2 on the previous page.) Flip a fair coin and observe the side that faces up. Because we expect that heads is as likely to come up as tails, we model this experiment with the probability distribution specified by $S = \{\mathrm{H}, \mathrm{T}\}$, $P(\mathrm{H}) = .5$, $P(\mathrm{T}) = .5$. Figure 4 on p. 463 suggests that the relative frequency of heads approaches .5 as the number of coin tosses gets large, so the fair coin model predicts the relative frequency for a large number of coin tosses quite well.
2. **Unfair Coin Model:** (See Quick Example 3 on the previous page.) Take $S = \{\mathrm{H}, \mathrm{T}\}$ and $P(\mathrm{H}) = .2$, $P(\mathrm{T}) = .8$. We can think of this distribution as a model for the experiment of flipping an unfair coin that is four times as likely to land with tails uppermost than heads.
3. **Fair Die Model:** Roll a fair die and observe the uppermost number. Because we expect to roll each specific number one sixth of the time, we model the experiment with the probability distribution specified by $S = \{1, 2, 3, 4, 5, 6\}$, $P(1) = 1/6$, $P(2) = 1/6, \dots, P(6) = 1/6$. This model predicts for example, that the relative frequency of throwing a 5 approaches $1/6$ as the number of times you roll the die gets large.
4. Roll a pair of fair dice (recall that there are a total of 36 outcomes if the dice are distinguishable). Then an appropriate model of the experiment has

$$S = \left\{\begin{array}{llllll} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6), \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6), \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6), \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6), \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6), \\ (6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6) \end{array}\right\}$$

   with each outcome being assigned a probability of $1/36$.
5. In the experiment in Quick Example 4, take E to be the event that the sum of the numbers that face up is 5, so

$$E = \{(1, 4), (2, 3), (3, 2), (4, 1)\}.$$

   By the properties of probability distributions,

$$P(E) = \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{4}{36} = \frac{1}{9}.$$

Notice that, in all of the Quick Examples above except for the unfair coin, all the outcomes are equally likely, and each outcome s has a probability of

$$P(s) = \frac{1}{\text{Total number of outcomes}} = \frac{1}{n(S)}.$$

More generally, in the last Quick Example we saw that adding the probabilities of the individual outcomes in an event E amounted to computing the ratio (Number of favorable outcomes)/(Total number of outcomes):

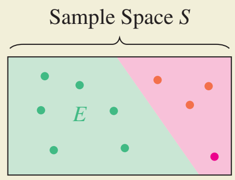
$$P(E) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}} = \frac{n(E)}{n(S)}.$$

Probability Model for Equally Likely Outcomes

In an experiment in which all outcomes are equally likely, we model the experiment by taking the probability of an event E to be

$$P(E) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}} = \frac{n(E)}{n(S)}.$$

Visualizing Probability for Equally Likely Outcomes



$$P(E) = \frac{n(E)}{n(S)} = \frac{6}{10} = .6$$

Note

Remember that this formula will work *only* when the outcomes are equally likely. If, for example, a die is *weighted*, then the outcomes may not be equally likely, and the formula above will not give an appropriate probability model.

Quick Examples

1. Toss a fair coin three times, so $S = \{$HHH, HHT, HTH, HTT, THH, THT, TTH, TTT$\}$. The probability that we throw exactly two heads is

$$P(E) = \frac{n(E)}{n(S)} = \frac{3}{8}.$$ There are eight equally likely outcomes and $E = \{$HHT, HTH, THH$\}$.

2. Roll a pair of fair dice. The probability that we roll a double (both dice show the same number) is

$$P(E) = \frac{n(E)}{n(S)} = \frac{6}{36} = \frac{1}{6}.$$ $E = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$

3. Randomly choose a person from a class of 40, in which 6 have red hair. If E is the event that a randomly selected person in the class has red hair, then

$$P(E) = \frac{n(E)}{n(S)} = \frac{6}{40} = .15.$$

EXAMPLE 1 Sales of Hybrid SUVs

(Compare Example 1 in Section 7.2.) In February 2008 total sales of hybrid SUVs amounted to only 4,840 vehicles. Of these, 1,940 were Toyota Highlanders, 1,150 were Lexus RX 400s, 1,510 were Ford Escapes, and 240 were other brands.*

a. What is the probability that a randomly selected hybrid SUV sold in the United States during February 2008 was either a Lexus RX 400 or a Ford Escape?

b. What is the probability that a randomly selected hybrid SUV sold in the United States during February 2008 was not a Toyota Highlander?

*Figures are rounded. Source: www.hybridsuv.com/hybrid-resources/hybrid-suv-sales.

Solution

a. The experiment suggested by the question consists of randomly choosing a hybrid SUV sold in the United States during February 2008 and determining its make. We are interested in the event E that the hybrid SUV was either a Lexus RX 400 or a Ford Escape. So,

$S =$ the set of hybrid SUVs sold in February 2008; $n(S) = 4{,}840$

$E =$ the set of Lexus RX 400s and Ford Escapes sold in February 2008;
$n(E) = 1{,}150 + 1{,}510 = 2{,}660$.

Are the outcomes equally likely in this experiment? Yes, because we are as likely to choose one SUV as another. Thus,

$$P(E) = \frac{n(E)}{n(S)} = \frac{2{,}660}{4{,}840} \approx .55.$$

b. Let the event F consist of those hybrid SUVs sold in February 2008 that were not Toyota Highlanders.

$$n(F) = 4{,}840 - 1{,}940 = 2{,}900$$

Hence,

$$P(F) = \frac{n(F)}{n(S)} = \frac{2{,}900}{4{,}840} \approx .60.$$

Q: *In Example 1 of Section 7.2 we had a similar example about hybrid SUVs, but we called the probabilities calculated there relative frequencies. Here they are probabilities. What is the difference?*

A: In Example 1 of Section 7.2, the data were based on the results of a survey, or sample, of only 120 hybrid SUVs (out of a total of about 4,840 sold in the United States during February 2008), and were therefore incomplete. (A statistician would say that we were given *sample data*.) It follows that any inference we draw from the 120 surveyed, such as the probability that a hybrid SUV sold in the United States is not a Toyota Highlander, is uncertain, and this is the cue that tells us that we are working with relative frequency, or estimated probability. Think of the survey as an experiment (choosing a hybrid SUV) repeated 120 times—exactly the setting for estimated probability.

In Example 1 above, on the other hand, the data do not describe how *some* hybrid SUV sales are broken down into the categories described, but they describe how *all* 4,840 hybrid SUV sales in the United States are broken down. (The statistician would say that we were given *population data* in this case, because the data describe the entire "population" of hybrid SUVs sold in the United States during February.)

EXAMPLE 2 Indistinguishable Dice

We recall from Section 7.1 that the sample space when we roll a pair of indistinguishable dice is

$$S = \left\{ \begin{array}{rrrrrr} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6), \\ & (2,2), & (2,3), & (2,4), & (2,5), & (2,6), \\ & & (3,3), & (3,4), & (3,5), & (3,6), \\ & & & (4,4), & (4,5), & (4,6), \\ & & & & (5,5), & (5,6), \\ & & & & & (6,6) \end{array} \right\}.$$

Construct a probability model for this experiment.

Solution Because there are 21 outcomes, it is tempting to say that the probability of each outcome should be taken to be 1/21. However, the outcomes are not all equally likely. For instance, the outcome (2, 3) is twice as likely as (2, 2), because (2, 3) can occur in two ways (it corresponds to the event {(2, 3), (3, 2)} for distinguishable dice). For purposes of calculating probability, it is easiest to use calculations for distinguishable dice.* Here are some examples.

* **NOTE** Note that any pair of real dice can be distinguished in principle because they possess slight differences, although we may regard them as indistinguishable by not attempting to distinguish them. Thus, the probabilities of events must be the same as for the corresponding events for distinguishable dice.

Outcome (indistinguishable dice)	(1, 1)	(1, 2)	(2, 2)	(1, 3)	(2, 3)	(3, 3)
Corresponding event (distinguishable dice)	{(1, 1)}	{(1, 2), (2, 1)}	{(2, 2)}	{(1, 3), (3, 1)}	{(2, 3), (3, 2)}	{(3, 3)}
Probability	$\frac{1}{36}$	$\frac{2}{36} = \frac{1}{18}$	$\frac{1}{36}$	$\frac{2}{36} = \frac{1}{18}$	$\frac{2}{36} = \frac{1}{18}$	$\frac{1}{36}$

If we continue this process for all 21 outcomes, we will find that they add to 1. Figure 5 illustrates the complete probability distribution:

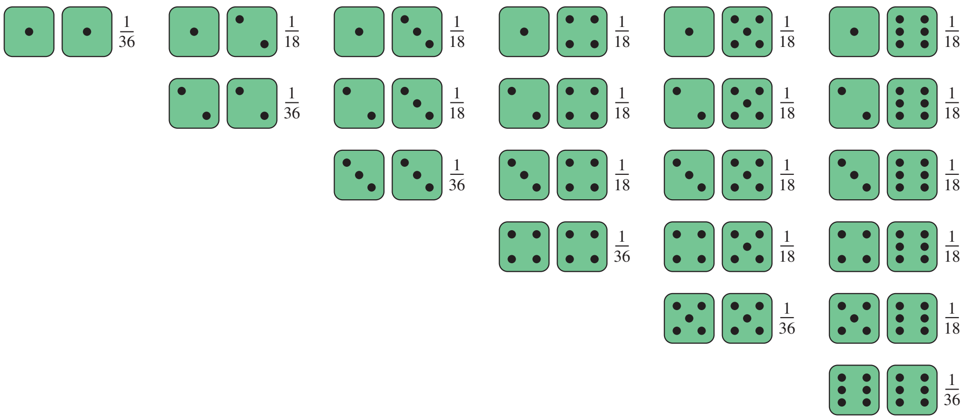


Figure 5

EXAMPLE 3 Weighted Dice

In order to impress your friends with your die-rolling skills, you have surreptitiously weighted your die in such a way that 6 is three times as likely to come up as any one of the other numbers. (All the other outcomes are equally likely.) Obtain a probability distribution for a roll of the die and use it to calculate the probability of an even number coming up.

Solution Let us label our unknowns (there appear to be two of them):

x = probability of rolling a 6

y = probability of rolling any one of the other numbers

We are first told that "6 is three times as likely to come up as any one of the other numbers." If we rephrase this in terms of our unknown probabilities we get, "the probability of rolling a 6 is three times the probability of rolling any one of the other numbers." In symbols,

$$x = 3y.$$

We must also use a piece of information not given to us, but one we know must be true: The sum of the probabilities of all the outcomes is 1:

$$x + y + y + y + y + y = 1$$

or

$$x + 5y = 1.$$

We now have two linear equations in two unknowns, and we solve for x and y. Substituting the first equation ($x = 3y$) in the second ($x + 5y = 1$) gives

$$8y = 1$$

or

$$y = \frac{1}{8}.$$

To get x, we substitute the value of y back into either equation and find

$$x = \frac{3}{8}.$$

Thus, the probability model we seek is the one shown in the following table.

Outcome	1	2	3	4	5	6
Probability	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{8}$

We can use the distribution to calculate the probability of an even number coming up by adding the probabilities of the favorable outcomes.

$$P(\{2,\ 4,\ 6\}) = \frac{1}{8} + \frac{1}{8} + \frac{3}{8} = \frac{5}{8}.$$

Thus there is a $5/8 = .625$ chance that an even number will come up.

Before we go on... We should check that the probability distribution in Example 3 satisfies the requirements: 6 is indeed three times as likely to come up as any other number. Also, the probabilities we calculated do add up to 1:

$$\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{3}{8} = 1. \blacksquare$$

Probability of Unions, Intersections, and Complements

So far, all we know about computing the probability of an event E is that $P(E)$ is the sum of the probabilities of the individual outcomes in E. Suppose, though, that we do not know the probabilities of the individual outcomes in E but we do know that $E = A \cup B$, where we happen to know $P(A)$ and $P(B)$. How do we compute the probability of $A \cup B$? We might be tempted to say that $P(A \cup B)$ is $P(A) + P(B)$, but let us look at an example using the probability distribution in Quick Example 5 at the beginning of this section:

Outcome	1	2	3	4	5	6
Probability	.3	.3	0	.1	.2	.1

For A let us take the event $\{2, 4, 5\}$, and for B let us take $\{2, 4, 6\}$. $A \cup B$ is then the event $\{2, 4, 5, 6\}$. We know that we can find the probabilities $P(A)$, $P(B)$, and $P(A \cup B)$ by adding the probabilities of all the outcomes in these events, so

$$P(A) = P(\{2, 4, 5\}) = .3 + .1 + .2 = .6$$
$$P(B) = P(\{2, 4, 6\}) = .3 + .1 + .1 = .5, \text{ and}$$
$$P(A \cup B) = P(\{2, 4, 5, 6\}) = .3 + .1 + .2 + .1 = .7.$$

Our first guess was wrong: $P(A \cup B) \neq P(A) + P(B)$. Notice, however, that the outcomes in $A \cap B$ are counted twice in computing $P(A) + P(B)$, but only once in computing $P(A \cup B)$:

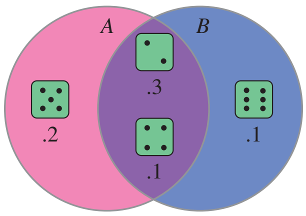
$$\begin{aligned} P(A) + P(B) &= P(\{2, 4, 5\}) + P(\{2, 4, 6\}) && A \cap B = \{2, 4\} \\ &= (.3 + .1 + .2) + (.3 + .1 + .1) && P(A \cap B) \text{ counted twice} \\ &= 1.1 \end{aligned}$$

whereas

$$\begin{aligned} P(A \cup B) &= P(\{2, 4, 5, 6\}) = .3 + .1 + .2 + .1 && P(A \cap B) \text{ counted once} \\ &= .7. \end{aligned}$$

Thus, if we take $P(A) + P(B)$ and then subtract the surplus $P(A \cap B)$, we get $P(A \cup B)$. In symbols,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$.7 = .6 + .5 - .4.$$



$P(A) = .6$ $P(B) = .5$
$P(A \cap B) = .4$
$P(A \cup B) = .6 + .5 - .4$

Figure 6

(see Figure 6). We call this formula the **addition principle**. One more thing: Notice that our original guess $P(A \cup B) = P(A) + P(B)$ would have worked if we had chosen A and B with no outcomes in common; that is, if $A \cap B = \emptyset$. When $A \cap B = \emptyset$, recall that we say that A and B are mutually exclusive.

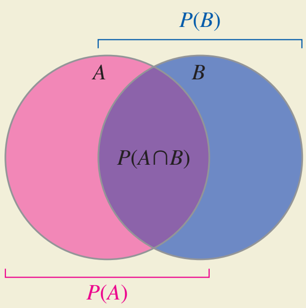
Addition Principle

If A and B are any two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Visualizing the Addition Principle

In the figure, the area of the union is obtained by adding the areas of A and B and then subtracting the overlap (because it is counted twice when we add the areas).



$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Addition Principle for Mutually Exclusive Events

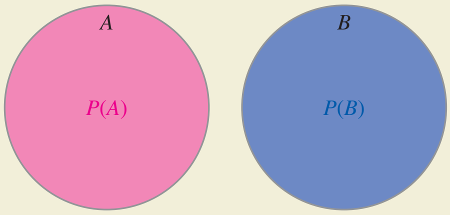
If $A \cap B = \emptyset$, we say that A and B are **mutually exclusive**, and we have

$$P(A \cup B) = P(A) + P(B).$$

Because $P(A \cap B) = 0$

Visualizing the Addition Principle for Mutually Exclusive Events

If A and B do not overlap, then the area of the union is obtained by adding the areas of A and B.



$P(A \cup B) = P(A) + P(B)$

This holds true also for more than two events: If $A_1, A_2, \ldots, A_n$ are mutually exclusive events (that is, the intersection of every pair of them is empty), then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n).$$

Addition principle for many mutually exclusive events

Quick Examples

1. There is a 10% chance of rain (R) tomorrow, a 20% chance of high winds (W), and a 5% chance of both. The probability of either rain or high winds (or both) is

$$\begin{aligned} P(R \cup W) &= P(R) + P(W) - P(R \cap W) \\ &= .10 + .20 - .05 = .25. \end{aligned}$$

2. The probability that you will be in Cairo at 6:00 AM tomorrow (C) is .3, while the probability that you will be in Alexandria at 6:00 AM tomorrow (A) is .2. Thus, the probability that you will be either in Cairo or Alexandria at 6:00 AM tomorrow is

$$\begin{aligned} P(C \cup A) &= P(C) + P(A) \\ &= .3 + .2 = .5. \end{aligned}$$

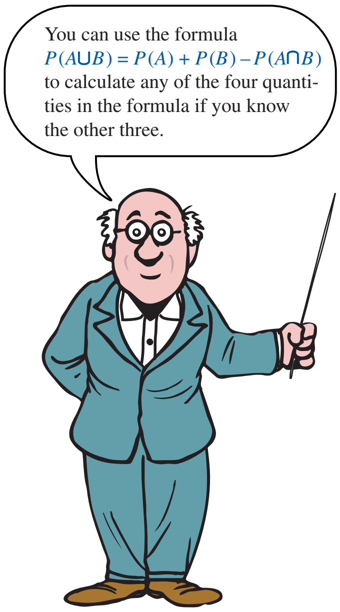
A and C are mutually exclusive.

3. When a pair of fair dice is rolled, the probability of the numbers that face up adding to 7 is $6/36$, the probability of their adding to 8 is $5/36$, and the probability of their adding to 9 is $4/36$. Thus, the probability of the numbers adding to 7, 8, or 9 is

$$\begin{aligned} P(\{7\} \cup \{8\} \cup \{9\}) &= P(7) + P(8) + P(9) \\ &= \frac{6}{36} + \frac{5}{36} + \frac{4}{36} = \frac{15}{36} = \frac{5}{12}. \end{aligned}$$

The events are mutually exclusive.*

* **NOTE** The sum of the numbers that face up cannot equal two different numbers at the same time.



EXAMPLE 4 School and Work

A survey conducted by the Bureau of Labor Statistics found that 67% of the high school graduating class of 2007 went on to college the following year, while 44% of the class was working.* Furthermore, 92% were either in college or working, or both.

a. What percentage went on to college and work at the same time?
b. What percentage went on to college but not work?

Solution We can think of the experiment of choosing a member of the high school graduating class of 2007 at random. The sample space is the set of all these graduates.

a. We are given information about two events:

A: A graduate went on to college; $P(A) = .67$

B: A graduate went on to work; $P(B) = .44$

We are also told that $P(A \cup B) = .92$. We are asked for the probability that a graduate went on to both college and work, $P(A \cap B)$. To find $P(A \cap B)$, we take advantage of the fact that the formula

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

can be used to calculate any one of the four quantities that appear in it as long as we know the other three. Substituting the quantities we know, we get

$$.92 = .67 + .44 - P(A \cap B)$$

so

$$P(A \cap B) = .67 + .44 - .92 = .19.$$

Thus, 19% of the graduates went on to college and work at the same time.

b. We are asked for the probability of a new event:

C: A graduate went on to college but not work.

C is the part of A outside of $A \cap B$, so $C \cup (A \cap B) = A$, and C and $A \cap B$ are mutually exclusive. (See Figure 7.)

Thus, applying the addition principle, we have

$$P(C) + P(A \cap B) = P(A).$$

From part (a), we know that $P(A \cap B) = .19$, so

$$P(C) + .19 = .67$$

giving

$$P(C) = .48.$$

In other words, 48% of the graduates went on to college but not work.

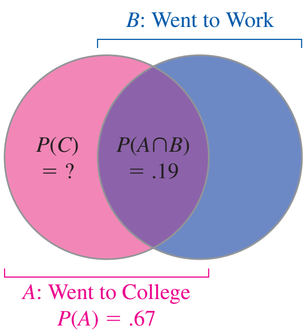


Figure 7

*"College Enrollment and Work Activity of 2007 High School Graduates," U.S. Bureau of Labor Statistics, April 2008, available at www.bls.gov/schedule/archives/all_nr.htm.

We can use the addition principle to deduce other useful properties of probability distributions.

More Principles of Probability Distributions

The following rules hold for any sample space S and any event A:

$P(S) = 1$ — The probability of *something* happening is 1.

$P(\emptyset) = 0$ — The probability of *nothing* happening is 0.

$P(A') = 1 - P(A).$ — The probability of A *not* happening is 1 minus the probability of A.

Note

We can also write the third equation as

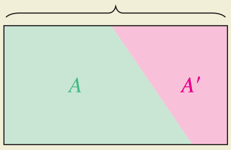
$$P(A) = 1 - P(A')$$

or

$$P(A) + P(A') = 1.$$

Visualizing the Rule for Complements

Think of A' as the portion of S outside of A. Adding the two areas gives the area of all of S, equal to 1.



Quick Examples

1. There is a 10% chance of rain (R) tomorrow. Therefore, the probability that it will *not* rain is

$$P(R') = 1 - P(R) = 1 - .10 = .90.$$

2. The probability that Eric Ewing will score at least two goals is .6. Therefore, the probability that he will score at most one goal is $1 - .6 = .4$.

Q: *Can you persuade me that all of these principles are true?*

A: Let us take them one at a time.

We know that $S = \{s_1, s_2, \ldots, s_n\}$ is the set of all outcomes, and so

$$\begin{aligned} P(S) &= P(\{s_1, s_2, \ldots, s_n\}) && \text{We add the probabilities of the outcomes to obtain the probability of an event.} \\ &= P(s_1) + P(s_2) + \cdots + P(s_n) \\ &= 1. && \text{By the definition of a probability distribution} \end{aligned}$$

Now, note that $S \cap \emptyset = \emptyset$, so that S and $\emptyset$ are mutually exclusive. Applying the addition principle gives

$$P(S) = P(S \cup \emptyset) = P(S) + P(\emptyset).$$

Subtracting $P(S)$ from both sides gives $0 = P(\emptyset)$.

If A is any event in S, then we can write

$$S = A \cup A'$$

where A and A' are mutually exclusive. (Why?) Thus, by the addition principle,

$$P(S) = P(A) + P(A').$$

Because $P(S) = 1$, we get

$$1 = P(A) + P(A')$$

or $$P(A') = 1 - P(A).$$

EXAMPLE 5 Subprime Mortgages

A home loan is either current, 30–59 days past due, 60–89 days past due, 90 or more days past due, in foreclosure, or repossessed by the lender. In November 2008, the probability that a randomly selected subprime home mortgage in California was not current was .51. The probability that a mortgage was not current, but neither in foreclosure nor repossessed, was .28.* Calculate the probabilities of the following events.

a. A California home mortgage was current.

b. A California home mortgage was in foreclosure or repossessed.

Solution

a. Let us write C for the event that a randomly selected subprime home mortgage in California was current. The event that the home mortgage was *not* current is its complement C', and we are given that $P(C') = .51$. We have

$$P(C) + P(C') = 1$$
$$P(C) + .51 = 1,$$
so $$P(C) = 1 - .51 = .49.$$

b. Take

F: A mortgage was in foreclosure or repossessed.

N: A mortgage was neither current, in foreclosure, nor repossessed.

We are given $P(N) = .28$. Further, the events F and N are mutually exclusive with union C', the set of all non-current mortgages. Hence,

$$P(C') = P(F) + P(N)$$
$$.51 = P(F) + .28$$

giving

$$P(F) = .51 - .28 = .23.$$

Thus, there was a 23% chance that a subprime home mortgage was either in foreclosure or repossessed.

* Source: Federal Reserve Bank of New York (www.newyorkfed.org/regional/subprime.html).

EXAMPLE 6 iPods, iPhones, and Macs

The following table shows sales, in millions of units, of iPods, iPhones, and Macintosh computers in the last three quarters of 2008.*

	iPods	iPhones	Macs	Total
2008 Q2	10.6	1.7	2.3	14.6
2008 Q3	11.0	0.7	2.5	14.2
2008 Q4	11.1	6.9	2.6	20.6
Total	32.7	9.3	7.4	49.4

If one of the items sold is selected at random, find the probabilities of the following events:

a. It is an iPod.

b. It was sold in the third quarter of 2008.

c. It is an iPod sold in the third quarter of 2008.

d. It is either an iPod or was sold in the third quarter of 2008.

e. It is not an iPod.

Solution

a. The sample space S is the set of all these items, and the outcomes consist of all the items sold in the given quarters. Because an item is being selected at random, all the outcomes are equally likely. If D is the event that the selected item is an iPod, then

$$P(D) = \frac{n(D)}{n(S)} = \frac{32.7}{49.4} \approx .662.$$

The event D is represented by the pink shaded region in the table:

	iPods	iPhones	Macs	Total
2008 Q2	10.6	1.7	2.3	14.6
2008 Q3	11.0	0.7	2.5	14.2
2008 Q4	11.1	6.9	2.6	20.6
Total	32.7	9.3	7.4	49.4

b. If A is the event that the selected item was sold in the third quarter of 2008, then

$$P(A) = \frac{n(A)}{n(S)} = \frac{14.2}{49.4} \approx .287.$$

In the table, A is represented as shown:

	iPods	iPhones	Macs	Total
2008 Q2	10.6	1.7	2.3	14.6
2008 Q3	11.0	0.7	2.5	14.2
2008 Q4	11.1	6.9	2.6	20.6
Total	32.7	9.3	7.4	49.4

*Figures are rounded. Source: Company reports (www.apple.com/investor/).

c. The event that the selected item is an iPod sold in the third quarter of 2008 is the event $D \cap A$.

$$P(D \cap A) = \frac{n(D \cap A)}{n(S)} = \frac{11.0}{49.4} \approx .223$$

In the table, $D \cap A$ is represented by the overlap of the regions representing D and A:

	iPods	iPhones	Macs	Total
2008 Q2	10.6	1.7	2.3	14.6
2008 Q3	11.0	0.7	2.5	14.2
2008 Q4	11.1	6.9	2.6	20.6
Total	32.7	9.3	7.4	49.4

d. The event that the selected item is either an iPod or was sold in the third quarter of 2008 is the event $D \cup A$, and is represented by the pink shaded area in the table:

	iPods	iPhones	Macs	Total
2008 Q2	10.6	1.7	2.3	14.6
2008 Q3	11.0	0.7	2.5	14.2
2008 Q4	11.1	6.9	2.6	20.6
Total	32.7	9.3	7.4	49.4

We can compute its probability in two ways:

1. Directly from the table:

$$P(D \cup A) = \frac{n(D \cup A)}{n(S)} = \frac{14.2 + 32.7 - 11.0}{49.4} \approx .727.$$

2. Using the addition principle:

$$\begin{aligned} P(D \cup A) &= P(D) + P(A) - P(D \cap A) \\ &\approx .662 + .287 - .223 = .726. \end{aligned}$$

(The discrepancy in the third decimal place is due to rounding error.)

e. The event that the selected item is not an iPod is the event D'. Its probability may be computed using the formula for the probability of the complement:

$$P(D') = 1 - P(D) \approx 1 - .662 = .338.$$

FAQs

Distinguishing Probability from Relative Frequency

Q: *Relative frequency and modeled probability using equally likely outcomes have essentially the same formula: (Number of favorable outcomes)/(Total number of outcomes). How do I know whether a given probability is one or the other?*

A: Ask yourself this: Has the probability been arrived at experimentally, by performing a number of trials and counting the number of times the event occurred? If so, the probability is estimated; that is, relative frequency. If, on the other hand, the probability was computed by analyzing the experiment under consideration rather than by performing actual trials of the experiment, it is a probability model.

Q: *Out of every 100 homes, 22 have broadband Internet service. Thus, the probability that a house has broadband service is .22. Is this probability estimated (relative frequency) or theoretical (a probability model)?*

A: That depends on how the ratio 22 out of 100 was arrived at. If it is based on a poll of *all* homes, then the probability is theoretical. If it is based on a survey of only a *sample* of homes, it is estimated (see the Q/A following Example 1).

7.3 EXERCISES

▼ more advanced ◆ challenging

T indicates exercises that should be solved using technology

1. Complete the following probability distribution table and then calculate the stated probabilities. HINT [See Quick Example 5, page 469.]

Outcome	a	b	c	d	e
Probability	.1	.05	.6	.05	

a. $P(\{a, c, e\})$
b. $P(E \cup F)$, where $E = \{a, c, e\}$ and $F = \{b, c, e\}$
c. $P(E')$, where E is as in part (b)
d. $P(E \cap F)$, where E and F are as in part (b)

2. Repeat the preceding exercise using the following table. HINT [See Quick Example 5, page 469.]

Outcome	a	b	c	d	e
Probability	.1		.65	.1	.05

In Exercises 3–8, calculate the (modeled) probability $P(E)$ using the given information, assuming that all outcomes are equally likely. HINT [See Quick Examples on page 471.]

3. $n(S) = 20,\ n(E) = 5$

4. $n(S) = 8,\ n(E) = 4$

5. $n(S) = 10,\ n(E) = 10$

6. $n(S) = 10,\ n(E) = 0$

7. $S = \{a, b, c, d\},\ E = \{a, b, d\}$

8. $S = \{1, 3, 5, 7, 9\},\ E = \{3, 7\}$

In Exercises 9–18, an experiment is given together with an event. Find the (modeled) probability of each event, assuming that the coins and dice are distinguishable and fair, and that what is observed are the faces or numbers uppermost. (Compare with Exercises 1–10 in Section 7.1.)

9. Two coins are tossed; the result is at most one tail.

10. Two coins are tossed; the result is one or more heads.

11. Three coins are tossed; the result is at most one head.

12. Three coins are tossed; the result is more tails than heads.

13. Two dice are rolled; the numbers add to 5.

14. Two dice are rolled; the numbers add to 9.

15. Two dice are rolled; the numbers add to 1.

16. Two dice are rolled; one of the numbers is even, the other is odd.

17. Two dice are rolled; both numbers are prime.[22]

18. Two dice are rolled; neither number is prime.

19. If two indistinguishable dice are rolled, what is the probability of the event $\{(4, 4), (2, 3)\}$? What is the corresponding event for a pair of distinguishable dice? HINT [See Example 2.]

20. If two indistinguishable dice are rolled, what is the probability of the event $\{(5, 5), (2, 5), (3, 5)\}$? What is the corresponding event for a pair of distinguishable dice? HINT [See Example 2.]

[22] A positive integer is prime if it is neither 1 nor a product of smaller integers.

21. A die is weighted in such a way that each of 2, 4, and 6 is twice as likely to come up as each of 1, 3, and 5. Find the probability distribution. What is the probability of rolling less than 4? HINT [See Example 3.]

22. Another die is weighted in such a way that each of 1 and 2 is three times as likely to come up as each of the other numbers. Find the probability distribution. What is the probability of rolling an even number?

23. A tetrahedral die has four faces, numbered 1–4. If the die is weighted in such a way that each number is twice as likely to land facing down as the next number (1 twice as likely as 2, 2 twice as likely as 3, and so on) what is the probability distribution for the face landing down?

24. A dodecahedral die has 12 faces, numbered 1–12. If the die is weighted in such a way that 2 is twice as likely to land facing up as 1, 3 is three times as likely to land facing up as 1, and so on, what is the probability distribution for the face landing up?

In Exercises 25–40, use the given information to find the indicated probability. HINT [See Quick Examples, page 476.]

25. $P(A) = .1$, $P(B) = .6$, $P(A \cap B) = .05$. Find $P(A \cup B)$.

26. $P(A) = .3$, $P(B) = .4$, $P(A \cap B) = .02$. Find $P(A \cup B)$.

27. $A \cap B = \emptyset$, $P(A) = .3$, $P(A \cup B) = .4$. Find $P(B)$.

28. $A \cap B = \emptyset$, $P(B) = .8$, $P(A \cup B) = .8$. Find $P(A)$.

29. $A \cap B = \emptyset$, $P(A) = .3$, $P(B) = .4$. Find $P(A \cup B)$.

30. $A \cap B = \emptyset$, $P(A) = .2$, $P(B) = .3$. Find $P(A \cup B)$.

31. $P(A \cup B) = .9$, $P(B) = .6$, $P(A \cap B) = .1$. Find $P(A)$.

32. $P(A \cup B) = 1.0$, $P(A) = .6$, $P(A \cap B) = .1$. Find $P(B)$.

33. $P(A) = .75$. Find $P(A')$.

34. $P(A) = .22$. Find $P(A')$.

35. A, B, and C are mutually exclusive. $P(A) = .3$, $P(B) = .4$, $P(C) = .3$. Find $P(A \cup B \cup C)$.

36. A, B, and C are mutually exclusive. $P(A) = .2$, $P(B) = .6$, $P(C) = .1$. Find $P(A \cup B \cup C)$.

37. A and B are mutually exclusive. $P(A) = .3$, $P(B) = .4$. Find $P((A \cup B)')$.

38. A and B are mutually exclusive. $P(A) = .4$, $P(B) = .4$. Find $P((A \cup B)')$.

39. $A \cup B = S$ and $A \cap B = \emptyset$. Find $P(A) + P(B)$.

40. $P(A \cup B) = .3$ and $P(A \cap B) = .1$. Find $P(A) + P(B)$.

In Exercises 41–46, determine whether the information shown is consistent with a probability distribution. If not, say why.

41. $P(A) = .2$; $P(B) = .1$; $P(A \cup B) = .4$

42. $P(A) = .2$; $P(B) = .4$; $P(A \cup B) = .2$

43. $P(A) = .2$; $P(B) = .4$; $P(A \cap B) = .2$

44. $P(A) = .2$; $P(B) = .4$; $P(A \cap B) = .3$

45. $P(A) = .1$; $P(B) = 0$; $P(A \cup B) = 0$

46. $P(A) = .1$; $P(B) = 0$; $P(A \cap B) = 0$

APPLICATIONS

47. ***Subprime Mortgages*** (Compare Exercise 27 in Section 7.2.) The following chart shows the (approximate) total number of subprime home mortgages in Texas in November 2008, broken down into four categories:[23]

Mortgage Status	Current	Past Due	In Foreclosure	Repossessed	Total
Number	136,330	53,310	8,750	5,090	203,480

(The four categories are mutually exclusive; for instance, "Past Due" refers to a mortgage whose payment status is past due but is not in foreclosure, and "In Foreclosure" refers to a mortgage that is in the process of being foreclosed but not yet repossessed.)

a. Find the probability that a randomly selected subprime mortgage in Texas during November 2008 was neither in foreclosure nor repossessed. HINT [See Example 1.]

b. What is the probability that a randomly selected subprime mortgage in Texas during November 2008 was not current?

48. ***Subprime Mortgages*** (Compare Exercise 28 in Section 7.2.) The following chart shows the (approximate) total number of subprime home mortgages in Florida in November 2008, broken down into four categories:[24]

Mortgage Status	Current	Past Due	In Foreclosure	Repossessed	Total
Number	130,400	73,260	72,380	17,000	293,040

(The four categories are mutually exclusive; for instance, "Past Due" refers to a mortgage whose payment status is past due but is not in foreclosure, and "In Foreclosure" refers to a mortgage that is in the process of being foreclosed but not yet repossessed.)

a. Find the probability that a randomly selected subprime mortgage in Florida during November 2008 was either in foreclosure or repossessed. HINT [See Example 1.]

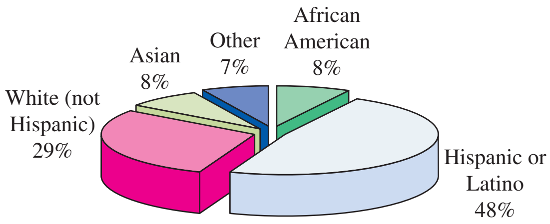
b. What is the probability that a randomly selected subprime mortgage in Florida during November 2008 was not repossessed?

49. ***Ethnic Diversity*** The following pie chart shows the ethnic makeup of California schools in the 2006–2007 academic year.[25]

[23] Data are rounded to the nearest 10 units. Source: Federal Reserve Bank of New York (www.newyorkfed.org/regional/subprime.html).

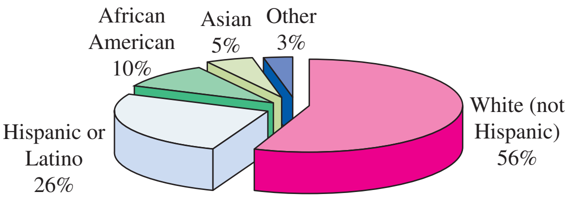
[24] Ibid.

[25] Source: CBEDS data collection, Educational Demographics, October 2006 (www.cde.ca.gov/resrc/factbook/).



Write down the probability distribution showing the probability that a randomly selected California student in 2006–2007 belonged to one of the ethnic groups named. What is the probability that a student is neither white nor Asian?

50. ***Ethnic Diversity*** (Compare Exercise 49.) The following pie chart shows the ethnic makeup of California schools in the 1981–1982 academic year.[26]



Write down the probability distribution showing the probability that a randomly selected California student in 1981–1982 belonged to one of the ethnic groups named. What is the probability that a student is neither Hispanic, Latino, nor African American?

51. ▼ ***Internet Investments in the 90s*** The following excerpt is from an article in *The New York Times* in July, 1999.[27]

> While statistics are not available for Web entrepreneurs who fail, the venture capitalists that finance such Internet start-up companies have a rule of thumb. For every 10 ventures that receive financing—and there are plenty who do not—2 will be stock market successes, which means spectacular profits for early investors; 3 will be sold to other concerns, which translates into more modest profits; and the rest will fail.

a. What is a sample space for the scenario?
b. Write down the associated probability distribution.
c. What is the probability that a start-up venture that receives financing will realize profits for early investors?

52. ◆ ***Internet Investments in the 90s*** The following excerpt is from an article in *The New York Times* in July, 1999.[28]

> Right now, the market for Web stocks is sizzling. Of the 126 initial public offerings of Internet stocks priced this year, 73 are trading above the price they closed on their first day of trading. . . . Still, 53 of the offerings have failed to live up to their fabulous first-day billings, and 17 [of these] are below the initial offering price.

Assume that, on the first day of trading, all stocks closed higher than their initial offering price.

a. What is a sample space for the scenario?
b. Write down the associated probability distribution. (Round your answers to two decimal places.)
c. What is the probability that an Internet stock purchased during the period reported ended either below its initial offering price or above the price it closed on its first day of trading? HINT [See Example 3.]

53. ***Market Share: Light Vehicles*** In 2003, 25% of all light vehicles sold (SUVs, pickups, passenger cars, and minivans) in the United States were SUVs and 15% were pickups. Moreover, a randomly chosen vehicle sold that year was five times as likely to be a passenger car as a minivan.[29] Find the associated probability distribution.

54. ***Market Share: Light Vehicles*** In 2000, 15% of all light vehicles (SUVs, pickups, passenger cars, and minivans) sold in the United States were pickups and 55% were passenger cars. Moreover, a randomly chosen vehicle sold that year was twice as likely to be an SUV as a minivan.[30] Find the associated probability distribution.

Gambling *In Exercises 55–62 are detailed some of the nefarious dicing practices of the Win Some/Lose Some Casino. In each case, find the probabilities of all the possible outcomes and also the probability that an odd number or an odd sum faces up.* HINT [See Example 3.]

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55. Some of the dice are specially designed so that 1 and 6 never come up, and all the other outcomes are equally likely.

56. Other dice are specially designed so that 1 comes up half the time, 6 never comes up, and all the other outcomes are equally likely.

[26] Source: CBEDS data collection, Educational Demographics, October 2006 (www.cde.ca.gov/resrc/factbook/).

[27] Article: "Not All Hit It Rich in the Internet Gold Rush," *The New York Times*, July 20, 1999, p. A1.

[28] Article: Ibid. Source for data: Comm-Scan/*The New York Times*, July 20, 1999, p. A1.

[29] Source: Environmental Protection Agency/*The New York Times*, June 28, 2003.

[30] Ibid.

57. Some of the dice are cleverly weighted so that each of 2, 3, 4, and 5 is twice as likely to come up as 1 is, and 1 and 6 are equally likely.

58. Other dice are weighted so that each of 2, 3, 4, and 5 is half as likely to come up as 1 is, and 1 and 6 are equally likely.

59. ▼ Some pairs of dice are magnetized so that each pair of mismatching numbers is twice as likely to come up as each pair of matching numbers.

60. ▼ Other pairs of dice are so strongly magnetized that mismatching numbers never come up.

61. ▼ Some dice are constructed in such a way that deuce (2) is five times as likely to come up as 4 and three times as likely to come up as each of 1, 3, 5, and 6.

62. ▼ Other dice are constructed in such a way that deuce is six times as likely to come up as 4 and four times as likely to come up as each of 1, 3, 5, and 6.

63. ***Astrology*** The astrology software package, Turbo Kismet,[31] works by first generating random number sequences and then interpreting them numerologically. When I ran it yesterday, it informed me that there was a 1/3 probability that I would meet a tall dark stranger this month, a 2/3 probability that I would travel this month, and a 1/6 probability that I would meet a tall dark stranger and also travel this month. What is the probability that I will either meet a tall dark stranger or that I will travel this month? HINT [See Quick Example 1, page 476 and Example 4.]

64. ***Astrology*** Another astrology software package, Java Kismet, is designed to help day traders choose stocks based on the position of the planets and constellations. When I ran it yesterday, it informed me that there was a .5 probability that Amazon.com will go up this afternoon, a .2 probability that Yahoo.com will go up this afternoon, and a .2 chance that both will go up this afternoon. What is the probability that either Amazon.com or Yahoo.com will go up this afternoon? HINT [See Quick Example 1 on page 476 and Example 4.]

65. ***Polls*** According to a *New York Times*/CBS poll released in March 2005, 61% of those polled ranked jobs or health care as the top domestic priority.[32] What is the probability that a randomly selected person polled did not rank either as the top domestic priority? HINT [See Example 5.]

66. ***Polls*** According to the *New York Times*/CBS poll of March 2005, referred to in Exercise 65, 72% of those polled ranked neither Iraq nor North Korea as the top foreign policy issue.[33] What is the probability that a randomly selected person polled ranked either Iraq or North Korea as the top foreign policy issue? HINT [See Example 5.]

67. ***Auto Sales*** In April 2008, the probability that a randomly chosen new automobile was manufactured by General Motors was .21, while the probability that it was manufactured by Toyota was .18.[34] What is the probability that a randomly chosen new automobile was manufactured by neither company?

68. ***Auto Sales*** In April 2008, the probability that a randomly chosen new automobile was manufactured by Ford was .15, while the probability that it was manufactured by Chrysler was .12.[35] What is the probability that a randomly chosen new automobile was manufactured by neither company?

Student Admissions *Exercises 69–84 are based on the following table, which shows the profile, by Math SAT I scores, of admitted students at UCLA for the Fall 2008 semester.*[36]

Math SAT I

	700–800	600–699	500–599	400–499	200–399	Total
Admitted	5,919	3,822	1,470	425	49	11,685
Not admitted	6,313	14,237	10,797	4,755	1,273	37,375
Total applicants	12,232	18,059	12,267	5,180	1,322	49,060

Determine the probabilities of the following events. (Round your answers to the nearest .01.) HINT [Example 6.]

69. An applicant was admitted.

70. An applicant had a Math SAT below 400.

71. An applicant had a Math SAT below 400 and was admitted.

72. An applicant had a Math SAT of 700 or above and was admitted.

73. An applicant was not admitted.

74. An applicant did not have a Math SAT below 400.

75. An applicant had a Math SAT in the range 500–599 or was admitted.

76. An applicant had a Math SAT of 700 or above or was admitted.

77. An applicant neither was admitted nor had a Math SAT in the range 500–599.

78. An applicant neither had a Math SAT of 700 or above nor was admitted.

79. ▼ An applicant who had a Math SAT below 400 was admitted.

80. ▼ An applicant who had a Math SAT of 700 or above was admitted.

81. ▼ An admitted student had a Math SAT of 700 or above.

82. ▼ An admitted student had a Math SAT below 400.

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[31] The name and concept were borrowed from a hilarious (as yet unpublished) novel by the science-fiction writer William Orr, who also happens to be a faculty member at Hofstra University.

[32] Source: *New York Times*, March 3, 2005, p. A20.

[33] Ibid.

[34] Source: *New York Times,* May 2, 2008. (www.nytimes.com/imagepages/2008/05/02/business/02autosgraphic.ready.html).

[35] Ibid.

[36] Source: University of California (www.admissions.ucla.edu/Prospect/Adm_fr/Frosh_Prof08.htm).

83. ▼ A rejected applicant had a Math SAT below 600.

84. ▼ A rejected applicant had a Math SAT of at least 600.

85. ▼ ***Social Security*** According to the *New York Times*/CBS poll of March, 2005, referred to in Exercise 65, 79% agreed that it should be the government's responsibility to provide a decent standard of living for the elderly, and 43% agreed that it would be a good idea to invest part of their Social Security taxes on their own.[37] What is the smallest percentage of people who could have agreed with both statements? What is the largest percentage of people who could have agreed with both statements?

86. ▼ ***Social Security*** According to the *New York Times*/CBS poll of March, 2005, referred to in Exercise 65, 49% agreed that Social Security taxes should be raised if necessary to keep the system afloat, and 43% agreed that it would be a good idea to invest part of their Social Security taxes on their own.[38] What is the largest percentage of people who could have agreed with at least one of these statements? What is the smallest percentage of people who could have agreed with at least one of these statements?

87. ▼ ***Greek Life*** The ΤΦΦ Sorority has a tough pledging program—it requires its pledges to master the Greek alphabet forward, backward, and "sideways." During the last pledge period, two-thirds of the pledges failed to learn it backward and three quarters of them failed to learn it sideways; 5 of the 12 pledges failed to master it either backward or sideways. Because admission into the sisterhood requires both backward and sideways mastery, what fraction of the pledges were disqualified on this basis?

88. ▼ ***Swords and Sorcery*** Lance the Wizard has been informed that tomorrow there will be a 50% chance of encountering the evil Myrmidons and a 20% chance of meeting up with the dreadful Balrog. Moreover, Hugo the Elf has predicted that there is a 10% chance of encountering both tomorrow. What is the probability that Lance will be lucky tomorrow and encounter neither the Myrmidons nor the Balrog?

89. ▼ ***Public Health*** A study shows that 80% of the population has been vaccinated against the Venusian flu, but 2% of the vaccinated population gets the flu anyway. If 10% of the total population gets this flu, what percent of the population either gets the vaccine or gets the disease?

90. ▼ ***Public Health*** A study shows that 75% of the population has been vaccinated against the Martian ague, but 4% of this group gets this disease anyway. If 10% of the total population gets this disease, what is the probability that a randomly selected person has been neither vaccinated nor has contracted Martian ague?

[37] Source: *New York Times*, March 3, 2005, p. A20.

[38] Ibid.

COMMUNICATION AND REASONING EXERCISES

91. Design an experiment based on rolling a fair die for which there are exactly three outcomes with the same probabilities.

92. Design an experiment based on rolling a fair die for which there are at least three outcomes with different probabilities.

93. ▼ Tony has had a "losing streak" at the casino—the chances of winning the game he is playing are 40%, but he has lost five times in a row. Tony argues that, because he should have won two times, the game must obviously be "rigged." Comment on his reasoning.

94. ▼ Maria is on a winning streak at the casino. She has already won four times in a row and concludes that her chances of winning a fifth time are good. Comment on her reasoning.

95. Complete the following sentence. The probability of the union of two events is the sum of the probabilities of the two events if _____.

96. A friend of yours asserted at lunch today that, according to the weather forecast for tomorrow, there is a 52% chance of rain and a 60% chance of snow. "But that's impossible!" you blurted out, "the percentages add up to more than 100%." Explain why you were wrong.

97. ▼ A certain experiment is performed a large number of times, and the event E has relative frequency equal to zero. This means that it should have modeled probability zero, right? HINT [See the definition of a probability model on p. 469.]

98. ▼ (Refer to the preceding exercise.) How can the modeled probability of winning the lotto be nonzero if you have never won it, despite playing 600 times? HINT [See the definition of a probability model on p. 469.]

99. ▼ Explain how the addition principle for mutually exclusive events follows from the general addition principle.

100. ▼ Explain how the property $P(A') = 1 - P(A)$ follows directly from the properties of a probability distribution.

101. ◆ It is said that lightning never strikes twice in the same spot. Assuming this to be the case, what should be the modeled probability that lightning will strike a given spot during a thunderstorm? Explain. HINT [See the definition of a probability model on p. 469.]

102. ◆ A certain event has modeled probability equal to zero. This means it will never occur, right? HINT [See the definition of a probability model on p. 469.]

103. ◆ Find a formula for the probability of the union of three (not necessarily mutually exclusive) events A, B, and C.

104. ◆ Four events A, B, C, and D have the following property: If any two events have an outcome in common, that outcome is common to all four events. Find a formula for the probability of their union.

7.4 Probability and Counting Techniques

We saw in the preceding section that, when all outcomes in a sample space are equally likely, we can use the following formula to model the probability of each event:

Modeling Probability: Equally Likely Outcomes

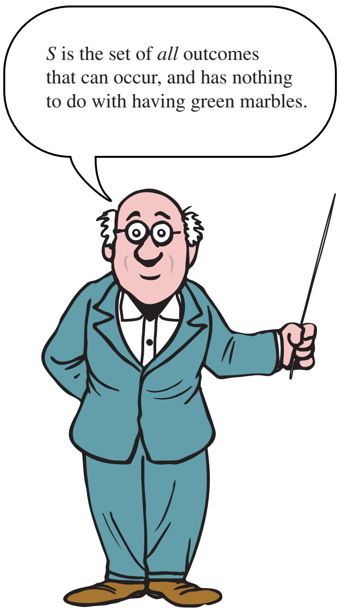
In an experiment in which all outcomes are equally likely, the probability of an event E is given by

$$P(E) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}} = \frac{n(E)}{n(S)}.$$

This formula is simple, but calculating $n(E)$ and $n(S)$ may not be. In this section, we look at some examples in which we need to use the counting techniques discussed in Chapter 6.

EXAMPLE 1 Marbles

A bag contains four red marbles and two green ones. Upon seeing the bag, Suzan (who has compulsive marble-grabbing tendencies) sticks her hand in and grabs three at random. Find the probability that she will get both green marbles.



Solution According to the formula, we need to know these numbers:

- The number of elements in the sample space S.
- The number of elements in the event E.

First of all, what is the sample space? The sample space is the set of all possible outcomes, and each outcome consists of a set of three marbles (in Suzan's hand). So, the set of outcomes is the set of all sets of three marbles chosen from a total of six marbles (four red and two green). Thus,

$$n(S) = C(6, 3) = 20.$$

Now what about E? This is the event that Suzan gets both green marbles. We must *rephrase this as a subset of S* in order to deal with it: "E is the collection of sets of three marbles such that one is red and two are green." Thus, $n(E)$ is the *number* of such sets, which we determine using a decision algorithm.

Step 1 Choose a red marble; $C(4, 1) = 4$ possible outcomes.

Step 2 Choose the two green marbles; $C(2, 2) = 1$ possible outcome.

We get $n(E) = 4 \times 1 = 4$. Now,

$$P(E) = \frac{n(E)}{n(S)} = \frac{4}{20} = \frac{1}{5}.$$

Thus, there is a one in five chance of Suzan's getting both the green marbles.

EXAMPLE 2 Investment Lottery

In order to "spice up" your investment portfolio, you decided to ignore your broker's cautious advice and select three stocks at random from the six most active stocks listed on the New York Stock Exchange shortly after the start of trading on January 23, 2009:*

Company	*Symbol*	*Price*	*Change*
General Electric	GE	\$12.74	−\$0.74
Bank of America	BAC	\$5.63	−\$0.03
SPDR S&P 500	SPY	\$82.75	\$0.00
Ultra Financials Pro	UYG	\$2.86	−\$0.16
Citigroup	C	\$3.11	\$0.00
Wyeth	WYE	\$42.81	+\$4.15

Find the probabilities of the following events:

a. Your portfolio included GE and BAC.

b. At most two of the stocks in your portfolio declined in value.

Solution First, the sample space is the set of all collections of three stocks chosen from the six. Thus,

$$n(S) = C(6, 3) = 20.$$

a. The event E of interest is the event that your portfolio includes GE and BAC. Thus, E is the set of all groups of three stocks that include GE and BAC. Because there is only one more stock left to choose,

$$n(E) = C(4, 1) = 4.$$

We now have

$$P(E) = \frac{n(E)}{n(S)} = \frac{4}{20} = \frac{1}{5} = .2.$$

b. Let F be the event that at most two of the stocks in your portfolio declined in value. Thus, F is the set of all portfolios of three stocks of which at most two declined in value. To calculate $n(F)$, we use the following decision algorithm, noting that three of the stocks declined in value and three did not.

Alternative 1: None of the stocks declined in value.

Step 1 Choose three stocks that did not decline in value; $C(3, 3) = 1$ possibility.

Alternative 2: One of the stocks declined in value.

Step 1 Choose one stock that declined in value; $C(3, 1) = 3$ possibilities.

Step 2 Choose two stocks that did not decline in value; $C(3, 2) = 3$ possibilities.

This gives $3 \times 3 = 9$ possibilities for this alternative.

Alternative 3: Two of the stocks declined in value.

Step 1 Choose two stocks that declined in value; $C(3, 2) = 3$ possibilities.

Step 2 Choose one stock that did not decline in value; $C(3, 1) = 3$ possibilities.

This gives $3 \times 3 = 9$ possibilities for this alternative.

*Prices shortly after the market opened on January 23, 2009. Source: Yahoo! Finance (http://finance.yahoo.com).

So, we have a total of $1 + 9 + 9 = 19$ possible portfolios. Thus,

$$n(F) = 19$$

and

$$P(F) = \frac{n(F)}{n(S)} = \frac{19}{20} = .95.$$

At most two of the stocks declined in value.

Complementary Events

At least three of the stocks declined in value.

Before we go on... When counting the number of outcomes in an event, the calculation is sometimes easier if we look at the *complement* of that event. In the case of part (b) of Example 2, the complement of the event F is

F': At least three of the stocks in your portfolio declined in value.

Because there are only three stocks in your portfolio, this is the same as the event that all three stocks in your portfolio declined in value. The decision algorithm for $n(F')$ is far simpler:

Step 1 Choose three stocks that declined in value: $C(3, 3) = 1$ possibility.

So, $n(F') = 1$, giving

$$n(F) = n(S) - n(F') = 20 - 1 = 19$$

as we calculated above. ■


Chris Ware/The Image Works

EXAMPLE 3 Poker Hands

You are dealt 5 cards from a well-shuffled standard deck of 52. Find the probability that you have a full house. (Recall that a full house consists of 3 cards of one denomination and 2 of another.)

Solution The sample space S is the set of all possible 5-card hands dealt from a deck of 52. Thus,

$$n(S) = C(52, 5) = 2{,}598{,}960.$$

If the deck is thoroughly shuffled, then each of these 5-card hands is equally likely. Now consider the event E, the set of all possible 5-card hands that constitute a full house. To calculate $n(E)$, we use a decision algorithm, which we show in the following compact form.

1. Choose first denomination.
2. Choose three cards of that denomination.
3. Choose second denomination.
4. Choose two cards of that denomination.

$$n(E) = C(13, 1) \times C(4, 3) \times C(12, 1) \times C(4, 2) = 3{,}744$$

Thus,

$$P(E) = \frac{n(E)}{n(S)} = \frac{3{,}744}{2{,}598{,}960} \approx .00144.$$

In other words, there is an approximately 0.144% chance that you will be dealt a full house.

EXAMPLE 4 More Poker Hands

You are playing poker, and you have been dealt the following hand:

J♠, J♦, J♥, 2♣, 10♠.

You decide to exchange the last two cards. The exchange works as follows: The two cards are discarded (not replaced in the deck), and you are dealt two new cards.

a. Find the probability that you end up with a full house.

b. Find the probability that you end up with four jacks.

c. What is the probability that you end up with either a full house or four jacks?

Solution

a. In order to get a full house, you must be dealt two of a kind. The sample space S is the set of all pairs of cards selected from what remains of the original deck of 52. You were dealt 5 cards originally, so there are $52 - 5 = 47$ cards left in the deck. Thus, $n(S) = C(47, 2) = 1{,}081$. The event E is the set of all pairs of cards that constitute two of a kind. Note that you cannot get two jacks because only one is left in the deck. Also, only three 2s and three 10s are left in the deck. We have

1. Choose a denomination other than Jacks, 2s, and 10s. **1.** Choose either 2s or 10s.

OR

$$n(E) = C(10, 1) \times C(4, 2) \quad + \quad C(2, 1) \times C(3, 2) = 66$$

2. Choose two cards of that denomination. **2.** Choose two cards of that denomination.

Thus,

$$P(E) = \frac{n(E)}{n(S)} = \frac{66}{1{,}081} \approx .0611.$$

b. We have the same sample space as in part (a). Let F be the set of all pairs of cards that include the missing jack of clubs. So,

1. Choose the jack of clubs.

2. Choose 1 card from the remaining 46.

$$n(F) = C(1, 1) \times C(46, 1) = 46.$$

Thus,

$$P(F) = \frac{n(F)}{n(S)} = \frac{46}{1{,}081} \approx .0426.$$

c. We are asked to calculate the probability of the event $E \cup F$. From the addition principle, we have

$$P(E \cup F) = P(E) + P(F) - P(E \cap F).$$

Because $E \cap F$ means "E and F," $E \cap F$ is the event that the pair of cards you are dealt are two of a kind and include the jack of clubs. But this is impossible because only one jack is left. Thus $E \cap F = \emptyset$, and so $P(E \cap F) = 0$. This gives us

$$P(E \cup F) = P(E) + P(F) \approx .0611 + .0426 = .1037.$$

In other words, there is slightly better than a one in ten chance that you will wind up with either a full house or four of a kind, given the original hand.

Before we go on... A more accurate answer to part (c) of Example 4 is $(66 + 46)/1{,}081 \approx .1036$; we lost some accuracy in rounding the answers to parts (a) and (b). ■

EXAMPLE 5 Committees

The University Senate bylaws at Hofstra University state the following:*

> The Student Affairs Committee shall consist of one elected faculty senator, one faculty senator-at-large, one elected student senator, five student senators-at-large (including one from the graduate school), two delegates from the Student Government Association, the President of the Student Government Association or his/her designate, and the President of the Graduate Student Organization. It shall be chaired by the elected student senator on the Committee and it shall be advised by the Dean of Students or his/her designate.

You are an undergraduate student and, even though you are not an elected student senator, you would very much like to serve on the Student Affairs Committee. The senators-at-large as well as the Student Government delegates are chosen by means of a random drawing from a list of candidates. There are already 13 undergraduate candidates for the position of senator-at-large, and 6 candidates for Student Government delegates, and you have been offered a position on the Student Government Association by the President (who happens to be a good friend of yours), should you wish to join it. (This would make you ineligible for a senator-at-large position.) What should you do?

Solution You have two options. Option 1 is to include your name on the list of candidates for the senator-at-large position. Option 2 is to join the Student Government Association (SGA) and add your name to its list of candidates. Let us look at the two options separately.

Option 1: Add your name to the senator-at-large list.
This will result in a list of 14 undergraduates for 4 undergraduate positions. The sample space is the set of all possible outcomes of the random drawing. Each outcome consists of a set of 4 lucky students chosen from 14. Thus,

$$n(S) = C(14, 4) = 1{,}001.$$

We are interested in the probability that you are among the chosen four. Thus, E is the set of sets of four that include you.

$$n(E) = \underbrace{C(1, 1)}_{\text{1. Choose yourself.}} \times \underbrace{C(13, 3)}_{\text{2. Choose 3 from the remaining 13.}} = 286.$$

* As of 2009. Source: Hofstra University Senate Bylaws.

So,

$$P(E) = \frac{n(E)}{n(S)} = \frac{286}{1{,}001} = \frac{2}{7} \approx .2857.$$

Option 2: Join the SGA and add your name to its list.
This results in a list of seven candidates from which two are selected. For this case, the sample space consists of all sets of two chosen from seven, so

$$n(S) = C(7, 2) = 21$$

and

$$n(E) = \underset{\text{1. Choose yourself.}}{C(1, 1)} \times \underset{\text{2. Choose one from the remaining six.}}{C(6, 1)} = 6.$$

Thus,

$$P(E) = \frac{n(E)}{n(S)} = \frac{6}{21} = \frac{2}{7} \approx .2857.$$

In other words, the probability of being selected is exactly the same for Option 1 as it is for Option 2! Thus, you can choose either option, and you will have slightly less than a 29% chance of being selected.

7.4 EXERCISES

▼ more advanced ◆ challenging

T indicates exercises that should be solved using technology

Recall from Example 1 that whenever Suzan sees a bag of marbles, she grabs a handful at random. In Exercises 1–10, she has seen a bag containing four red marbles, three green ones, two white ones, and one purple one. She grabs five of them. Find the probabilities of the following events, expressing each as a fraction in lowest terms. HINT [See Example 1.]

1. She has all the red ones.
2. She has none of the red ones.
3. She has at least one white one.
4. She has at least one green one.
5. She has two red ones and one of each of the other colors.
6. She has two green ones and one of each of the other colors.
7. She has at most one green one.
8. She has no more than one white one.
9. She does not have all the red ones.
10. She does not have all the green ones.

Dogs of the Dow *The phrase "Dogs of the Dow" refers to the stocks listed on the Dow with the highest dividend yield. Exercises 11–16 are based on the following table, which shows the top ten stocks of the "Dogs of the Dow" list in January 2009:*[39] HINT [See Example 2.]

Symbol	*Company*	*Yield*
BAC	Bank of America	9.09%
GE	General Electric	7.65%
PFE	Pfizer	7.23%
DD	DuPont	6.48%
AA	Alcoa	6.04%
T	AT&T	5.75%
VZ	Verizon	5.43%
MRK	Merck	5.00%
JPM	JP Morgan Chase	4.82%
KFT	Kraft	4.32%

[39] Source: www.dogsofthedow.com.

11. If you selected two of these stocks at random, what is the probability that both the stocks in your selection had yields of 7% or more?

12. If you selected three of these stocks at random, what is the probability that all three of the stocks in your selection had yields of 7% or more?

13. If you selected four of these stocks at random, what is the probability that your selection included the company with the highest yield and excluded the company with the lowest yield?

14. If you selected four of these stocks at random, what is the probability that your selection included BAC and GE but excluded PFE and DD?

15. ▼ If your portfolio included 100 shares of PFE and you then purchased 100 shares each of any two companies on the list at random, find the probability that you ended up with a total of 200 shares of PFE.

16. ▼ If your portfolio included 100 shares of PFE and you then purchased 100 shares each of any three companies on the list at random, find the probability that you ended up with a total of 200 shares of PFE.

17. ***Tests*** A test has three parts. Part A consists of eight true-false questions, Part B consists of five multiple choice questions with five choices each, and Part C requires you to match five questions with five different answers one-to-one. Assuming that you make random guesses in filling out your answer sheet, what is the probability that you will earn 100% on the test? (Leave your answer as a formula.)

18. ***Tests*** A test has three parts. Part A consists of four true-false questions, Part B consists of four multiple choice questions with five choices each, and Part C requires you to match six questions with six different answers one-to-one. Assuming that you make random choices in filling out your answer sheet, what is the probability that you will earn 100% on the test? (Leave your answer as a formula.)

Poker *In Exercises 19–24, you are asked to calculate the probability of being dealt various poker hands. (Recall that a poker player is dealt 5 cards at random from a standard deck of 52.) Express each of your answers as a decimal rounded to four decimal places, unless otherwise stated.* HINT [See Example 3.]

19. Two of a kind: Two cards with the same denomination and three cards with other denominations (different from each other and that of the pair). Example: K♣, K♥, 2♠, 4♦, J♠

20. Three of a kind: Three cards with the same denomination and two cards with other denominations (different from each other and that of the three). Example: Q♣, Q♥, Q♠, 4♦, J♠

21. Two pair: Two cards with one denomination, two with another, and one with a third. Example: 3♣, 3♥, Q♠, Q♥, 10♠

22. Straight Flush: Five cards of the same suit with consecutive denominations but not a royal flush (a royal flush consists of the 10, J, Q, K, and A of one suit). Round the answer to one significant digit. Examples: A♣, 2♣, 3♣, 4♣, 5♣, or 9♦, 10♦, J♦, Q♦, K♦, or A♥, 2♥, 3♥, 4♥, 5♥, but *not* 10♦, J♦, Q♦, K♦, A♦

23. Flush: Five cards of the same suit, but not a straight flush or royal flush. Example: A♣, 5♣, 7♣, 8♣, K♣

24. Straight: Five cards with consecutive denominations, but not all of the same suit. Examples: 9♦, 10♦, J♣, Q♥, K♦, and 10♥, J♦, Q♦, K♦, A♦.

25. ***The Monkey at the Typewriter*** Suppose that a monkey is seated at a computer keyboard and randomly strikes the 26 letter keys and the space bar. Find the probability that its first 39 characters (including spaces) will be "to be or not to be that is the question". (Leave your answer as a formula.)

26. ***The Cat on the Piano*** A standard piano keyboard has 88 different keys. Find the probability that a cat, jumping on 4 keys in sequence and at random (possibly with repetition), will strike the first four notes of Beethoven's Fifth Symphony. (Leave your answer as a formula.)

27. ***(Based on a question from the GMAT)*** Tyler and Gebriella are among seven contestants from which four semifinalists are to be selected at random. Find the probability that neither Tyler nor Gebriella is selected.

28. ***(Based on a question from the GMAT)*** Tyler and Gebriella are among seven contestants from which four semifinalists are to be selected at random. Find the probability that Tyler but not Gebriella is selected.

29. ▼ ***Lotteries*** The Sorry State Lottery requires you to select five different numbers from 0 through 49. (Order is not important.) You are a Big Winner if the five numbers you select agree with those in the drawing, and you are a Small-Fry Winner if four of your five numbers agree with those in the drawing. What is the probability of being a Big Winner? What is the probability of being a Small-Fry Winner? What is the probability that you are either a Big Winner or a Small-Fry winner?

30. ▼ ***Lotto*** The Sad State Lottery requires you to select a sequence of three different numbers from 0 through 49. (Order is important.) You are a winner if your sequence agrees with that in the drawing, and you are a booby prize winner if your selection of numbers is correct, but in the wrong order. What is the probability of being a winner? What is the probability of being a booby prize winner? What is the probability that you are either a winner or a booby prize winner?

31. ▼ ***Transfers*** Your company is considering offering 400 employees the opportunity to transfer to its new headquarters in Ottawa and, as personnel manager, you decide that it would be fairest if the transfer offers are decided by means of a lottery. Assuming that your company currently employs 100 managers, 100 factory workers, and 500 miscellaneous staff, find the following probabilities, leaving the answers as formulas:

a. All the managers will be offered the opportunity.
b. You will be offered the opportunity.

32. ▼ ***Transfers*** (Refer back to the preceding exercise.) After thinking about your proposed method of selecting employees for the opportunity to move to Ottawa, you decide it might be a better idea to select 50 managers, 50 factory workers, and 300 miscellaneous staff, all chosen at random. Find the probability that you will be offered the opportunity. (Leave your answer as a formula.)

33. ▼ ***Lotteries*** In a New York State daily lottery game, a sequence of three digits (not necessarily different) in the range 0–9 are selected at random. Find the probability that all three are different.

34. ▼ ***Lotteries*** Refer back to the preceding exercise. Find the probability that two of the three digits are the same.

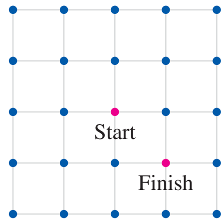
35. ▼ ***Sports*** The following table shows the results of the Big Eight Conference for the 1988 college football season.[40]

Team	*Won*	*Lost*
Nebraska (NU)	7	0
Oklahoma (OU)	6	1
Oklahoma State (OSU)	5	2
Colorado (CU)	4	3
Iowa State (ISU)	3	4
Missouri (MU)	2	5
Kansas (KU)	1	6
Kansas State (KSU)	0	7

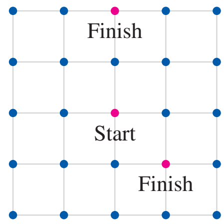
This is referred to as a "perfect progression." Assuming that the "Won" score for each team is chosen at random in the range 0–7, find the probability that the results form a perfect progression.[41] (Leave your answer as a formula.)

36. ▼ ***Sports*** Refer back to Exercise 35. Find the probability of a perfect progression with Nebraska scoring seven wins and zero losses. (Leave your answer as a formula.)

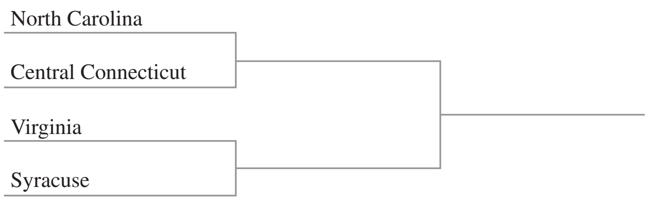
37. ▼ ***Graph Searching*** A graph consists of a collection of **nodes** (the dots in the figure) connected by **edges** (line segments from one node to another). A **move on a graph** is a move from one node to another along a single edge. Find the probability of going from Start to Finish in a sequence of two random moves in the graph shown. (All directions are equally likely.)



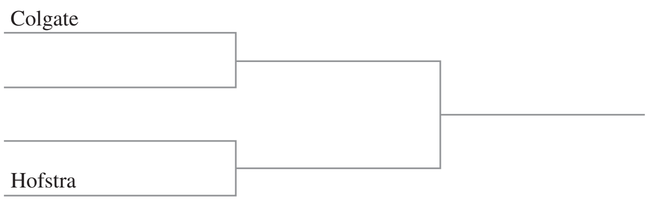
38. ▼ ***Graph Searching*** Refer back to Exercise 37. Find the probability of going from Start to one of the Finish nodes in a sequence of two random moves in the following figure. (All directions are equally likely.)



39. ▼ ***Tournaments*** What is the probability that North Carolina will beat Central Connecticut but lose to Virginia in the following (fictitious) soccer tournament? (Assume that all outcomes are equally likely.)



40. ▼ ***Tournaments*** In a (fictitious) soccer tournament involving the four teams San Diego State, De Paul, Colgate, and Hofstra, find the probability that Hofstra will play Colgate in the finals and win. (Assume that all outcomes are equally likely and that the teams not listed in the first round slots are placed at random.)



41. ◆ ***Product Design*** Your company has patented an electronic digital padlock which a user can program with his or her own four-digit code. (Each digit can be 0 through 9, and repetitions are allowed.) The padlock is designed to open either if the correct code is keyed in or—and this is helpful for forgetful people—if exactly one of the digits is incorrect. What is the probability that a randomly chosen sequence of four digits will open a programmed padlock?

[40] Source: On the probability of a perfect progression, *The American Statistician,* August 1991, vol. 45, no. 3, p. 214.

[41] Even if all the teams are equally likely to win each game, the chances of a perfect progression actually coming up are a little more difficult to estimate, because the number of wins by one team impacts directly on the number of wins by the others. For instance, it is impossible for all eight teams to show a score of seven wins and zero losses at the end of the season—someone must lose! It is, however, not too hard to come up with a counting argument to estimate the total number of win-loss scores actually possible.

42. ◆ ***Product Design*** Assume that you already know the first digit of the combination for the lock described in Exercise 41. Find the probability that a random guess of the remaining three digits will open the lock. HINT [See Example 5.]

43. ◆ ***Committees*** An investigatory Committee in the Kingdom of Utopia consists of a chief investigator (a Royal Party member), an assistant investigator (a Birthday Party member), two at-large investigators (either party), and five ordinary members (either party). Royal Party member Larry Sifford is hoping to avoid serving on the committee, unless he is the Chief Investigator and Otis Taylor, a Birthday Party member, is the Assistant Investigator. The committee is to be selected at random from a pool of 12 candidates (including Larry Sifford and Otis Taylor), half of whom are Royal Party and half of whom are Birthday Party.

   a. How many different committees are possible? HINT [See Example 5.]
   b. How many committees are possible in which Larry's hopes are fulfilled? (This includes the possibility that he's not on the committee at all.)
   c. What is the probability that he'll be happy with a randomly selected committee?

44. ◆ ***Committees*** A committee is to consist of a chair, three hagglers, and four do-nothings. The committee is formed by choosing randomly from a pool of 10 people and assigning them to the various "jobs."

   a. How many different committees are possible? HINT [See Example 5.]
   b. Norman is eager to be the chair of the committee. What is the probability that he will get his wish?
   c. Norman's girlfriend Norma is less ambitious and would be happy to hold any position on the committee provided Norman is also selected as a committee member. What is the probability that she will get her wish and serve on the committee?
   d. Norma does not get along with Oona (who is also in the pool of prospective members) and would be most unhappy if Oona were to chair the committee. Find the probability that all her wishes will be fulfilled: she and Norman are on the committee and it is not chaired by Oona.

COMMUNICATION AND REASONING EXERCISES

45. What is wrong with the following argument? A bag contains two blue marbles and two red ones; two are drawn at random. Because there are four possibilities—(red, red), (blue, blue), (red, blue) and (blue, red)—the probability that both are red is $1/4$.

46. What is wrong with the following argument? When we roll two indistinguishable dice, the number of possible outcomes (unordered groups of two not necessarily distinct numbers) is 21 and the number of outcomes in which both numbers are the same is 6. Hence, the probability of throwing a double is $6/21 = 2/7$.

47. ▼ Suzan grabs two marbles out of a bag of five red marbles and four green ones. She could do so in two ways: She could take them out one at a time, so that there is a first and a second marble, or she could grab two at once so that there is no order. Does the method she uses to grab the marbles affect the probability that she gets two red marbles?

48. ▼ If Suzan grabs two marbles, one at a time, out of a bag of five red marbles and four green ones, find an event with a probability that depends on the order in which the two marbles are drawn.

49. Create an interesting application whose solution requires finding a probability using combinations.

50. Create an interesting application whose solution requires finding a probability using permutations.

7.5 Conditional Probability and Independence

Cyber Video Games, Inc., ran a television ad in advance of the release of its latest game, "Ultimate Hockey." As Cyber Video's director of marketing, you would like to assess the ad's effectiveness, so you ask your market research team to survey video game players. The results of its survey of 2000 video game players are summarized in the following table:

	Saw Ad	Did Not See Ad	Total
Purchased Game	100	200	300
Did Not Purchase Game	200	1,500	1,700
Total	300	1,700	2,000

The market research team concludes in its report that the ad is highly persuasive, and recommends using the company that produced the ad for future projects.

But wait, how could the ad possibly have been persuasive? Only 100 people who saw the ad purchased the game, while 200 people purchased the game without seeing the ad at all! At first glance, it looks as though potential customers are being *put off* by the ad. But let us analyze the figures a little more carefully.

First, let us restrict attention to those players who saw the ad (first column of data: "Saw Ad") and compute the estimated probability that a player *who saw the ad* purchased Ultimate Hockey.

	Saw Ad
Purchased Game	100
Did Not Purchase Game	200
Total	300

To compute this probability, we calculate

$$\text{Probability that someone who saw the ad purchased the game}$$
$$= \frac{\text{Number of people who saw the ad and bought the game}}{\text{Total number of people who saw the ad}} = \frac{100}{300} \approx .33.$$

In other words, 33% of game players who saw the ad went ahead and purchased the game. Let us compare this with the corresponding probability for those players who did *not* see the ad (second column of data "Did Not See Ad"):

	Did Not See Ad
Purchased Game	200
Did Not Purchase Game	1,500
Total	1,700

$$\text{Probability that someone who did not see the ad purchased the game}$$
$$= \frac{\text{Number of people who did not see the ad and bought the game}}{\text{Total number of people who did not see the ad}} = \frac{200}{1{,}700} \approx .12.$$

In other words, only 12% of game players who did not see the ad purchased the game, whereas 33% of those who *did* see the ad purchased the game. Thus, it appears that the ad *was* highly persuasive.

Here's some terminology. In this example there were two related events of importance:

A: A video game player purchased Ultimate Hockey.

B: A video game player saw the ad.

The first probability we computed was the estimated probability that a video game player purchased Ultimate Hockey *given that* he or she saw the ad. We call the latter probability the (estimated) **probability of A, given B**, and we write it as $P(A \mid B)$. We call $P(A \mid B)$ a **conditional probability**—it is the probability of A under the condition that B occurred. Put another way, it is the probability of A occurring if the sample space is reduced to just those outcomes in B.

$$P(\text{Purchased game } \textit{given that} \text{ saw the ad}) = P(A \mid B) \approx .33$$

The second probability we computed was the estimated probability that a video game player purchased Ultimate Hockey *given that* he or she did not see the ad, or the **probability of A, given B'**.

$$P(\text{Purchased game } \textit{given that} \text{ did not see the ad}) = P(A \mid B') \approx .12$$

Calculating Conditional Probabilities

How do we calculate conditional probabilities? In the example above we used the ratio

$$P(A \mid B) = \frac{\text{Number of people who saw the ad and bought the game}}{\text{Total number of people who saw the ad}}.$$

The numerator is the frequency of $A \cap B$, and the denominator is the frequency of B:

$$P(A \mid B) = \frac{fr(A \cap B)}{fr(B)}.$$

Now, we can write this formula in another way:

$$P(A \mid B) = \frac{fr(A \cap B)}{fr(B)} = \frac{fr(A \cap B)/N}{fr(B)/N} = \frac{P(A \cap B)}{P(B)}.$$

We therefore have the following definition, which applies to general probability distributions.

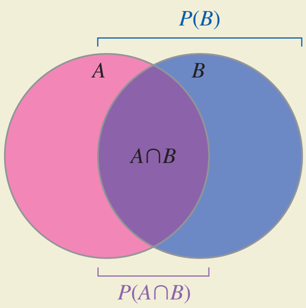
Conditional Probability

If A and B are events with $P(B) \neq 0$, then the probability of A given B is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

Visualizing Conditional Probability

In the figure, $P(A \mid B)$ is represented by the fraction of B that is covered by A.



$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Quick Examples

1. If there is a 50% chance of rain (R) and a 10% chance of both rain and lightning (L), then the probability of lightning, given that it rains, is

$$P(L \mid R) = \frac{P(L \cap R)}{P(R)} = \frac{.10}{.50} = .20.$$

Here are two more ways to express the result:

- If it rains, the probability of lightning is .20.
- Assuming that it rains, there is a 20% chance of lightning.

2. Referring to the Cyber Video data at the beginning of this section, the probability that a video game player did not purchase the game (A'), given that she did not see the ad (B'), is

$$P(A' \mid B') = \frac{P(A' \cap B')}{P(B')} = \frac{1{,}500/2{,}000}{1{,}700/2{,}000} = \frac{15}{17} \approx .88.$$

Q: *Returning to the video game sales survey, how do we compute the* ordinary probability *of A, not "given" anything?*

A: We look at the event A that a randomly chosen game player purchased Ultimate Hockey *regardless of whether or not he or she saw the ad*. In the "Purchased Game" row we see that a total of 300 people purchased the game out of a total of 2,000 surveyed. Thus, the (estimated) probability of A is

$$P(A) = \frac{fr(A)}{N} = \frac{300}{2{,}000} = .15.$$

We sometimes refer to $P(A)$ as the **unconditional** probability of A to distinguish it from conditional probabilities like $P(A|B)$ and $P(A|B')$.

Now, let's see some more examples involving conditional probabilities.

EXAMPLE 1 Dice

If you roll a fair die twice and observe the numbers that face up, find the probability that the sum of the numbers is 8, given that the first number is 3.

Solution We begin by recalling that the sample space when we roll a fair die twice is the set $S = \{(1, 1), (1, 2), \ldots, (6, 6)\}$ containing the 36 different equally likely outcomes.

The two events under consideration are

A: The sum of the numbers is 8.

B: The first number is 3.

We also need

$A \cap B$: The sum of the numbers is 8 and the first number is 3.

But this can only happen in one way: $A \cap B = \{(3, 5)\}$. From the formula, then,

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/36}{6/36} = \frac{1}{6}.$$

Before we go on... There is another way to think about Example 1. When we say that the first number is 3, we are restricting the sample space to the six outcomes (3, 1), (3, 2), . . . , (3, 6), all still equally likely. Of these six, only one has a sum of 8, so the probability of the sum being 8, given that the first number is 3, is $1/6$. ■

Notes

1. Remember that, in the expression $P(A \mid B)$, A is the event whose probability you want, given that you know the event B has occurred.
2. From the formula, notice that $P(A \mid B)$ is not defined if $P(B) = 0$. Could $P(A \mid B)$ make any sense if the event B were impossible? ■

EXAMPLE 2 School and Work

A survey of the high school graduating class of 2007,* conducted by the Bureau of Labor Statistics, found that, if a graduate went on to college, there was a 36% chance that he or she would work at the same time. On the other hand, there was a 67% chance that a randomly selected graduate would go on to college. What is the probability that a graduate went to college and work at the same time?

Solution To understand what the question asks and what information is given, it is helpful to rephrase everything using the standard wording "*the probability that* ___" and "*the probability that* ___ *given that* ___." Now we have, "The probability that a graduate worked, given that the graduate went on to college, equals .36. (See Figure 8.) The probability that a graduate went on to college is .67." The events in question are as follows:

If a graduate went on to college, there was a 36% chance that he or she would work.

Rephrase by filling in the blanks:

The probability that ______ given that ______ equals ____.

***The probability that* a graduate worked, *given that* the graduate went on to college, *equals* .36.**

$P(\text{Worked} \mid \text{Went to college}) = .36$

Figure 8

W: A high school graduate went on to work.

C: A high school graduate went on to college.

From our rephrasing of the question we can write:

$$P(W \mid C) = .36. \qquad P(C) = .67. \qquad \text{Find } P(W \cap C).$$

The definition

$$P(W \mid C) = \frac{P(W \cap C)}{P(C)}$$

can be used to find $P(W \cap C)$:

$$\begin{aligned} P(W \cap C) &= P(W \mid C)P(C) \\ &= (.36)(.67) \approx .24. \end{aligned}$$

Thus there is a 24% chance that a member of the high school graduating class of 2008 went on to college and work at the same time.

*"College Enrollment and Work Activity of 2007 High School Graduates," U.S. Bureau of Labor Statistics, April 2008, available at www.bls.gov/schedule/archives/all_nr.htm.

The Multiplication Principle and Trees

In Example 2, we saw that the formula

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

can be used to calculate $P(A \cap B)$ if we rewrite the formula in the following form, known as the **multiplication principle for conditional probability**:

Multiplication Principle for Conditional Probability

If A and B are events, then

$$P(A \cap B) = P(A \mid B)P(B).$$

Quick Example

If there is a 50% chance of rain (R) and a 20% chance of a lightning (L) if it rains, then the probability of both rain and lightning is

$$P(R \cap L) = P(L \mid R)P(R) = (.20)(.50) = .10.$$

The multiplication principle is often used in conjunction with **tree diagrams**. Let's return to Cyber Video Games, Inc., and its television ad campaign. Its marketing survey was concerned with the following events:

A: A video game player purchased Ultimate Hockey.

B: A video game player saw the ad.

We can illustrate the various possibilities by means of the two-stage "tree" shown in Figure 9.

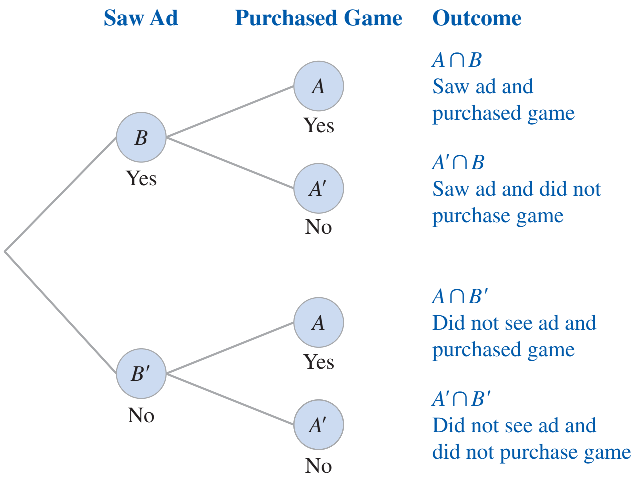


Figure 9

Consider the outcome $A \cap B$. To get there from the starting position on the left, we must first travel up to the B node. (In other words, B must occur.) Then we must travel up the branch from the B node to the A node. We are now going to associate a probability with each branch of the tree: the probability of traveling along that branch *given that we have gotten to its beginning node.* For instance, the probability of traveling up the branch from the starting position to the B node is $P(B) = 300/2{,}000 = .15$ (see the data in the survey). The probability of going up the branch from the B node to the A node is the probability that A occurs, given that B has occurred. In other words, it is the *conditional* probability $P(A \mid B) \approx .33$. (We calculated this probability at the beginning of the section.) The probability of the outcome $A \cap B$ can then be computed using the multiplication principle:

$$P(A \cap B) = P(B)P(A \mid B) \approx (.15)(.33) \approx .05.$$

In other words, *to obtain the probability of the outcome* $A \cap B$, *we multiply the probabilities on the branches leading to that outcome* (Figure 10).

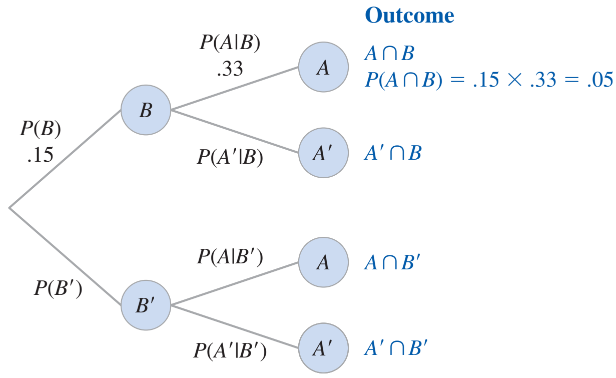


Figure 10

The same argument holds for the remaining three outcomes, and we can use the table given at the beginning of this section to calculate all the conditional probabilities shown in Figure 11.

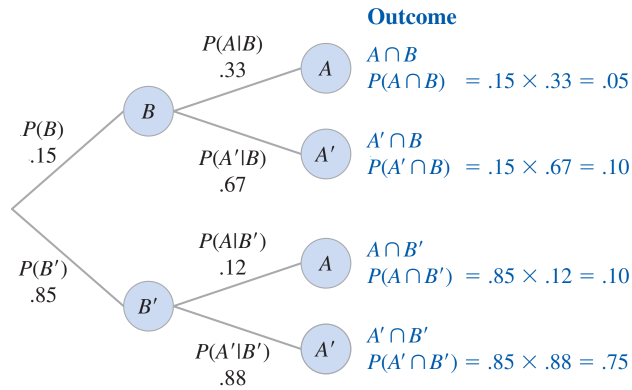


Figure 11

Note The sum of the probabilities on the branches leaving any node is always 1 (why?). This observation often speeds things up because after we have labeled one branch (or, all but one, if a node has more than two branches leaving it), we can easily label the remaining one. ■

EXAMPLE 3 Unfair Coins

An experiment consists of tossing two coins. The first coin is fair, while the second coin is twice as likely to land with heads facing up as it is with tails facing up. Draw a tree diagram to illustrate all the possible outcomes, and use the multiplication principle to compute the probabilities of all the outcomes.

Solution A quick calculation shows that the probability distribution for the second coin is $P(\mathrm{H}) = 2/3$ and $P(\mathrm{T}) = 1/3$. (How did we get that?) Figure 12 shows the tree diagram and the calculations of the probabilities of the outcomes.

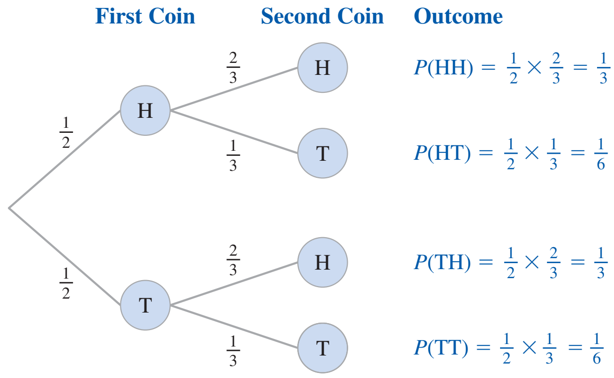


Figure 12

Independence

Let us go back once again to Cyber Video Games, Inc., and its ad campaign. How did we assess the ad's effectiveness? We considered the following events.

A: A video game player purchased Ultimate Hockey.

B: A video game player saw the ad.

We used the survey data to calculate $P(A)$, the probability that a video game player purchased Ultimate Hockey, and $P(A \mid B)$, the probability that a video game player *who saw the ad* purchased Ultimate Hockey. When these probabilities are compared, one of three things can happen.

Case 1 $P(A \mid B) > P(A)$
This is what the survey data actually showed: A video game player was more likely to purchase Ultimate Hockey if he or she saw the ad. This indicates that the ad is effective; seeing the ad had a positive effect on a player's decision to purchase the game.

Case 2 $P(A \mid B) < P(A)$
If this had happened, then a video game player would have been *less* likely to purchase Ultimate Hockey if he or she saw the ad. This would have indicated that the ad had "backfired"; it had, for some reason, put potential customers off. In this case, just as in the first case, the event B would have had an effect—a negative one—on the event A.

Case 3 $P(A \mid B) = P(A)$
In this case seeing the ad would have had absolutely no effect on a potential customer's buying Ultimate Hockey. Put another way, the probability of A occurring *does not depend* on whether B occurred or not. We say in a case like this that the events A and B are **independent**.

In general, we say that two events A and B are independent if $P(A \mid B) = P(A)$. When this happens, we have

$$P(A) = P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

so

$$P(A \cap B) = P(A)P(B).$$

Conversely, if $P(A \cap B) = P(A)P(B)$, then, assuming $P(B) \neq 0$,* $P(A) = P(A \cap B)/P(B) = P(A \mid B)$. Thus, saying that $P(A) = P(A \mid B)$ is the same as saying that $P(A \cap B) = P(A)P(B)$. Also, we can switch A and B in this last formula and conclude that saying that $P(A \cap B) = P(A)P(B)$ is the same as saying that $P(B \mid A) = P(B)$.

* **NOTE** We shall only discuss the independence of two events in cases where their probabilities are both nonzero.

Independent Events

The events A and B are **independent** if

$$P(A \cap B) = P(A)P(B).$$

Equivalent formulas (assuming neither A nor B is impossible) are

$$P(A \mid B) = P(A)$$

and $$P(B \mid A) = P(B).$$

If two events A and B are not independent, then they are **dependent**.

The property $P(A \cap B) = P(A)P(B)$ can be extended to three or more independent events. If, for example, A, B, and C are three mutually independent events (that is, each one of them is independent of each of the other two and of their intersection), then, among other things,

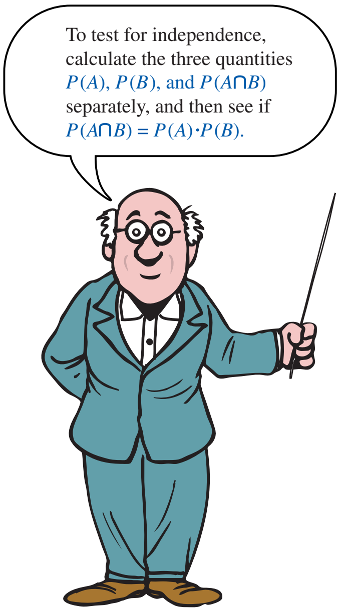
$$P(A \cap B \cap C) = P(A)P(B)P(C).$$

Quick Examples

1. If A and B are independent, and if A has a probability of .2 and B has a probability of .3, then $A \cap B$ has a probability of (.2)(.3) = .06.

2. Let us assume that the phase of the moon has no effect on whether or not my newspaper is delivered. The probability of a full moon (M) on a randomly selected day is about .034, and the probability that my newspaper will be delivered (D) on the random day is .20. Therefore, the probability that it is a full moon and my paper is delivered is

$$P(M \cap D) = P(M)P(D) = (.034)(.20) = .0068.$$



Testing for Independence

To check whether two events A and B are independent, we compute $P(A)$, $P(B)$, and $P(A \cap B)$. If $P(A \cap B) = P(A)P(B)$, the events are independent; otherwise, they are dependent. Sometimes it is obvious that two events, by their nature, are independent, so a test is not necessary. For example, the event that a die you roll comes up 1 is clearly independent of whether or not a coin you toss comes up heads.

Quick Examples

1. Roll two distinguishable dice (one red, one green) and observe the numbers that face up.

A: The red die is even; $P(A) = \frac{18}{36} = \frac{1}{2}$.

B: The dice have the same parity*; $P(B) = \frac{18}{36} = \frac{1}{2}$.

$A \cap B$: Both dice are even; $P(A \cap B) = \frac{9}{36} = \frac{1}{4}$.

$P(A \cap B) = P(A)P(B)$, and so A and B are independent.

* **NOTE** Two numbers have the **same parity** if both are even or both are odd. Otherwise, they have **opposite parity**.

2. Roll two distinguishable dice and observe the numbers that face up.

A: The sum of the numbers is 6; $P(A) = \dfrac{5}{36}$.

B: Both numbers are odd; $P(B) = \dfrac{9}{36} = \dfrac{1}{4}$.

$A \cap B$: The sum is 6, and both are odd; $P(A \cap B) = \dfrac{3}{36} = \dfrac{1}{12}$.

$P(A \cap B) \neq P(A)P(B)$, and so A and B are dependent.

EXAMPLE 4 Weather Prediction

According to the weather service, there is a 50% chance of rain in New York and a 30% chance of rain in Honolulu. Assuming that New York's weather is independent of Honolulu's, find the probability that it will rain in at least one of these cities.

Solution We take A to be the event that it will rain in New York and B to be the event that it will rain in Honolulu. We are asked to find the probability of $A \cup B$, the event that it will rain in at least one of the two cities. We use the addition principle:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

We know that $P(A) = .5$ and $P(B) = .3$. But what about $P(A \cap B)$? Because the events A and B are independent, we can compute

$$\begin{aligned} P(A \cap B) &= P(A)P(B) \\ &= (.5)(.3) = .15. \end{aligned}$$

Thus,

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= .5 + .3 - .15 \\ &= .65. \end{aligned}$$

So, there is a 65% chance that it will rain either in New York or in Honolulu (or in both).

EXAMPLE 5 Roulette

You are playing roulette, and have decided to leave all 10 of your \$1 chips on black for five consecutive rounds, hoping for a sequence of five blacks which, according to the rules, will leave you with \$320. There is a 50% chance of black coming up on each spin, ignoring the complicating factor of zero or double zero. What is the probability that you will be successful?

Solution Because the roulette wheel has no memory, each spin is independent of the others. Thus, if A_1 is the event that black comes up the first time, A_2 the event that it comes up the second time, and so on, then

$$P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) = P(A_1)P(A_2)P(A_3)P(A_4)P(A_5) = \left(\frac{1}{2}\right)^5 = \frac{1}{32}.$$

The next example is a version of a well known "brain teaser" that forces one to think carefully about conditional probability.

EXAMPLE 6 Legal Argument

A man was arrested for attempting to smuggle a bomb on board an airplane. During the subsequent trial, his lawyer claimed that, by means of a simple argument, she would prove beyond a shadow of a doubt that her client was not only innocent of any crime, but was in fact contributing to the safety of the other passengers on the flight. This was her eloquent argument: "Your Honor, first of all, my client had absolutely no intention of setting off the bomb. As the record clearly shows, the detonator was unarmed when he was apprehended. In addition—and your Honor is certainly aware of this—there is a small but definite possibility that there will be a bomb on any given flight. On the other hand, the chances of there being *two* bombs on a flight are so remote as to be negligible. There is in fact no record of this having *ever* occurred. Thus, because my client had already brought one bomb on board (with no intention of setting it off) and because we have seen that the chances of there being a second bomb on board were vanishingly remote, it follows that the flight was far safer as a result of his action! I rest my case." This argument was so elegant in its simplicity that the judge acquitted the defendant. Where is the flaw in the argument? (Think about this for a while before reading the solution.)

Solution The lawyer has cleverly confused the phrases "two bombs on board" and "a second bomb on board." To pinpoint the flaw, let us take B to be the event that there is one bomb on board a given flight, and let A be the event that there are two independent bombs on board. Let us assume for argument's sake that $P(B) = 1/1{,}000{,}000 = .000\,001$. Then the probability of the event A is

$$(.000\,001)(.000\,001) = .000\,000\,000\,001.$$

This *is* vanishingly small, as the lawyer contended. It was at this point that the lawyer used a clever maneuver: She assumed in concluding her argument that the probability of having two bombs on board was the same as the probability of having a *second* bomb on board. But to say that there is a *second* bomb on board is to imply that there already is one bomb on board. This is therefore a *conditional* event: the event that there are two bombs on board, *given that there is already one bomb on board.* Thus, the probability that there is a second bomb on board is the probability that there are two bombs on board, given that there is already one bomb on board, which is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{.000\,000\,000\,001}{.000\,001} = .000\,001.$$

In other words, it is the same as the probability of there being a single bomb on board to begin with! Thus the man's carrying the bomb onto the plane did not improve the flight's safety at all.*

* **NOTE** If we want to be picky, there was a *slight* decrease in the probability of a second bomb because there was one less seat for a potential second bomb bearer to occupy. In terms of our analysis, this is saying that the event of one passenger with a bomb and the event of a second passenger with a bomb are not completely independent.

FAQ

Probability of what given what?

Q: *How do I tell if a statement in an application is talking about conditional probability or unconditional probability? And if it is talking about conditional probability, how do I determine what to use as A and B in $P(A \mid B)$?*

A: Look carefully at the wording of the statement. If there is some kind of qualification or restriction to a smaller set than the entire sample space, then it is probably talking about conditional probability, as in the following examples:

> *60% of veterans vote Republican while 40% of the entire voting population vote Republican.*
> Here the sample space can be taken to be the entire voting population.
> Reworded (see Example 2): *The probability of voting Republican* (R) *is 60% given that the person is a veteran* (V); $P(R \mid V) = .60$, *whereas the probability of voting Republican is .40:* $P(R) = .40$.
>
> *The likelihood of being injured if in an accident is 80% for a driver not wearing a seatbelt but it is 50% for all drivers.*
> Here, the sample space can be taken to be the set of drivers involved in an accident—these are the only drivers discussed.
> Reworded: *The probability of a driver being injured* (I) *is .80 given that the driver is not wearing a seatbelt* (B); $P(I \mid B) = .80$ *whereas, for all drivers, the probability of being injured is* .50: $P(I) = .50$.

7.5 EXERCISES

▼ more advanced ◆ challenging

T indicates exercises that should be solved using technology

In Exercises 1–10, compute the indicated quantity.

1. $P(B) = .5$, $P(A \cap B) = .2$. Find $P(A \mid B)$.

2. $P(B) = .6$, $P(A \cap B) = .3$. Find $P(A \mid B)$.

3. $P(A \mid B) = .2$, $P(B) = .4$. Find $P(A \cap B)$.

4. $P(A \mid B) = .1$, $P(B) = .5$. Find $P(A \cap B)$.

5. $P(A \mid B) = .4$, $P(A \cap B) = .3$. Find $P(B)$.

6. $P(A \mid B) = .4$, $P(A \cap B) = .1$. Find $P(B)$.

7. $P(A) = .5$, $P(B) = .4$. A and B are independent. Find $P(A \cap B)$.

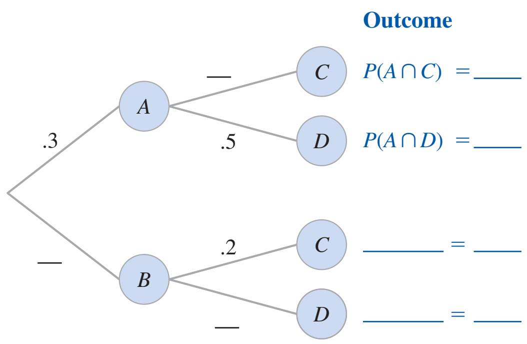
8. $P(A) = .2$, $P(B) = .2$. A and B are independent. Find $P(A \cap B)$.

9. $P(A) = .5$, $P(B) = .4$. A and B are independent. Find $P(A \mid B)$.

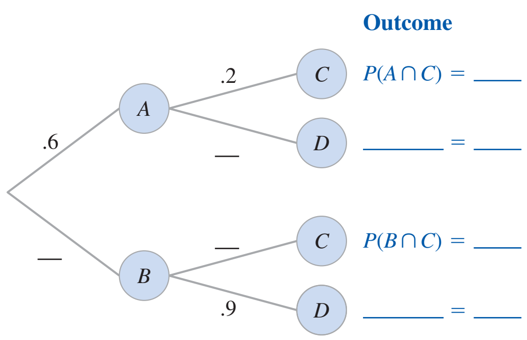
10. $P(A) = .3$, $P(B) = .6$. A and B are independent. Find $P(B \mid A)$.

In Exercises 11–14, supply the missing quantities.

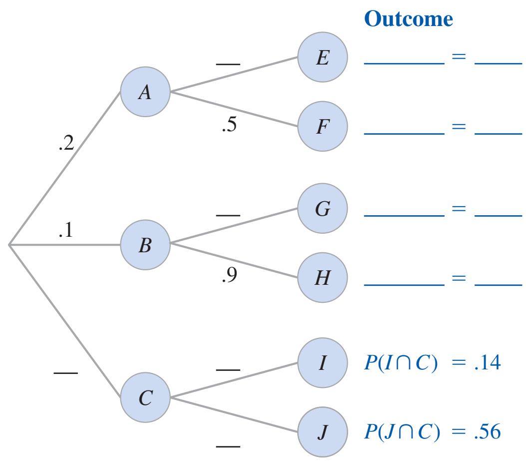
11.



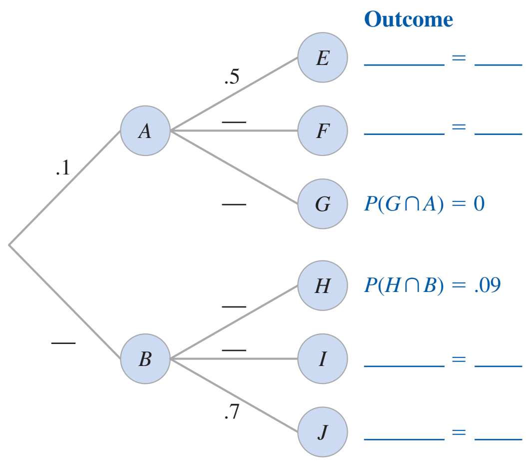
12.



13.



14.



In Exercises 15–20, find the conditional probabilities of the indicated events when two fair dice (one red and one green) are rolled. HINT [See Example 1.]

15. The sum is 5, given that the green one is not a 1.

16. The sum is 6, given that the green one is either 4 or 3.

17. The red one is 5, given that the sum is 6.

18. The red one is 4, given that the green one is 4.

19. The sum is 5, given that the dice have opposite parity.

20. The sum is 6, given that the dice have opposite parity.

Exercises 21–26 require the use of counting techniques from the last chapter. A bag contains three red marbles, two green ones, one fluorescent pink one, two yellow ones, and two orange ones. Suzan grabs four at random. Find the probabilities of the indicated events.

21. She gets all the red ones, given that she gets the fluorescent pink one.

22. She gets all the red ones, given that she does not get the fluorescent pink one.

23. She gets none of the red ones, given that she gets the fluorescent pink one.

24. She gets one of each color other than fluorescent pink, given that she gets the fluorescent pink one.

25. She gets one of each color other than fluorescent pink, given that she gets at least one red one.

26. She gets at least two red ones, given that she gets at least one green one.

In Exercises 27–30, say whether the given pairs of events are independent, mutually exclusive, or neither.

27. A: Your new skateboard design is a success.
B: Your new skateboard design is a failure.

28. A: Your new skateboard design is a success.
B: There is life in the Andromeda galaxy.

29. A: Your new skateboard design is a success.
B: Your competitor's new skateboard design is a failure.

30. A: Your first coin flip results in heads.
B: Your second coin flip results in heads.

In Exercises 31–36, two dice (one red and one green) are rolled, and the numbers that face up are observed. Test the given pairs of events for independence. HINT [See Quick Example on Testing for Independence, page 503.]

31. A: The red die is 1, 2, or 3; B: The green die is even.

32. A: The red die is 1; B: The sum is even.

33. A: Exactly one die is 1; B: The sum is even.

34. A: Neither die is 1 or 6; B: The sum is even.

35. A: Neither die is 1; B: Exactly one die is 2.

36. A: Both dice are 1; B: Neither die is 2.

37. If a coin is tossed 11 times, find the probability of the sequence H, T, T, H, H, H, T, H, H, T, T. HINT [See Example 5.]

38. If a die is rolled four times, find the probability of the sequence 4, 3, 2, 1. HINT [See Example 5.]

In Exercises 39–44, fill in the blanks using the named events. HINT [See Example 2 and also the FAQ on page 506.]

39. 10% of all Anchovians detest anchovies (D), whereas 30% of all married Anchovians (M) detest them. $P(\_\_\_) = \_\_\_$; $P(\_\_\_ \mid \_\_\_) = \_\_\_$

40. 95% of all music composers can read music (M), whereas 99% of all classical music composers (C) can read music. $P(\_\_\_) = \_\_\_$; $P(\_\_\_ \mid \_\_\_) = \_\_\_$

41. 30% of all lawyers who lost clients (L) were antitrust lawyers (A), whereas 10% of all antitrust lawyers lost clients. $P(\_\_\_ \mid \_\_\_) = \_\_\_$; $P(\_\_\_ \mid \_\_\_) = \_\_\_$

42. 2% of all items bought on my auction site (B) were works of art (A), whereas only 1% of all works of art on the site were bought. $P(\_\_\_ \mid \_\_\_) = \_\_\_$; $P(\_\_\_ \mid \_\_\_) = \_\_\_$

43. 55% of those who go out in the midday sun (M) are Englishmen (E) whereas only 5% of those who do not go out in the midday sun are Englishmen. $P(\_\_\_ \mid \_\_\_) = \_\_\_$; $P(\_\_\_ \mid \_\_\_) = \_\_\_$

44. 80% of those who have a Mac now (M) will purchase a Mac next time (X) whereas 20% of those who do not have a Mac now will purchase a Mac next time. $P(\_\_\_ \mid \_\_\_) = \_\_\_$; $P(\_\_\_ \mid \_\_\_) = \_\_\_$

APPLICATIONS

45. ***Personal Bankruptcy*** In 2004, the probability that a person in the United States would declare personal bankruptcy was .006. The probability that a person in the United States would declare personal bankruptcy and had recently experienced a "big three" event (loss of job, medical problem, or divorce or separation) was .005.[42] What was the probability that a person had recently experienced one of the "big three" events, given that she had declared personal bankruptcy? (Round your answer to one decimal place.)

46. ***Personal Bankruptcy*** In 2004, the probability that a person in the United States would declare personal bankruptcy was .006. The probability that a person in the United States would declare personal bankruptcy and had recently overspent credit cards was .0004.[43] What was the probability that a person had recently overspent credit cards given that he had declared personal bankruptcy?

47. ***Listings on* eBay** During 2008, approximately 2,750 million new listings were posted worldwide on **eBay**, of which 1,130 million were in the United States. A total of 730 million of the worldwide new listings, including 275 million in the United States, were posted during the fourth quarter of 2008.[44]

a. Find the probability that a new listing on **eBay** was in the United States, given that the item was posted during the fourth quarter.

b. Find the probability that a new listing on **eBay** was posted in the fourth quarter, given that the item was posted in the United States.

48. ***Listings on* eBay** Refer to the data given in Exercise 41.

a. Find the probability that a new listing on **eBay** was outside the United States, given that the item was posted during the fourth quarter.

b. Find the probability that a new listing on **eBay** was posted in the fourth quarter, given that the item was posted outside the United States.

49. ***Social Security*** According to a *New York Times*/CBS poll released in March, 2005, 79% agreed that it should be the government's responsibility to provide a decent standard of living for the elderly, and 43% agreed that it would be a good idea to invest part of their Social Security taxes on their own.[45] If agreement with one of these propositions is independent of agreement with the other, what is the probability that a person agreed with both propositions? (Round your answer to two decimal places.) HINT [See Quick Examples on Independence on page 503.]

50. ***Social Security*** According to the *New York Times*/CBS poll of March, 2005, referred to in Exercise 43, 49% agreed that Social Security taxes should be raised if necessary to keep the system afloat, and 43% agreed that it would be a good idea to invest part of their Social Security taxes on their own.[46] If agreement with one of these propositions is independent of agreement with the other, what is the probability that a person agreed with both propositions? (Round your answer to two decimal places.) HINT [See Quick Examples on Independence on page 503.]

51. ***Marketing*** A market survey shows that 40% of the population used Brand X laundry detergent last year, 5% of the population gave up doing its laundry last year, and 4% of the population used Brand X and then gave up doing laundry last year. Are the events of using Brand X and giving up doing laundry independent? Is a user of Brand X detergent more or less likely to give up doing laundry than a randomly chosen person?

52. ***Marketing*** A market survey shows that 60% of the population used Brand Z computers last year, 5% of the population quit their jobs last year, and 3% of the population used Brand Z computers and then quit their jobs. Are the events of using Brand Z computers and quitting your job independent? Is a user of Brand Z computers more or less likely to quit a job than a randomly chosen person?

53. ***Road Safety*** In 1999, the probability that a randomly selected vehicle would be involved in a deadly tire-related accident was approximately 3×10^{-6}, whereas the probability that a tire-related accident would prove deadly was .02.[47] What was the probability that a vehicle would be involved in a tire-related accident?

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[42] Probabilities are approximate. Source: *New York Times*, March 13, 2005, p, WK3.

[43] Ibid. The .0004 figure is an estimate by the authors.

[44] Source: http://news.ebay.com/about.cfm.

[45] Source: *New York Times*, March 3, 2005, p. A20.

[46] Ibid.

[47] The original data reported three tire-related deaths per million vehicles. Source: *New York Times* analysis of National Traffic Safety Administration crash data/Polk Company vehicle registration data/ *New York Times*, Nov. 22, 2000, p. C5.

54. ***Road Safety*** In 1998, the probability that a randomly selected vehicle would be involved in a deadly tire-related accident was approximately 2.8×10^{-6}, while the probability that a tire-related accident would prove deadly was .016.[48] What was the probability that a vehicle would be involved in a tire-related accident?

Publishing *Exercises 55–62 are based on the following table, which shows the results of a survey of 100 authors by a publishing company.*

	New Authors	Established Authors	Total
Successful	5	25	30
Unsuccessful	15	55	70
Total	20	80	100

Compute the following conditional probabilities:

55. An author is established, given that she is successful.

56. An author is successful, given that he is established.

57. An author is unsuccessful, given that he is a new author.

58. An author is a new author, given that she is unsuccessful.

59. An author is unsuccessful, given that she is established.

60. An author is established, given that he is unsuccessful.

61. An unsuccessful author is established.

62. An established author is successful.

In Exercises 63–68, draw an appropriate tree diagram and use the multiplication principle to calculate the probabilities of all the outcomes. HINT [See Example 3.]

63. ***Sales*** Each day, there is a 40% chance that you will sell an automobile. You know that 30% of all the automobiles you sell are two-door models, and the rest are four-door models.

64. ***Product Reliability*** You purchase Brand X floppy discs one quarter of the time and Brand Y floppies the rest of the time. Brand X floppy discs have a 1% failure rate, while Brand Y floppy discs have a 3% failure rate.

65. ***Car Rentals*** Your auto rental company rents out 30 small cars, 24 luxury sedans, and 46 slightly damaged "budget" vehicles. The small cars break down 14% of the time, the luxury sedans break down 8% of the time, and the "budget" cars break down 40% of the time.

66. ***Travel*** It appears that there is only a one in five chance that you will be able to take your spring vacation to the Greek Islands. If you are lucky enough to go, you will visit either Corfu (20% chance) or Rhodes. On Rhodes, there is a 20% chance of meeting a tall dark stranger, while on Corfu, there is no such chance.

67. ***Weather Prediction*** There is a 50% chance of rain today and a 50% chance of rain tomorrow. Assuming that the event that it rains today is independent of the event that it rains tomorrow, draw a tree diagram showing the probabilities of all outcomes. What is the probability that there will be no rain today or tomorrow?

68. ***Weather Prediction*** There is a 20% chance of snow today and a 20% chance of snow tomorrow. Assuming that the event that it snows today is independent of the event that it snows tomorrow, draw a tree diagram showing the probabilities of all outcomes. What is the probability that it will snow by the end of tomorrow?

Education and Employment *Exercises 69–78 are based on the following table, which shows U.S. employment figures for 2007, broken down by educational attainment.*[49] *All numbers are in millions, and represent civilians aged 25 years and over. Those classed as "not in labor force" were not employed nor actively seeking employment. Round all answers to two decimal places.*

	Employed	Unemployed	Not in Labor Force	Total
Less Than High School Diploma	11.5	1.6	14.2	27.3
High School Diploma Only	36.9	1.7	22.4	61.0
Some College or Associate's Degree	34.6	1.3	14.0	49.9
Bachelor's Degree or Higher	44.1	0.9	12.1	57.1
Total	127.1	5.5	62.7	195.3

69. Find the probability that a person was employed, given that the person had a bachelor's degree or higher.

70. Find the probability that a person was employed, given that the person had attained less than a high school diploma.

71. Find the probability that a person had a bachelor's degree or higher, given that the person was employed.

72. Find the probability that a person had attained less than a high school diploma, given that the person was employed.

73. ▼ Find the probability that a person who had not completed a bachelor's degree or higher was not in the labor force.

74. ▼ Find the probability that a person who had completed at least a high school diploma was not in the labor force.

75. ▼ Find the probability that a person who had completed a bachelor's degree or higher and was in the labor force was employed.

[48] The original data reported 2.8 tire-related deaths per million vehicles. Source: Ibid.

[49] Source: Bureau of Labor Statistics (www.bls.gov/).

76. ▼ Find the probability that a person who had completed less than a high school diploma and was in the labor force was employed.

77. ▼ Your friend claims that an unemployed person is more likely to have a high school diploma only than an employed person. Respond to this claim by citing actual probabilities.

78. ▼ Your friend claims that a person not in the labor force is more likely to have less than a high school diploma than an employed person. Respond to this claim by citing actual probabilities.

79. ▼ ***Airbag Safety*** According to a study conducted by the Harvard School of Public Health, a child seated in the front seat who was wearing a seatbelt was 31% more likely to be killed in an accident if the car had an air bag that deployed than if it did not.[50] Let the sample space S be the set of all accidents involving a child seated in the front seat wearing a seatbelt. Let K be the event that the child was killed and let D be the event that the airbag deployed. Fill in the missing terms and quantities: $P(\_\_\_ \mid \_\_\_) = \_\_\_ \times P(\_\_\_ \mid \_\_\_)$. HINT [When we say "A is 31% more likely than B" we mean that the probability of A is 1.31 times the probability of B.]

80. ▼ ***Airbag Safety*** According to the study cited in Exercise 73, a child seated in the front seat not wearing a seatbelt was 84% more likely to be killed in an accident if the car had an air bag that deployed than if it did not.[51] Let the sample space S be the set of all accidents involving a child seated in the front seat not wearing a seatbelt. Fill in the missing terms and quantities: $P(\_\_\_ \mid \_\_\_) = \_\_\_ \times P(\_\_\_ \mid \_\_\_)$. HINT [When we say "A is 84% more likely than B" we mean that the probability of A is 1.84 times the probability of B.]

81. ▼ ***Productivity*** A company wishes to enhance productivity by running a one-week training course for its employees. Let T be the event that an employee participated in the course, and let I be the event that an employee's productivity improved the week after the course was run.

a. Assuming that the course has a positive effect on productivity, how are $P(I \mid T)$ and $P(I)$ related?

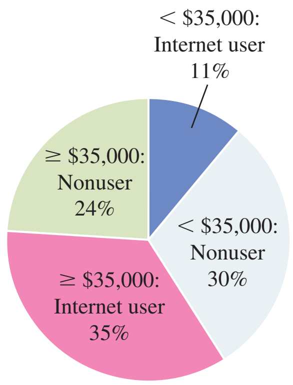
b. If T and I are independent, what can one conclude about the training course?

82. ▼ ***Productivity*** Consider the events T and I in the preceding exercise.

a. Assuming that everyone who improved took the course but that not everyone took the course, how are $P(T \mid I)$ and $P(T)$ related?

b. If half the employees who improved took the course and half the employees took the course, are T and I independent?

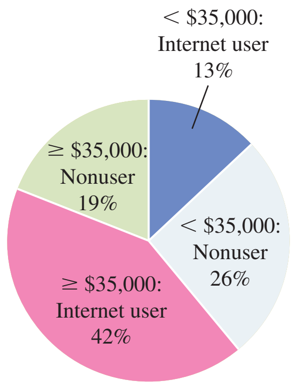
83. ▼ ***Internet Use in 2000*** The following pie chart shows the percentage of the population that used the Internet in 2000, broken down further by family income, and based on a survey taken in August 2000.[52]



a. Determine the probability that a randomly chosen person was an Internet user, given that his or her family income was at least $35,000.

b. Based on the data, was a person more likely to be an Internet user if his or her family income was less than $35,000 or $35,000 or more? (Support your answer by citing the relevant conditional probabilities.)

84. ▼ ***Internet Use in 2001*** Repeat Exercise 77 using the following pie chart, which shows the results of a similar survey taken in September 2001.[53]



[50] The study was conducted by Dr. Segul-Gomez at the Harvard School of Public Health. Source: *New York Times*, December 1, 2000, p. F1.

[51] Ibid.

[52] Source: *Falling Through the Net: Toward Digital Inclusion, A Report on Americans' Access to Technology Tools*, U.S. Department of Commerce, October, 2000. Available at www.ntia.doc.gov/ntiahome/fttn00/contents00.html.

[53] Source: *A Nation Online: How Americans Are Expanding Their Use of the Internet*, U.S. Department of Commerce, February, 2002. Available at www.ntia.doc.gov/ntiahome/dn/index.html.

Auto Theft *Exercises 85–90 are based on the following table, which shows the probability that an owner of the given model would report his or her vehicle stolen in a one-year period.*[54]

Brand	Jeep Wrangler	Suzuki Sidekick (two-door)	Toyota Land Cruiser	Geo Tracker (two-door)	Acura Integra (two-door)
Probability	.0170	.0154	.0143	.0142	.0123
Brand	**Mitsubishi Montero**	**Acura Integra (four-door)**	**BMW 3-series (two-door)**	**Lexus GS300**	**Honda Accord (two-door)**
Probability	.0108	.0103	.0077	.0074	.0070

In an experiment in which a vehicle is selected, consider the following events:

R: The vehicle was reported stolen.
J: The vehicle was a Jeep Wrangler.
$A2$: The vehicle was an Acura Integra (two-door).
$A4$: The vehicle was an Acura Integra (four-door).
A: The vehicle was an Acura Integra (either two-door or four-door).

85. ▼ Fill in the blanks: $P(\_\_\_ \mid \_\_\_) = .0170$.

86. ▼ Fill in the blanks: $P(\_\_\_ \mid A4) = \_\_\_$.

87. ▼ Which of the following is true?

(A) There is a 1.43% chance that a vehicle reported stolen was a Toyota Land Cruiser.
(B) Of all the vehicles reported stolen, 1.43% of them were Toyota Land Cruisers.
(C) Given that a vehicle was reported stolen, there is a .0143 probability that it was a Toyota Land Cruiser.
(D) Given that a vehicle was a Toyota Land Cruiser, there was a 1.43% chance that it was reported stolen.

88. ▼ Which of the following is true?

(A) $P(R \mid A) = .0123 + .0103 = .0226.$
(B) $P(R' \mid A2) = 1 - .0123 = .9877.$
(C) $P(A2 \mid A) = .0123/(.0123 + .0103) \approx .544.$
(D) $P(R \mid A2') = 1 - .0123 = .9877.$

89. ▼ It is now January, and I own a BMW 3-series and a Lexus GS300. Because I house my vehicles in different places, the event that one of my vehicles gets stolen does not depend on the event that the other gets stolen. Compute each probability to six decimal places.

a. Both my vehicles will get stolen this year.
b. At least one of my vehicles will get stolen this year.

90. ▼ It is now December, and I own a Mitsubishi Montero and a Jeep Wrangler. Because I house my vehicles in different places, the event that one of my vehicles gets stolen does not depend on the event that the other gets stolen. I have just returned from a one-year trip to the Swiss Alps.

a. What is the probability that my Montero, but not my Wrangler, has been stolen?
b. Which is more likely: the event that my Montero was stolen or the event that *only* my Montero was stolen?

91. ▼ ***Drug Tests*** If 90% of the athletes who test positive for steroids in fact use them, and 10% of all athletes use steroids and test positive, what percentage of athletes test positive?

92. ▼ ***Fitness Tests*** If 80% of candidates for the soccer team pass the fitness test, and only 20% of all athletes are soccer team candidates who pass the test, what percentage of the athletes are candidates for the soccer team?

93. ▼ ***Food Safety*** According to a University of Maryland study of 200 samples of ground meats,[55] the probability that a sample was contaminated by salmonella was .20. The probability that a salmonella-contaminated sample was contaminated by a strain resistant to at least three antibiotics was .53. What was the probability that a ground meat sample was contaminated by a strain of salmonella resistant to at least three antibiotics?

94. ▼ ***Food Safety*** According to the study mentioned in Exercise 87,[56] the probability that a ground meat sample was contaminated by salmonella was .20. The probability that a salmonella-contaminated sample was contaminated by a strain resistant to at least one antibiotic was .84. What was the probability that a ground meat sample was contaminated by a strain of salmonella resistant to at least one antibiotic?

95. ◆ ***Food Safety*** According to a University of Maryland study of 200 samples of ground meats,[57] the probability that one of the samples was contaminated by salmonella was .20. The probability that a salmonella-contaminated sample was contaminated by a strain resistant to at least one antibiotic was .84, and the probability that a salmonella-contaminated sample was contaminated by a strain resistant to at least three antibiotics was .53. Find the probability that a ground meat sample that was contaminated by an antibiotic-resistant strain was contaminated by a strain resistant to at least three antibiotics.

96. ◆ ***Food Safety*** According to a University of Maryland study of 200 samples of ground meats,[58] the probability that a ground meat sample was contaminated by a strain of

[54] Data are for insured vehicles, for 1995 to 1997 models except Wrangler, which is for 1997 models only. Source: Highway Loss Data Institute/*The New York Times,* March 28, 1999, p. WK3.

[55] As cited in the *New York Times*, October 16, 2001, p. A12.

[56] Ibid.

[57] Ibid.

[58] Ibid.

salmonella resistant to at least three antibiotics was .11. The probability that someone infected with any strain of salmonella will become seriously ill is .10. What is the probability that someone eating a randomly-chosen ground meat sample will not become seriously ill with a strain of salmonella resistant to at least three antibiotics?

COMMUNICATION AND REASONING EXERCISES

97. Name three events, each independent of the others, when a fair coin is tossed four times.

98. Name three pairs of independent events when a pair of distinguishable and fair dice is rolled and the numbers that face up are observed.

99. You wish to ascertain the probability of an event E, but you happen to know that the event F has occurred. Is the probability you are seeking $P(E)$ or $P(E|F)$? Give the reason for your answer.

100. Your television advertising campaign seems to have been very persuasive: 10,000 people who saw the ad purchased your product, while only 2,000 people purchased the product without seeing the ad. Explain how additional data could show that your ad campaign was, in fact, unpersuasive.

101. You are having trouble persuading your friend Iliana that conditional probability is different from unconditional probability. She just said: "Look here, Saul, the probability of throwing a double-six is $1/36$, and that's that! That probability is not affected by anything, including the 'given' that the sum is larger than 7." How do you persuade her otherwise?

102. Your other friend Giuseppe is spreading rumors that the conditional probability $P(E|F)$ is always bigger than $P(E)$. Is he right? (If he is, explain why; if not, give an example to prove him wrong.)

103. ▼ If $A \subseteq B$ and $P(B) \neq 0$, why is $P(A \mid B) = \dfrac{P(A)}{P(B)}$?

104. ▼ If $B \subseteq A$ and $P(B) \neq 0$, why is $P(A \mid B) = 1$?

105. ▼ Your best friend thinks that it is impossible for two mutually exclusive events with nonzero probabilities to be independent. Establish whether or not he is correct.

106. ▼ Another of your friends thinks that two mutually exclusive events with nonzero probabilities can never be dependent. Establish whether or not she is correct.

107. ◆ Show that if A and B are independent, then so are A' and B' (assuming none of these events has zero probability). [Hint: $A' \cap B'$ is the complement of $A \cup B$.]

108. ◆ Show that if A and B are independent, then so are A and B' (assuming none of these events has zero probability). [Hint: $P(B' \mid A) + P(B \mid A) = 1$.]

7.6 Bayes' Theorem and Applications

Should schools test their athletes for drug use? A problem with drug testing is that there are always false positive results, so one can never be certain that an athlete who tests positive is in fact using drugs. Here is a typical scenario.

EXAMPLE 1 Steroids Testing

Gamma Chemicals advertises its anabolic steroid detection test as being 95% effective at detecting steroid use, meaning that it will show a positive result on 95% of all anabolic steroid users. It also states that its test has a false positive rate of 6%. This means that the probability of a nonuser testing positive is .06. Estimating that about 10% of its athletes are using anabolic steroids, Enormous State University (ESU) begins testing its football players. The quarterback, Hugo V. Huge, tests positive and is promptly dropped from the team. Hugo claims that he is not using anabolic steroids. How confident can we be that he is not telling the truth?

Solution There are two events of interest here: the event T that a person tests positive, and the event A that the person tested uses anabolic steroids. Here are the probabilities we are given:

$$P(T \mid A) = .95$$
$$P(T \mid A') = .06$$
$$P(A) = .10$$

We are asked to find $P(A \mid T)$, the probability that someone who tests positive is using anabolic steroids. We can use a tree diagram to calculate $P(A \mid T)$. The trick to setting up the tree diagram is to use as the first branching the events with *unconditional* probabilities we know. Because the only unconditional probability we are given is $P(A)$, we use A and A' as our first branching (Figure 13).

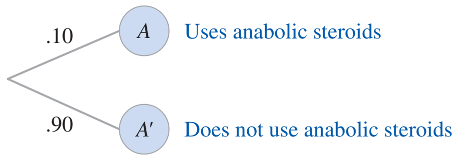


Figure 13

For the second branching, we use the outcomes of the drug test: positive (T) or negative (T'). The probabilities on these branches are conditional probabilities because they depend on whether or not an athlete uses steroids. (See Figure 14.) (We fill in the probabilities that are not supplied by remembering that the sum of the probabilities on the branches leaving any node must be 1.)

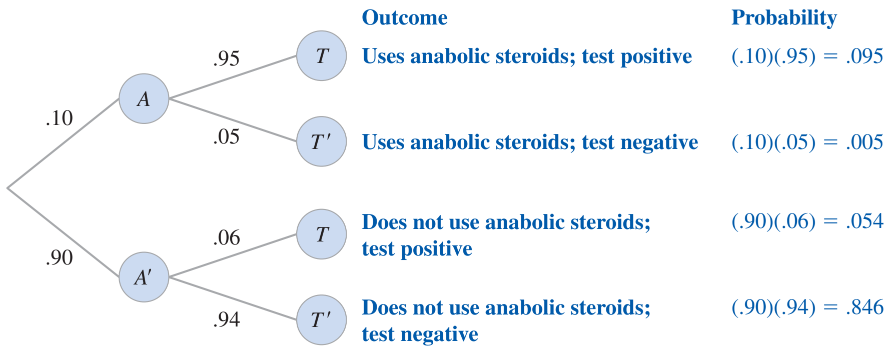


Figure 14

We can now calculate the probability we are asked to find:

$$P(A \mid T) = \frac{P(A \cap T)}{P(T)} = \frac{P(\text{Uses anabolic steroids and tests positive})}{P(\text{Tests positive})}$$

$$= \frac{P(\text{Using } A \text{ and } T \text{ branches})}{\text{Sum of } P(\text{Using branches ending in } T)}.$$

From the tree diagram, we see that $P(A \cap T) = .095$. To calculate $P(T)$, the probability of testing positive, notice that there are two outcomes on the tree diagram that reflect a positive test result. The probabilities of these events are .095 and .054. Because these two events are mutually exclusive (an athlete either uses steroids or does not, but not both), the probability of a test being positive (ignoring whether or not steroids are used) is the sum of these probabilities, .149. Thus,

$$P(A \mid T) = \frac{.095}{.095 + .054} = \frac{.095}{.149} \approx .64.$$

Thus there is a 64% chance that a randomly selected athlete who tests positive, like Hugo, is using steroids. In other words, we can be 64% confident that Hugo is lying.

Before we go on... Note that the correct answer in Example 1 is 64%, *not* the 94% we might suspect from the test's false positive rating. In fact, we can't answer the question asked without knowing the percentage of athletes who actually use steroids. For instance, if *no* athletes at all use steroids, then Hugo must be telling the truth, and so the test result has no significance whatsoever. On the other hand, if *all* athletes use steroids, then Hugo is definitely lying, regardless of the outcome of the test.

False positive rates are determined by testing a large number of samples known not to contain drugs and computing estimated probabilities. False negative rates are computed similarly by testing samples known to contain drugs. However, the accuracy of the tests depends also on the skill of those administering them. False positives were a significant problem when drug testing started to become common, with estimates of false positive rates for common immunoassay tests ranging from 10–30% on the high end,[59] but the accuracy has improved since then. Because of the possibility of false positive results, positive immunoassay tests need to be confirmed by the more expensive and much more reliable gas chromatograph/mass spectrometry (GC/MS) test. See also the NCAA's Drug-Testing Program Handbook, available at www.ncaa.org/. (The section on Institutional Drug Testing addresses the problem of false positives.) ■

Bayes' Theorem

The calculation we used to answer the question in Example 1 can be recast as a formula known as **Bayes' theorem**. Figure 15 shows a general form of the tree we used in Example 1.

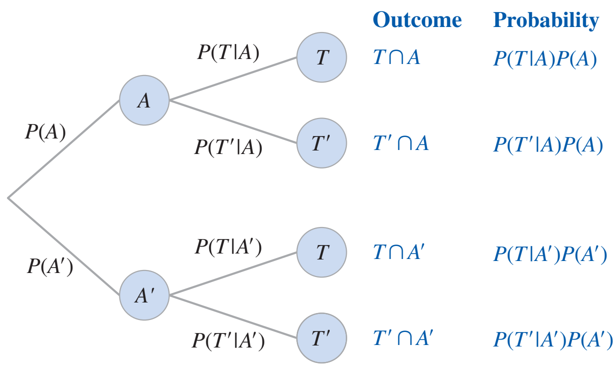


Figure 15

We first calculated

$$P(A \mid T) = \frac{P(A \cap T)}{P(T)}$$

as follows. We first calculated the numerator $P(A \cap T)$ using the multiplication principle:

$$P(A \cap T) = P(T \mid A)P(A).$$

We then calculated the denominator $P(T)$ by using the addition principle for mutually exclusive events together with the multiplication principle:

$$\begin{aligned} P(T) &= P(A \cap T) + P(A' \cap T) \\ &= P(T \mid A)P(A) + P(T \mid A')P(A'). \end{aligned}$$

[59] *Drug Testing in the Workplace,* ACLU Briefing Paper, 1996.

Substituting gives

$$P(A \mid T) = \frac{P(T \mid A)P(A)}{P(T \mid A)P(A) + P(T \mid A')P(A')}.$$

This is the short form of Bayes' theorem.

Bayes' Theorem (Short Form)

If A and T are events, then

Bayes' Formula

$$P(A \mid T) = \frac{P(T \mid A)P(A)}{P(T \mid A)P(A) + P(T \mid A')P(A')}.$$

Using a Tree

$$P(A \mid T) = \frac{P(\text{Using } A \text{ and } T \text{ branches})}{\text{Sum of } P(\text{Using branches ending in } T)}$$

Quick Example

Let us calculate the probability that an athlete from Example 1 who tests positive is actually using steroids if only 5% of ESU athletes are using steroids. Thus,

$$P(T \mid A) = .95$$
$$P(T \mid A') = .06$$
$$P(A) = .05$$
$$P(A') = .95$$

and so

$$P(A \mid T) = \frac{P(T \mid A)P(A)}{P(T \mid A)P(A) + P(T \mid A')P(A')} = \frac{(.95)(.05)}{(.95)(.05) + (.06)(.95)} \approx .45.$$

In other words, it is actually more likely that such an athlete does *not* use steroids than he does.*

* **NOTE** Without knowing the results of the test we would have said that there was a probability of $P(A) = 0.05$ that the athlete is using steroids. The positive test result raises the probability to $P(A \mid T) = 0.45$, but the test gives too many false positives for us to be any more certain than that that the athlete is actually using steroids.

Remembering the Formula

Although the formula looks complicated at first sight, it is not hard to remember if you notice the pattern. Or, you could re-derive it yourself by thinking of the tree diagram.

The next example illustrates that we can use either a tree diagram or the Bayes' theorem formula.

EXAMPLE 2 Lie Detectors

The Sherlock Lie Detector Company manufactures the latest in lie detectors, and the Count-Your-Pennies (CYP) store chain is eager to use them to screen their employees for theft. Sherlock's advertising claims that the test misses a lie only once in every 100 instances. On the other hand, an analysis by a consumer group reveals 20% of people who are telling the truth fail the test anyway.* The local police department estimates that 1 out of every 200 employees has engaged in theft. When the CYP store first screened

* **NOTE** The reason for this is that many people show physical signs of distress when asked accusatory questions. Many people are nervous around police officers even if they have done nothing wrong.

its employees, the test indicated Mrs. Prudence V. Good was lying when she claimed that she had never stolen from CYP. What is the probability that she was lying and had in fact stolen from the store?

Solution We are asked for the probability that Mrs. Good was lying, and in the preceding sentence we are told that the lie detector test showed her to be lying. So, we are looking for a conditional probability: the probability that she is lying, given that the lie detector test is positive. Now we can start to give names to the events:

L: A subject is lying.

T: The test is positive (indicated that the subject was lying).

We are looking for $P(L \mid T)$. We know that 1 out of every 200 employees engages in theft; let us assume that no employee admits to theft while taking a lie detector test, so the probability $P(L)$ that a test subject is lying is $1/200$. We also know the false negative and false positive rates $P(T' \mid L)$ and $P(T \mid L')$.

Using a tree diagram
Figure 16 shows the tree diagram.

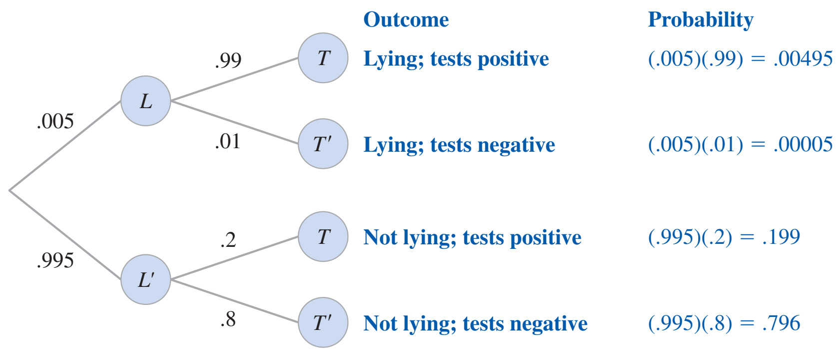


Figure 16

We see that

$$P(L \mid T) = \frac{P(\text{Using } L \text{ and } T \text{ branches})}{\text{Sum of } P(\text{Using branches ending in } T)}$$

$$= \frac{.00495}{.00495 + .199} \approx .024.$$

This means that there was only a 2.4% chance that poor Mrs. Good was lying and had stolen from the store!

Using Bayes' Theorem
We have

$P(L) = .005$

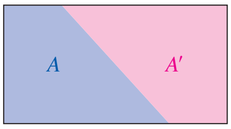
$P(T' \mid L) = .01$, from which we obtain

$P(T \mid L) = .99$

$P(T \mid L') = .2$

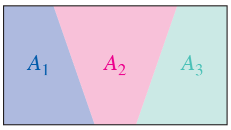
and so

$$P(L \mid T) = \frac{P(T \mid L)P(L)}{P(T \mid L)P(L) + P(T \mid L')P(L')} = \frac{(.99)(.005)}{(.99)(.005) + (.2)(.995)} \approx .024.$$



A and A' form a partition of S.

Figure 17



A_1, A_2, and A_3 form a partition of S.

Figure 18

Expanded Form of Bayes' Theorem

We have seen the "short form" of Bayes' theorem. What is the "long form?" To motivate an expanded form of Bayes' theorem, look again at the formula we've been using:

$$P(A \mid T) = \frac{P(T \mid A)P(A)}{P(T \mid A)P(A) + P(T \mid A')P(A')}.$$

The events A and A' form a **partition** of the sample space S; that is, their union is the whole of S and their intersection is empty (Figure 17).

The expanded form of Bayes' theorem applies to a partition of S into three or more events, as shown in Figure 18.

By saying that the events A_1, A_2, and A_3 form a partition of S, we mean that their union is the whole of S and the intersection of any two of them is empty, as in the figure. When we have a partition into three events as shown, the formula gives us $P(A_1 \mid T)$ in terms of $P(T \mid A_1)$, $P(T \mid A_2)$, $P(T \mid A_3)$, $P(A_1)$, $P(A_2)$, and $P(A_3)$.

Bayes' Theorem (Expanded Form)

If the events A_1, A_2, and A_3 form a partition of the sample space S, then

$$P(A_1 \mid T) = \frac{P(T \mid A_1)P(A_1)}{P(T \mid A_1)P(A_1) + P(T \mid A_2)P(A_2) + P(T \mid A_3)P(A_3)}$$

As for why this is true, and what happens when we have a partition into *four or more* events, we will wait for the exercises. In practice, as was the case with a partition into two events, we can often compute $P(A_1 \mid T)$ by constructing a tree diagram.

EXAMPLE 3 School and Work

A survey* conducted by the Bureau of Labor Statistics found that approximately 24% of the high school graduating class of 2007 went on to a two-year college, 43% went on to a four-year college, and the remaining 33% did not go on to college. Of those who went on to a two-year college, 49% worked at the same time, 28% of those going on to a four-year college worked, and 61% of those who did not go on to college worked. What percentage of those working had not gone on to college?

Solution We can interpret these percentages as probabilities if we consider the experiment of choosing a member of the high school graduating class of 2007 at random. The events we are interested in are these:

R_1: A graduate went on to a two-year college.

R_2: A graduate went on to a four-year college.

R_3: A graduate did not go to college.

A: A graduate went on to work.

*"College Enrollment and Work Activity of High School Graduates," U.S. Bureau of Labor Statistics (www.bls.gov/news.release/hsgec.htm).

The three events R_1, R_2, and R_3 partition the sample space of all graduates into three events. We are given the following probabilities:

$$P(R_1) = .24 \qquad P(R_2) = .43 \qquad P(R_3) = .33$$
$$P(A \mid R_1) = .49 \quad P(A \mid R_2) = .28 \quad P(A \mid R_3) = .61.$$

We are asked to find the probability that a graduate who went on to work did not go to college, so we are looking for $P(R_3 \mid A)$. Bayes' formula for these events is

$$P(R_3 \mid A) = \frac{P(A \mid R_3)P(R_3)}{P(A \mid R_1)P(R_1) + P(A \mid R_2)P(R_2) + P(A \mid R_3)P(R_3)}$$
$$= \frac{(.61)(.33)}{(.49)(.24) + (.28)(.43) + (.61)(.33)} \approx .46.$$

Thus we conclude that 46% of all those working had not gone on to college.

Before we go on... We could also solve Example 3 using a tree diagram. As before, the first branching corresponds to the events with unconditional probabilities that we know: R_1, R_2, and R_3. You should complete the tree and check that you obtain the same result as above. ■

7.6 EXERCISES

▼ more advanced ◆ challenging

T indicates exercises that should be solved using technology

In Exercises 1–8, use Bayes' theorem or a tree diagram to calculate the indicated probability. Round all answers to four decimal places. HINT [See Quick Examples on page 515.]

1. $P(A \mid B) = .8$, $P(B) = .2$, $P(A \mid B') = .3$. Find $P(B \mid A)$.
2. $P(A \mid B) = .6$, $P(B) = .3$, $P(A \mid B') = .5$. Find $P(B \mid A)$.
3. $P(X \mid Y) = .8$, $P(Y') = .3$, $P(X \mid Y') = .5$. Find $P(Y \mid X)$.
4. $P(X \mid Y) = .6$, $P(Y') = .4$, $P(X \mid Y') = .3$. Find $P(Y \mid X)$.
5. Y_1, Y_2, Y_3 form a partition of S. $P(X \mid Y_1) = .4$, $P(X \mid Y_2) = .5$, $P(X \mid Y_3) = .6$, $P(Y_1) = .8$, $P(Y_2) = .1$. Find $P(Y_1 \mid X)$.
6. Y_1, Y_2, Y_3 form a partition of S. $P(X \mid Y_1) = .2$, $P(X \mid Y_2) = .3$, $P(X \mid Y_3) = .6$, $P(Y_1) = .3$, $P(Y_2) = .4$. Find $P(Y_1 \mid X)$.
7. Y_1, Y_2, Y_3 form a partition of S. $P(X \mid Y_1) = .4$, $P(X \mid Y_2) = .5$, $P(X \mid Y_3) = .6$, $P(Y_1) = .8$, $P(Y_2) = .1$. Find $P(Y_2 \mid X)$.
8. Y_1, Y_2, Y_3 form a partition of S. $P(X \mid Y_1) = .2$, $P(X \mid Y_2) = .3$, $P(X \mid Y_3) = .6$, $P(Y_1) = .3$, $P(Y_2) = .4$. Find $P(Y_2 \mid X)$.

APPLICATIONS

9. ***Music Downloading*** According to a study on the effect of music downloading on spending on music, 11% of all Internet users had decreased their spending on music.[60] We estimate that 40% of all music fans used the Internet at the time of the study.[61] If 20% of non-Internet users had decreased their spending on music, what percentage of those who had decreased their spending on music were Internet users? HINT [See Examples 1 and 2.]
10. ***Music Downloading*** According to the study cited in the preceding exercise, 36% of experienced file-sharers with broadband access had decreased their spending on music. Let us estimate that 3% of all music fans were experienced file-sharers with broadband access at the time of the study.[62] If 20% of the other music fans had decreased their spending on music, what percentage of those who had decreased their spending on music were experienced file-sharers with broadband access? HINT [See Examples 1 and 2.]
11. ***Weather*** It snows in Greenland an average of once every 25 days, and when it does, glaciers have a 20% chance of growing. When it does not snow in Greenland, glaciers have only a 4% chance of growing. What is the probability that it is snowing in Greenland when glaciers are growing?

[60] Regardless of whether they used the Internet to download music. Source: *New York Times*, May 6, 2002, p. C6.

[61] According to the U.S. Department of Commerce, 51% of all U.S. households had computers in 2001.

[62] Around 15% of all online households had broadband access in 2001 according to a *New York Times* article (Dec. 24, 2001, p. C1).

12. ***Weather*** It rains in Spain an average of once every 10 days, and when it does, hurricanes have a 2% chance of happening in Hartford. When it does not rain in Spain, hurricanes have a 1% chance of happening in Hartford. What is the probability that it rains in Spain when hurricanes happen in Hartford?

13. ***Side Impact Hazard*** In 2004, 45.4% of all light vehicles were cars, and the rest were pickups or SUVs. The probability that a severe side-impact crash would prove deadly to a driver depended on the type of vehicle he or she was driving at the time, as shown in the table:[63]

Car	1.0
Light Truck or SUV	.3

What is the probability that the victim of a deadly side-impact accident was driving a car?

14. ***Side Impact Hazard*** In 2004, 27.3% of all light vehicles were light trucks, and the rest were cars or SUVs. The probability that a severe side-impact crash would prove deadly to a driver depended on the type of vehicle he or she was driving at the time, as shown in the table:[64]

Light Truck	.2
Car or SUV	.7

What is the probability that the victim of a deadly side-impact accident was driving a car or SUV?

15. ***Athletic Fitness Tests*** Any athlete who fails the Enormous State University's women's soccer fitness test is automatically dropped from the team. Last year, Mona Header failed the test, but claimed that this was due to the early hour. (The fitness test is traditionally given at 5 AM on a Sunday morning.) In fact, a study by the ESU Physical Education Department suggested that 50% of athletes fit enough to play on the team would fail the soccer test, although no unfit athlete could possibly pass the test. It also estimated that 45% of the athletes who take the test are fit enough to play soccer. Assuming these estimates are correct, what is the probability that Mona was justifiably dropped?

16. ***Academic Testing*** Professor Frank Nabarro insists that all senior physics majors take his notorious physics aptitude test. The test is so tough that anyone *not* going on to a career in physics has no hope of passing, whereas 60% of the seniors who do go on to a career in physics still fail the test. Further, 75% of all senior physics majors in fact go on to a career in physics. Assuming that you fail the test, what is the probability that you will not go on to a career in physics? HINT [See Example 3.]

17. ***Side Impact Hazard*** (Compare Exercise 13.) In 2004, 27.3% of all light vehicles were light trucks, 27.3% were SUVs, and 45.4% were cars. The probability that a severe side-impact crash would prove deadly to a driver depended on the type of vehicle he or she was driving at the time, as shown in the table:[65]

Light Truck	.210
SUV	.371
Car	1.000

What is the probability that the victim of a deadly side-impact accident was driving an SUV?

18. ***Side Impact Hazard*** In 1986, 23.9% of all light vehicles were pickups, 5.0% were SUVs, and 71.1% were cars. Refer to Exercise 17 for the probabilities that a severe side-impact crash would prove deadly. What is the probability that the victim of a deadly side-impact accident was driving a car? HINT [See Example 3.]

19. ***University Admissions*** In fall 2008, UCLA admitted 22% of its California resident applicants, 28% of its applicants from other U.S. states, and 22% of its international student applicants. Of all its applicants, 84% were California residents, 10% were from other U.S. states, and 6% were international students.[66] What percentage of all admitted students were California residents? (Round your answer to the nearest 1%.)

20. ***University Admissions*** In fall 2002, UCLA admitted 26% of its California resident applicants, 18% of its applicants from other U.S. states, and 13% of its international student applicants. Of all its applicants, 86% were California residents, 11% were from other U.S. states, and 3% were international students.[67] What percentage of all admitted students were California residents? (Round your answer to the nearest 1%.)

21. ***Internet Use*** In 2000, 86% of all Caucasians in the United States, 77% of all African-Americans, 77% of all Hispanics, and 85% of residents not classified into one of these groups used the Internet for e-mail.[68] At that time, the U.S. population was 69% Caucasian, 12% African-American, and 13% Hispanic. What percentage of U.S. residents who used the Internet for e-mail were Hispanic?

[63] A "serious" side-impact accident is defined as one in which the driver of a car would be killed. Source: National Highway Traffic Safety Administration/*New York Times*, May 30, 2004, p. BU 9.

[64] Ibid.

[65] Ibid.

[66] Source: University of California (www.admissions.ucla.edu/Prospect/Adm_fr/Frosh_Prof08.htm).

[67] Source: UCLA Web site, May 2002 (www.admissions.ucla.edu/Prospect/Adm_fr/Frosh_Prof.htm).

[68] Source: NTIA and ESA, U.S. Department of Commerce, using August 2000 U.S. Bureau of The Census Current Population Survey Supplement.

22. ***Internet Use*** In 2000, 59% of all Caucasians in the United States, 57% of all African-Americans, 58% of all Hispanics, and 54% of residents not classified into one of these groups used the Internet to search for information.[69] At that time, the U.S. population was 69% Caucasian, 12% African-American, and 13% Hispanic. What percentage of U.S. residents who used the Internet for information search were African-American?

23. ▼ ***Market Surveys*** A *New York Times* survey of homeowners showed that 86% of those with swimming pools were married couples, and the other 14% were single.[70] It also showed that 15% of all homeowners had pools.

a. Assuming that 90% of all homeowners without pools are married couples, what percentage of homes owned by married couples have pools?

b. Would it better pay pool manufacturers to go after single homeowners or married homeowners? Explain.

24. ▼ ***Crime and Preschool***. Another *New York Times* survey of needy and disabled youths showed that 51% of those who had no preschool education were arrested or charged with a crime by the time they were 19, whereas only 31% who had preschool education wound up in this category.[71] The survey did not specify what percentage of the youths in the survey had preschool education, so let us take a guess at that and estimate that 20% of them had attended preschool.

a. What percentage of the youths arrested or charged with a crime had no preschool education?

b. What would this figure be if 80% of the youths had attended preschool? Would youths who had preschool education be more likely to be arrested or charged with a crime than those who did not? Support your answer by quoting probabilities.

25. ▼ ***Grade Complaints*** Two of the mathematics professors at Enormous State are Professor A (known for easy grading) and Professor F (known for tough grading). Last semester, roughly three quarters of Professor F's class consisted of former students of Professor A; these students apparently felt encouraged by their (utterly undeserved) high grades. (Professor F's own former students had fled in droves to Professor A's class to try to shore up their grade point averages.) At the end of the semester, as might have been predicted, all of Professor A's former students wound up with a C– or lower. The rest of the students in the class—former students of Professor F who had decided to "stick it out"—fared better, and two-thirds of them earned higher than a C–. After discovering what had befallen them, all the students who earned C– or lower got together and decided to send a delegation to the Department Chair to complain that their grade point averages had been ruined by this callous and heartless beast! The contingent was to consist of 10 representatives selected at random from among them. How many of the 10 would you estimate to have been former students of Professor A?

26. ▼ ***Weather Prediction*** A local TV station employs Desmorelda, "Mistress of the Zodiac," as its weather forecaster. Now, when it rains, Sagittarius is in the shadow of Jupiter one-third of the time, and it rains on 4 out of every 50 days. Sagittarius falls in Jupiter's shadow on only 1 in every 5 rainless days. The powers that be at the station notice a disturbing pattern to Desmorelda's weather predictions. It seems that she always predicts that it will rain when Sagittarius is in the shadow of Jupiter. What percentage of the time is she correct? Should they replace her?

27. ▼ ***Employment in the 1980s*** In a 1987 survey of married couples with earnings, 95% of all husbands were employed. Of all employed husbands, 71% of their wives were also employed.[72] Noting that either the husband or wife in a couple with earnings had to be employed, find the probability that the husband of an employed woman was also employed.

28. ▼ ***Employment in the 1980s*** Repeat the preceding exercise in the event that 50% of all husbands were employed.

29. ▼ ***Juvenile Delinquency*** According to a study at the Oregon Social Learning Center, boys who had been arrested by age 14 were 17.9 times more likely to become chronic offenders than those who had not.[73] Use these data to estimate the percentage of chronic offenders who had been arrested by age 14 in a city where 0.1% of all boys have been arrested by age 14. (Hint: Use Bayes' formula rather than a tree.) HINT [See Example 3.]

30. ▼ ***Crime*** According to the same study at the Oregon Social Learning Center, chronic offenders were 14.3 times more likely to commit violent offenses than people who were not chronic offenders.[74] In a neighborhood where 2 in every 1,000 residents is a chronic offender, estimate the probability that a violent offender is also a chronic offender. (Hint: Use Bayes' formula rather than a tree.) HINT [See Example 3.]

31. ▼ ***Benefits of Exercise*** According to a study in *The New England Journal of Medicine*,[75] 202 of a sample of 5,990 middle-aged men had developed diabetes. It also found that men who were very active (burning about 3,500 calories daily) were half as likely to develop diabetes compared with men who were sedentary. Assume that one-third of all middle-aged men are very active, and the rest are classified as sedentary. What is the probability that a middle-aged man with diabetes is very active?

32. ▼ ***Benefits of Exercise*** Repeat Exercise 31, assuming that only 1 in 10 middle-aged men is very active.

---

[69] Source: NTIA and ESA, U.S. Department of Commerce, using August 2000 U.S. Bureau of The Census Current Population Survey Supplement.

[70] Source: "All about Swimming Pools," *The New York Times*, September 13, 1992.

[71] Source: "Governors Develop Plan to Help Preschool Children," *The New York Times*, August 2, 1992.

[72] Source: *Statistical Abstract of the United States*, 111th Ed., 1991, U.S. Dept. of Commerce/U.S. Bureau of Labor Statistics. Figures rounded to the nearest 1%.

[73] Based on a study, by Marion S. Forgatch, of 319 boys from high-crime neighborhoods in Eugene, Oregon. Source: W. Wayt Gibbs, "Seeking the Criminal Element," *Scientific American*, March 1995, pp. 101–107.

[74] Ibid.

[75] As cited in an article in *The New York Times* on July 18, 1991.

33. ◆ ***Airbag Safety*** According to a study conducted by the Harvard School of Public Health, a child seated in the front seat who was wearing a seatbelt was 31% more likely to be killed in an accident if the car had an air bag that deployed than if it did not.[76] Airbags deployed in 25% of all accidents. For a child seated in the front seat wearing a seatbelt, what is the probability that the airbag deployed in an accident in which the child was killed? (Round your answer to two decimal places.) HINT [When we say "A is 31% more likely than B" we mean that the probability of A is 1.31 times the probability of B.]

34. ◆ ***Airbag Safety*** According to the study cited in Exercise 33, a child seated in the front seat who was not wearing a seatbelt was 84% more likely to be killed in an accident if the car had an air bag that deployed than if it did not.[77] Airbags deployed in 25% of all accidents. For a child seated in the front seat not wearing a seatbelt, what is the probability that the airbag deployed in an accident in which the child was killed? (Round your answer to two decimal places.) HINT [When we say "A is 84% more likely than B" we mean that the probability of A is 1.84 times the probability of B.]

COMMUNICATION AND REASONING EXERCISES

35. Your friend claims that the probability of A given B is the same as the probability of B given A. How would you convince him that he is wrong?

36. Complete the following sentence. To use Bayes' formula to compute $P(E \mid F)$, you need to be given ______.

37. ▼ Give an example in which a steroids test gives a false positive only 1% of the time, and yet if an athlete tests positive, the chance that he or she has used steroids is under 10%.

38. ▼ Give an example in which a steroids test gives a false positive 30% of the time, and yet if an athlete tests positive, the chance that he or she has used steroids is over 90%.

39. ▼ Use a tree to derive the expanded form of Bayes' Theorem for a partition of the sample space S into three events R_1, R_2, and R_3.

40. ▼ Write down an expanded form of Bayes' Theorem that applies to a partition of the sample space S into four events R_1, R_2, R_3, and R_4.

41. ◆ ***Politics*** The following letter appeared in the *New York Times*.[78]

> To the Editor:
>
> It stretches credulity when William Safire contends (column, Jan. 11) that 90 percent of those who agreed with his Jan. 8 column, in which he called the First Lady, Hillary Rodham Clinton, "a congenital liar," were men and 90 percent of those who disagreed were women.
>
> Assuming these percentages hold for Democrats as well as Republicans, only 10 percent of Democratic men disagreed with him. Is Mr. Safire suggesting that 90 percent of Democratic men supported him? How naive does he take his readers to be?
>
> A. D.
> New York, Jan. 12, 1996

Comment on the letter writer's reasoning.

42. ◆ ***Politics*** Refer back to the preceding exercise. If the letter writer's conclusion was correct, what percentage of all Democrats would have agreed with Safire's column?

[76] The study was conducted by Dr. Segul-Gomez at the Harvard School of Public Health. Source: *The New York Times*, December 1, 2000, p. F1.

[77] Ibid.

[78] The original letter appeared in *The New York Times*, January 16, 1996, p. A16. The authors have edited the first phrase of the second paragraph slightly for clarity; the original sentence read: "Assuming the response was equally divided between Democrats and Republicans, . . . "

7.7 Markov Systems

Many real-life situations can be modeled by processes that pass from state to state with given probabilities. A simple example of such a **Markov system** is the fluctuation of a gambler's fortune as he or she continues to bet. Other examples come from the study of trends in the commercial world and the study of neural networks and artificial intelligence. The mathematics of Markov systems is an interesting combination of probability and matrix arithmetic.

Here is a basic example we shall use many times: A market analyst for Gamble Detergents is interested in whether consumers prefer powdered laundry detergents or liquid detergents. Two market surveys taken one year apart revealed that 20% of powdered detergent users had switched to liquid one year later, while the rest were still using powder. Only 10% of liquid detergent users had switched to powder one year later, with the rest still using liquid.

We analyze this example as follows: Every year a consumer may be in one of two possible **states**: He may be a powdered detergent user or a liquid detergent user. Let us number these states: A consumer is in state 1 if he uses powdered detergent and in state 2 if he uses liquid. There is a basic **time step** of one year. If a consumer happens to be in state 1 during a given year, then there is a probability of 20% = .2 (the chance that a randomly chosen powder user will switch to liquid) that he will be in state 2 the next year. We write

$$p_{12} = .2$$

to indicate that the probability of going *from* state 1 *to* state 2 in one time step is .2. The other 80% of the powder users are using powder the next year. We write

$$p_{11} = .8$$

to indicate that the probability of *staying* in state 1 from one year to the next is .8.* What if a consumer is in state 2? Then the probability of going to state 1 is given as 10% = .1, so the probability of remaining in state 2 is .9. Thus,

$$p_{21} = .1$$

and

$$p_{22} = .9.$$

We can picture this system as in Figure 19, which shows the **state transition diagram** for this example. The numbers p_{ij}, which appear as labels on the arrows, are the **transition probabilities**.

*** NOTE** Notice that these are actually *conditional* probabilities. For instance, p_{12} is the probability that the system (the consumer in this case) will go into state 2, *given that the system* (the consumer) *is in state 1*.

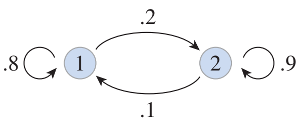


Figure 19

Markov System, States, and Transition Probabilities

A **Markov system*** (or **Markov process** or **Markov chain**) is a system that can be in one of several specified **states**. There is specified a certain **time step**, and at each step the system will randomly change states or remain where it is. The probability of going from state i to state j is a fixed number p_{ij}, called the **transition probability**.

*** NOTE** Named after the Russian mathematician A.A. Markov (1856–1922) who first studied these "non-deterministic" processes.

Quick Example

The Markov system depicted in Figure 19 has two states: state 1 and state 2. The transition probabilities are as follows:

$$\begin{aligned}
p_{11} &= \text{Probability of going from state 1 to state 1} = .8\\
p_{12} &= \text{Probability of going from state 1 to state 2} = .2\\
p_{21} &= \text{Probability of going from state 2 to state 1} = .1\\
p_{22} &= \text{Probability of going from state 2 to state 2} = .9.
\end{aligned}$$

Notice that, because the system must go somewhere at each time step, the transition probabilities originating at a particular state always add up to 1. For example, in the transition diagram above, when we add the probabilities originating at state 1, we get $.8 + .2 = 1$.

The transition probabilities may be conveniently arranged in a matrix.

Transition Matrix

The **transition matrix** associated with a given Markov system is the matrix P whose ijth entry is the transition probability p_{ij}, the transition probability of going *from* state i *to* state j. In other words, the entry in position ij is the *label on the arrow going from state i to state j* in a state transition diagram.

Thus, the transition matrix for a system with two states would be set up as follows:

$$\begin{array}{cc} & \textbf{To:} \\ & \begin{array}{cc} \mathbf{1} & \mathbf{2} \end{array} \\ \textbf{From:} \begin{array}{c} \mathbf{1} \\ \mathbf{2} \end{array} & \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} . \end{array} \quad \begin{array}{l} \text{Arrows Originating in State 1} \\ \text{Arrows Originating in State 2} \end{array}$$

Quick Example

In the system pictured in Figure 19, the transition matrix is

$$P = \begin{bmatrix} .8 & .2 \\ .1 & .9 \end{bmatrix}.$$

Note Notice that because the sum of the transition probabilities that originate at any state is 1, *the sum of the entries in any row of a transition matrix is 1.* ■

Now, let's start doing some calculations.

EXAMPLE 1 Laundry Detergent Switching

Consider the Markov system found by Gamble Detergents at the beginning of this section. Suppose that 70% of consumers are now using powdered detergent, while the other 30% are using liquid.

a. What will be the distribution one year from now? (That is, what percentage will be using powdered and what percentage liquid detergent?)

b. Assuming that the probabilities remain the same, what will be the distribution two years from now? Three years from now?

Solution

a. First, let us think of the statement that 70% of consumers are using powdered detergent as telling us a probability: The probability that a randomly chosen consumer uses powdered detergent is .7. Similarly, the probability that a randomly chosen consumer uses liquid detergent is .3. We want to find the corresponding probabilities one year from now. To do this, consider the tree diagram in Figure 20.

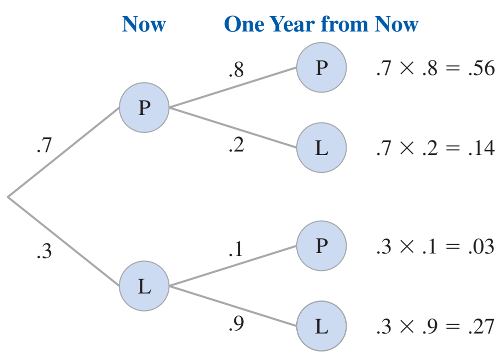


Figure 20

using Technology

Technology can be used to compute the distribution vectors in Example 1:

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Define `[A]` as the initial distribution and `[B]` as the transition matrix. Then compute [A]*[B] for the distribution after 1 step. `Ans*[B]` gives the distribution after successive steps.
[More details on page 542.]

Excel

Use the MMULT command to multiply the associated matrices. [More details on page 544.]

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→ Matrix Algebra Tool

You can enter the transition matrix P and the initial distribution vector v in the input area, and compute the various distribution vectors using the following formulas:

`v*P`	1 step
`v*P^2`	2 steps
`v*P^3`	3 steps.

The first branching shows the probabilities now, while the second branching shows the (conditional) transition probabilities. So, if we want to know the probability that a consumer is using powdered detergent one year from now, it will be

$$\text{Probability of using powder after one year} = .7 \times .8 + .3 \times .1 = .59.$$

On the other hand, we have:

$$\text{Probability of using liquid after one year} = .7 \times .2 + .3 \times .9 = .41.$$

Now, here's the crucial point: *These are exactly the same calculations as in the matrix product*

$$\underset{\text{Initial distribution}}{\underset{\uparrow}{[.7 \quad .3]}} \underset{\text{Transition matrix}}{\underset{\uparrow}{\begin{bmatrix} .8 & .2 \\ .1 & .9 \end{bmatrix}}} = \underset{\text{Distribution after 1 step}}{\underset{\uparrow}{[.59 \quad .41]}}.$$

Thus, to get the distribution of detergent users after one year, all we have to do is multiply the **initial distribution vector** $[.7 \quad .3]$ by the transition matrix P. The result is $[.59 \quad .41]$, the **distribution vector after one step**.

b. Now what about the distribution after *two* years? If we assume that the same fraction of consumers switch or stay put in the second year as in the first, we can simply repeat the calculation we did above, using the new distribution vector:

$$\underset{\text{Distribution after 1 step}}{\underset{\uparrow}{[.59 \quad .41]}} \underset{\text{Transition matrix}}{\underset{\uparrow}{\begin{bmatrix} .8 & .2 \\ .1 & .9 \end{bmatrix}}} = \underset{\text{Distribution after 2 steps}}{\underset{\uparrow}{[.513 \quad .487]}}.$$

Thus, after two years we can expect 51.3% of consumers to be using powdered detergent and 48.7% to be using liquid detergent. Similarly, after three years we have

$$[.513 \quad .487] \begin{bmatrix} .8 & .2 \\ .1 & .9 \end{bmatrix} = [.4591 \quad .5409].$$

So, after three years 45.91% of consumers will be using powdered detergent and 54.09% will be using liquid. Slowly but surely, liquid detergent seems to be winning.

➡ **Before we go on...** Note that the sum of the entries is 1 in each of the distribution vectors in Example 1. In fact, these vectors are giving the probability distributions for each year of finding a randomly chosen consumer using either powdered or liquid detergent. A vector having non-negative entries adding up to 1 is called a **probability vector**. ■

Distribution Vector after *m* Steps

A **distribution vector** is a probability vector giving the probability distribution for finding a Markov system in its various possible states. If v is a distribution vector, then the distribution vector one step later will be vP. The distribution m steps later will be

$$\text{Distribution after } m \text{ steps} = v \cdot P \cdot P \cdot \ldots \cdot P \ (m \text{ times}) = vP^m.$$

Quick Example

If $P = \begin{bmatrix} 0 & 1 \\ .5 & .5 \end{bmatrix}$ and $v = [.2 \quad .8]$, then we can calculate the following distribution vectors:

$$vP = [.2 \quad .8]\begin{bmatrix} 0 & 1 \\ .5 & .5 \end{bmatrix} = [.4 \quad .6]$$ Distribution after one step

$$vP^2 = (vP)P = [.4 \quad .6]\begin{bmatrix} 0 & 1 \\ .5 & .5 \end{bmatrix} = [.3 \quad .7]$$ Distribution after two steps

$$vP^3 = (vP^2)P = [.3 \quad .7]\begin{bmatrix} 0 & 1 \\ .5 & .5 \end{bmatrix} = [.35 \quad .65].$$ Distribution after three steps

What about the matrix P^m that appears above? Multiplying a distribution vector v times P^m gives us the distribution m steps later, so we can think of P^m and the m-step transition matrix. More explicitly, consider the following example.

EXAMPLE 2 Powers of the Transition Matrix

Continuing the example of detergent switching, suppose that a consumer is now using powdered detergent. What are the probabilities that the consumer will be using powdered or liquid detergent two years from now? What if the consumer is now using liquid detergent?

Solution To record the fact that we know that the consumer is using powdered detergent, we can take as our initial distribution vector $v = [1 \quad 0]$. To find the distribution two years from now, we compute vP^2. To make a point, we do the calculation slightly differently:

$$\begin{aligned} vP^2 &= [1 \quad 0]\begin{bmatrix} .8 & .2 \\ .1 & .9 \end{bmatrix}\begin{bmatrix} .8 & .2 \\ .1 & .9 \end{bmatrix} \\ &= [1 \quad 0]\begin{bmatrix} .66 & .34 \\ .17 & .83 \end{bmatrix} \\ &= [.66 \quad .34]. \end{aligned}$$

So, the probability that our consumer is using powdered detergent two years from now is .66, while the probability of using liquid detergent is .34. The point to notice is that these are the entries in the first row of P^2. Similarly, if we consider a consumer now using liquid detergent, we should take the initial distribution vector to be $v = [0 \quad 1]$ and compute

$$vP^2 = [0 \quad 1]\begin{bmatrix} .66 & .34 \\ .17 & .83 \end{bmatrix} = [.17 \quad .83].$$

Thus, the bottom row gives the probabilities that a consumer, now using liquid detergent, will be using either powdered or liquid detergent two years from now.

In other words, the ijth entry of P^2 gives the probability that a consumer, starting in state i, will be in state j after two time steps.

What is true in Example 2 for two time steps is true for any number of time steps:

Powers of the Transition Matrix

P^m $(m = 1, 2, 3, \ldots)$ is the ***m*-step transition matrix**. The *ij*th entry in P^m is the probability of a transition from state i to state j in m steps.

Quick Example

If $P = \begin{bmatrix} 0 & 1 \\ .5 & .5 \end{bmatrix}$ then

$$P^2 = P \cdot P = \begin{bmatrix} .5 & .5 \\ .25 & .75 \end{bmatrix} \qquad \text{2-step transition matrix}$$

$$P^3 = P \cdot P^2 = \begin{bmatrix} .25 & .75 \\ .375 & .625 \end{bmatrix}. \qquad \text{3-step transition matrix}$$

The probability of going from state 1 to state 2 in two steps = (1, 2)-entry of $P^2 = .5$.

The probability of going from state 1 to state 2 in three steps = (1, 2)-entry of $P^3 = .75$.

What happens if we follow our laundry detergent-using consumers for many years?

EXAMPLE 3 Long-Term Behavior

Suppose that 70% of consumers are now using powdered detergent while the other 30% are using liquid. Assuming that the transition matrix remains valid the whole time, what will be the distribution 1, 2, 3, . . . , and 50 years later?

Solution Of course, to do this many matrix multiplications, we're best off using technology. We already did the first three calculations in an earlier example.

Distribution after 1 year: $[.7 \quad .3]\begin{bmatrix} .8 & .2 \\ .1 & .9 \end{bmatrix} = [.59 \quad .41]$.

Distribution after 2 years: $[.59 \quad .41]\begin{bmatrix} .8 & .2 \\ .1 & .9 \end{bmatrix} = [.513 \quad .487]$.

Distribution after 3 years: $[.513 \quad .487]\begin{bmatrix} .8 & .2 \\ .1 & .9 \end{bmatrix} = [.4591 \quad .5409]$.

. . .

Distribution after 48 years: $[.33333335 \quad .66666665]$.

Distribution after 49 years: $[.33333334 \quad .66666666]$.

Distribution after 50 years: $[.33333334 \quad .66666666]$.

Thus, the distribution after 50 years is approximately $[.33333334 \quad .66666666]$.

Something interesting seems to be happening in Example 3. The distribution seems to be getting closer and closer to

$$[.333333\ldots \quad .666666\ldots] = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

Let's call this distribution vector v_∞. Notice two things about v_∞:

- v_∞ is a probability vector.
- If we calculate $v_\infty P$, we find

$$v_\infty P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} .8 & .2 \\ .1 & .9 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix} = v_\infty.$$

In other words,

$$v_\infty P = v_\infty.$$

We call a probability vector v with the property that $vP = v$ a **steady-state (probability) vector**.

Q: *Where does the name steady-state vector come from?*

A: If $vP = v$, then v is a distribution that will not change from time step to time step. In the example above, because $[1/3 \;\; 2/3]$ is a steady-state vector, if $1/3$ of consumers use powdered detergent and $2/3$ use liquid detergent one year, then the proportions will be the same the next year. Individual consumers may still switch from year to year, but as many will switch from powder to liquid as switch from liquid to powder, so the number using each will remain constant.

But how do we find a steady-state vector?

EXAMPLE 4 Calculating the Steady-State Vector

Calculate the steady-state probability vector for the transition matrix in the preceding examples:

$$P = \begin{bmatrix} .8 & .2 \\ .1 & .9 \end{bmatrix}.$$

Solution We are asked to find

$$v_\infty = [x \quad y].$$

This vector must satisfy the equation

$$v_\infty P = v_\infty$$

or

$$[x \;\; y] \begin{bmatrix} .8 & .2 \\ .1 & .9 \end{bmatrix} = [x \;\; y].$$

Doing the matrix multiplication gives

$$[.8x + .1y \quad .2x + .9y] = [x \;\; y].$$

Equating corresponding entries gives

$$\begin{aligned} .8x + .1y &= x \\ .2x + .9y &= y \end{aligned}$$

or

$$\begin{aligned} -.2x + .1y &= 0 \\ .2x - .1y &= 0. \end{aligned}$$

Now these equations are really the same equation. (Do you see that?) There is one more thing we know, though: Because $[x \quad y]$ is a probability vector, its entries must add up to 1. This gives one more equation:

$$x + y = 1.$$

Taking this equation together with one of the two equations above gives us the following system:

$$\begin{aligned} x + \quad y &= 1 \\ -.2x + .1y &= 0. \end{aligned}$$

We now solve this system using any of the techniques we learned for solving systems of linear equations. We find that the solution is $x = 1/3$, and $y = 2/3$, so the steady-state vector is

$$v_\infty = [x \quad y] = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

as suggested in Example 3.

The method we just used works for any size transition matrix and can be summarized as follows.

Calculating the Steady-State Distribution Vector

To calculate the steady-state probability vector for a Markov System with transition matrix P, we solve the system of equations given by

$$\begin{aligned} &x + y + z + \cdots = 1 \\ &[x \quad y \quad z \ \ldots]P = [x \quad y \quad z \ \ldots] \end{aligned}$$

where we use as many unknowns as there are states in the Markov system. The steady-state probability vector is then

$$v_\infty = [x \quad y \quad z \ \ldots].$$

Q: *Is there always a steady-state distribution vector?*

A: Yes, although the explanation why is more involved than we can give here.

Q: *In Example 3, we started with a distribution vector v and found that vP^m got closer and closer to v_∞ as m got larger. Does that always happen?*

A: It does if the Markov system is **regular**, as we define on the next page, but may not for other kinds of systems. Again, we shall not prove this fact here.

using Technology

Technology can be used to compute the steady state vector in Example 4.

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Define `[A]` as the coefficient matrix of the system of equations being solved, and `[B]` as the column matrix of the right-hand sides. Then compute `[A]`$^{-1}$`[B]`
[More details on page 543.]

Excel

Use the MMULT and MINVERSE commands to solve the necessary system of equations.
[More details on page 545.]

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You can use the Pivot and Gauss-Jordan Tool to solve the system of equations that gives you the steady-state vector.

Regular Markov Systems

A **regular** Markov system is one for which some power of its transition matrix P has no zero entries. If a Markov system is regular, then

1. It has a unique steady-state probability vector v_∞, and

2. If v is any probability vector whatsoever, then vP^m approaches v_∞ as m gets large. We say that the **long-term behavior** of the system is to have distribution (close to) v_∞.

Interpreting the Steady-State Vector

In a regular Markov system, the entries in the steady-state probability vector give the long-term probabilities that the system will be in the corresponding states, or the fractions of time one can expect to find the Markov system in the corresponding states.

Quick Examples

1. The system with transition matrix $P = \begin{bmatrix} .8 & .2 \\ .1 & .9 \end{bmatrix}$ is regular because $P(= P^1)$ has no zero entries.

2. The system with transition matrix $P = \begin{bmatrix} 0 & 1 \\ .5 & .5 \end{bmatrix}$ is regular because

$P^2 = \begin{bmatrix} .5 & .5 \\ .25 & .75 \end{bmatrix}$ has no zero entries.

3. The system with transition matrix $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is *not* regular:

$P^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $P^3 = P$ again, so the powers of P alternate between these two matrices. Thus, every power of P has zero entries. Although this system has a steady-state vector, namely $[.5 \quad .5]$, if we take $v = [1 \quad 0]$, then $vP = [0 \quad 1]$ and $vP^2 = v$, so the distribution vectors vP^m just alternate between these two vectors, not approaching v_∞.

We finish with one more example.

EXAMPLE 5 Gambler's Ruin

A timid gambler, armed with her annual bonus of \$20, decides to play roulette using the following scheme. At each spin of the wheel, she places \$10 on red. If red comes up, she wins an additional \$10; if black comes up, she loses her \$10. For the sake of simplicity, assume that she has a probability of $1/2$ of winning. (In the real game, the probability is slightly lower—a fact that many gamblers forget.) She keeps playing until she has either gotten up to \$30 or lost it all. In either case, she then packs up and leaves. Model this situation as a Markov system and find the associated transition matrix. What can we say about the long-term behavior of this system?

Solution We must first decide on the states of the system. A good choice is the gambler's financial state, the amount of money she has at any stage of the game. According to her rules, she can be broke, have \$10, \$20, or \$30. Thus, there are four states: $1 = \$0$, 2 = \$10, 3 = \$20, and 4 = \$30. Because she bets \$10 each time, she moves down \$10 if she loses (with probability $1/2$) and up \$10 if she wins (with probability also $1/2$), until she reaches one of the extremes. The transition diagram is shown in Figure 21.

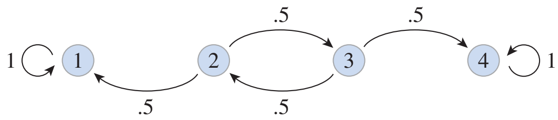

Figure 21

Note that once the system enters state 1 or state 4, it does not leave; with probability 1, it stays in the same state.* We call such states **absorbing states**. We can now write down the transition matrix:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ .5 & 0 & .5 & 0 \\ 0 & .5 & 0 & .5 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(Notice all the 0 entries, corresponding to possible transitions that we did not draw in the transition diagram because they have 0 probability of occurring. We usually leave out such arrows.) Is this system regular? Take a look at P^2:

$$P^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ .5 & .25 & 0 & .25 \\ .25 & 0 & .25 & .5 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Notice that the first and last rows haven't changed. After two steps, there is still no chance of leaving states 1 or 4. In fact, no matter how many powers we take, no matter how many steps we look at, there will still be no way to leave either of those states, and the first and last rows will still have plenty of zeros. This system is not regular.

Nonetheless, we can try to find a steady-state probability vector. If we do this (and you should set up the system of linear equations and solve it), we find that there are infinitely many steady-state probability vectors, namely all vectors of the form $[x \quad 0 \quad 0 \quad 1-x]$ for $0 \le x \le 1$. (You can check directly that these are all steady-state vectors.) As with a regular system, if we start with any distribution, the system will tend toward one of these steady-state vectors. In other words, eventually the gambler will either lose all her money or leave the table with \$30.

But which outcome is more likely, and with what probability? One way to approach this question is to try computing the distribution after many steps. The distribution that represents the gambler starting with \$20 is $v = [0 \quad 0 \quad 1 \quad 0]$. Using technology, it's easy to compute vP^n for some large values of n:

$$vP^{10} \approx [.333008 \quad 0 \quad .000976 \quad .666016]$$
$$vP^{50} \approx [.333333 \quad 0 \quad 0 \quad .666667].$$

So, it looks like the probability that she will leave the table with \$30 is approximately $2/3$, while the probability that she loses it all is $1/3$.

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→ Online Utilities
→ Markov Simulation
where you can enter the transition matrix and watch a visual simulation showing the system hopping from state to state. Run the simulation a number of times to get an experimental estimate of the time to absorption. (Take the average time to absorption over many runs.) Also, try varying the transition probabilities to study the effect on the time to absorption.

---

*These states are like "Roach Motels" ("Roaches check in, but they don't check out")!

What if she started with only \$10? Then our initial distribution would be $v = [0 \;\; 1 \;\; 0 \;\; 0]$ and

$$vP^{10} \approx [.666016 \quad .000976 \quad 0 \quad .333008]$$
$$vP^{50} \approx [.666667 \quad 0 \quad 0 \quad .333333].$$

So, this time, the probability of her losing everything is about $2/3$ while the probability of her leaving with \$30 is $1/3$. There is a way of calculating these probabilities exactly using matrix arithmetic; however, it would take us too far afield to describe it here.

Before we go on... Another interesting question is, how long will it take the gambler in Example 5 to get to \$30 or lose it all? This is called the *time to absorption* and can be calculated using matrix arithmetic. ■

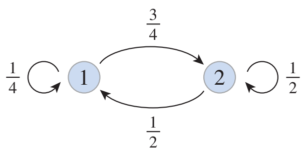
7.7 EXERCISES

▼ more advanced ◆ challenging

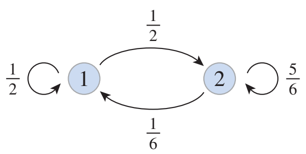
T indicates exercises that should be solved using technology

In Exercises 1–10, write down the transition matrix associated with each state transition diagram.

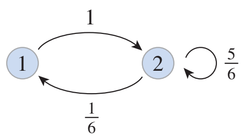
1.



2.



3.



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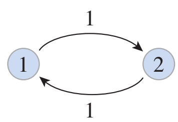
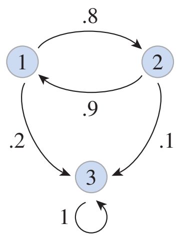
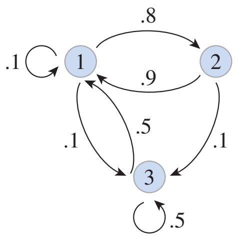


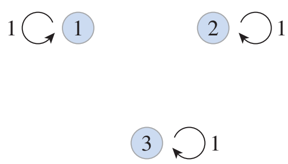
5.



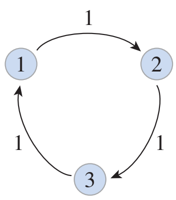
6.



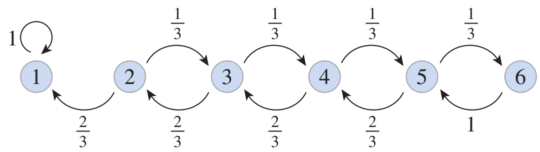
7.



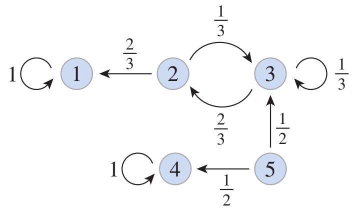
8.



9.



10.



In each of Exercises 11–24, you are given a transition matrix P and initial distribution vector v. Find

(a) the two-step transition matrix and

(b) the distribution vectors after one, two, and three steps.

HINT [See Quick Examples on pages 525 and 526.]

11. $P = \begin{bmatrix} .5 & .5 \\ 0 & 1 \end{bmatrix}$, $v = [1 \quad 0]$

12. $P = \begin{bmatrix} 1 & 0 \\ .5 & .5 \end{bmatrix}$, $v = [0 \quad 1]$

13. $P = \begin{bmatrix} .2 & .8 \\ .4 & .6 \end{bmatrix}$, $v = [.5 \quad .5]$

14. $P = \begin{bmatrix} 1/3 & 2/3 \\ 1/2 & 1/2 \end{bmatrix}$, $v = [1/4 \quad 3/4]$

15. $P = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}$, $v = [2/3 \quad 1/3]$

16. $P = \begin{bmatrix} 0 & 1 \\ 1/4 & 3/4 \end{bmatrix}$, $v = [1/5 \quad 4/5]$

17. $P = \begin{bmatrix} 3/4 & 1/4 \\ 3/4 & 1/4 \end{bmatrix}$, $v = [1/2 \quad 1/2]$

18. $P = \begin{bmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix}$, $v = [1/7 \quad 6/7]$

19. $P = \begin{bmatrix} .5 & .5 & 0 \\ 0 & 1 & 0 \\ 0 & .5 & .5 \end{bmatrix}$, $v = [1 \quad 0 \quad 0]$

20. $P = \begin{bmatrix} .5 & 0 & .5 \\ 1 & 0 & 0 \\ 0 & .5 & .5 \end{bmatrix}$, $v = [0 \quad 1 \quad 0]$

21. $P = \begin{bmatrix} 0 & 1 & 0 \\ 1/3 & 1/3 & 1/3 \\ 1 & 0 & 0 \end{bmatrix}$, $v = [1/2 \quad 0 \quad 1/2]$

22. $P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$, $v = [0 \quad 0 \quad 1]$

23. $P = \begin{bmatrix} .1 & .9 & 0 \\ 0 & 1 & 0 \\ 0 & .2 & .8 \end{bmatrix}$, $v = [.5 \quad 0 \quad .5]$

24. $P = \begin{bmatrix} .1 & .1 & .8 \\ .5 & 0 & .5 \\ .5 & 0 & .5 \end{bmatrix}$, $v = [0 \quad 1 \quad 0]$

In each of Exercises 25–36, you are given a transition matrix P. Find the steady-state distribution vector. HINT [See Example 4.]

25. ▼ $P = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}$

26. ▼ $P = \begin{bmatrix} 0 & 1 \\ 1/4 & 3/4 \end{bmatrix}$

27. ▼ $P = \begin{bmatrix} 1/3 & 2/3 \\ 1/2 & 1/2 \end{bmatrix}$

28. ▼ $P = \begin{bmatrix} .2 & .8 \\ .4 & .6 \end{bmatrix}$

29. ▼ $P = \begin{bmatrix} .1 & .9 \\ .6 & .4 \end{bmatrix}$

30. ▼ $P = \begin{bmatrix} .2 & .8 \\ .7 & .3 \end{bmatrix}$

31. ▼ $P = \begin{bmatrix} .5 & 0 & .5 \\ 1 & 0 & 0 \\ 0 & .5 & .5 \end{bmatrix}$

32. ▼ $P = \begin{bmatrix} 0 & .5 & .5 \\ .5 & .5 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

33. ▼ $P = \begin{bmatrix} 0 & 1 & 0 \\ 1/3 & 1/3 & 1/3 \\ 1 & 0 & 0 \end{bmatrix}$

34. ▼ $P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$

35. ▼ $P = \begin{bmatrix} .1 & .9 & 0 \\ 0 & 1 & 0 \\ 0 & .2 & .8 \end{bmatrix}$

36. ▼ $P = \begin{bmatrix} .1 & .1 & .8 \\ .5 & 0 & .5 \\ .5 & 0 & .5 \end{bmatrix}$

APPLICATIONS

37. ***Marketing*** A market survey shows that half the owners of Sorey State Boogie Boards became disenchanted with the product and switched to C&T Super Professional Boards the next surf season, while the other half remained loyal to Sorey State. On the other hand, three quarters of the C&T Boogie Board users remained loyal to C&T, while the rest switched to Sorey State. Set these data up as a Markov transition matrix, and calculate the probability that a Sorey State Board user will be using the same brand two seasons later. HINT [See Example 1.]

38. ***Major Switching*** At Suburban Community College, 10% of all business majors switched to another major the next semester, while the remaining 90% continued as business majors. Of all nonbusiness majors, 20% switched to a business major the following semester, while the rest did not. Set up these data as a Markov transition matrix, and calculate the probability that a business major will no longer be a business major in two semesters' time. HINT [See Example 1.]

39. ***Pest Control*** In an experiment to test the effectiveness of the latest roach trap, the "Roach Resort," 50 roaches were placed in the vicinity of the trap and left there for an hour. At the end of the hour, it was observed that 30 of them had "checked in," while the rest were still scurrying around. (Remember that "once a roach checks in, it never checks out.")

a. Set up the transition matrix P for the system with decimal entries, and calculate P^2 and P^3.
b. If a roach begins outside the "Resort," what is the probability of it "checking in" by the end of 1 hour? 2 hours? 3 hours?
c. What do you expect to be the long-term impact on the number of roaches? HINT [See Example 5.]

40. ***Employment*** You have worked for the Department of Administrative Affairs (DAA) for 27 years, and you still have little or no idea exactly what your job entails. To make your life a little more interesting, you have decided on the following course of action. Every Friday afternoon, you will use your desktop computer to generate a random digit from 0 to 9 (inclusive). If the digit is a zero, you will immediately quit your job, never to return. Otherwise, you will return to work the following Monday.

a. Use the states (1) employed by the DAA and (2) not employed by the DAA to set up a transition probability matrix P with decimal entries, and calculate P^2 and P^3.
b. What is the probability that you will still be employed by the DAA after each of the next three weeks?
c. What are your long-term prospects for employment at the DAA? HINT [See Example 5.]

41. ▼ ***Risk Analysis*** An auto insurance company classifies each motorist as "high-risk" if the motorist has had at least one moving violation during the past calendar year and "low risk" if the motorist has had no violations during the past calendar year. According to the company's data, a high-risk motorist has a 50% chance of remaining in the high-risk category the next year and a 50% chance of moving to the low-risk category. A low-risk motorist has a 10% chance of moving to the high-risk category the next year and a 90% chance of remaining in the low-risk category. In the long term, what percentage of motorists fall in each category?

42. ▼ ***Debt Analysis*** A credit card company classifies its cardholders as falling into one of two credit ratings: "good" and "poor." Based on its rating criteria, the company finds that a cardholder with a good credit rating has an 80% chance of remaining in that category the following year and a 20% chance of dropping into the poor category. A cardholder with a poor credit rating has a 40% chance of moving into the good rating the following year and a 60% chance of remaining in the poor category. In the long term, what percentage of cardholders fall in each category?

43. ▼ ***Textbook Adoptions*** College instructors who adopt this book are (we hope!) twice as likely to continue to use the book the following semester as they are to drop it, whereas nonusers are nine times as likely to remain nonusers the following year as they are to adopt this book.

a. Determine the probability that a nonuser will be a user in two years.

b. In the long term, what proportion of college instructors will be users of this book?

44. ▼ ***Confidence Level*** Tommy the Dunker's performance on the basketball court is influenced by his state of mind: If he scores, he is twice as likely to score on the next shot as he is to miss, whereas if he misses a shot, he is three times as likely to miss the next shot as he is to score.

a. If Tommy has missed a shot, what is the probability that he will score two shots later?

b. In the long term, what percentage of shots are successful?

45. ▼ ***Debt Analysis*** As the manager of a large retailing outlet, you have classified all credit customers as falling into one of the following categories: Paid Up, Outstanding 0–90 Days, Bad Debts. Based on an audit of your company's records, you have come up with the following table, which gives the probabilities that a single credit customer will move from one category to the next in the period of 1 month.

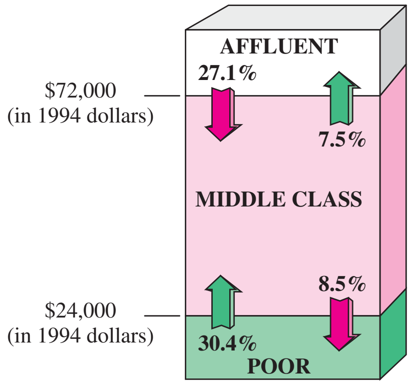
		To		
		Paid Up	**0–90 Days**	**Bad Debts**
From	**Paid Up**	.5	.5	0
	0–90 Days	.5	.3	.2
	Bad Debts	0	.5	.5

How do you expect the company's credit customers to be distributed in the long term?

46. ▼ ***Debt Analysis*** Repeat the preceding exercise using the following table:

		To		
		Paid Up	**0–90 Days**	**Bad Debts**
From	**Paid Up**	.8	.2	0
	0–90 Days	.5	.3	.2
	Bad Debts	0	.5	.5

47. ▼ ***Income Brackets*** The following diagram shows the movement of U.S. households among three income groups—affluent, middle class, and poor—over the 11-year period 1980–1991.[79]

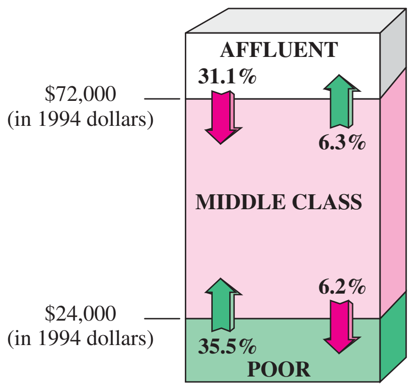


a. Use the transitions shown in the diagram to construct a transition matrix (assuming zero probabilities for the transitions between affluent and poor).

b. Assuming that the trend shown was to continue, what percent of households classified as affluent in 1980 were predicted to become poor in 2002? (Give your answer to the nearest 0.1%.)

c. T According to the model, what percentage of all U.S. households will be in each income bracket in the long term? (Give your answer to the nearest 0.1%.)

48. ▼ ***Income Brackets*** The following diagram shows the movement of U.S. households among three income groups—affluent, middle class, and poor—over the 12-year period 1967–1979.[80]



[79] The figures were based on household after-tax income. The study was conducted by G. J. Duncan of Northwestern University and T. Smeeding of Syracuse University and based on annual surveys of the personal finances of 5,000 households since the late 1960s. (The surveys were conducted by the University of Michigan.) Source: *The New York Times*, June 4, 1995, p. E4.

[80] Ibid.

a. Use the transitions shown in the diagram to construct a transition matrix (assuming zero probabilities for the transitions between affluent and poor).

b. Assuming that the trend shown had continued, what percent of households classified as affluent in 1967 would have been poor in 1991? (Give your answer to the nearest 0.1%.)

c. According to the model, what percentage of all U.S. households will be in each income bracket in the long term? (Give your answer to the nearest 0.1%.)

49. ***Income Distribution*** A University of Michigan study shows the following one-generation transition probabilities among four major income groups.[81]

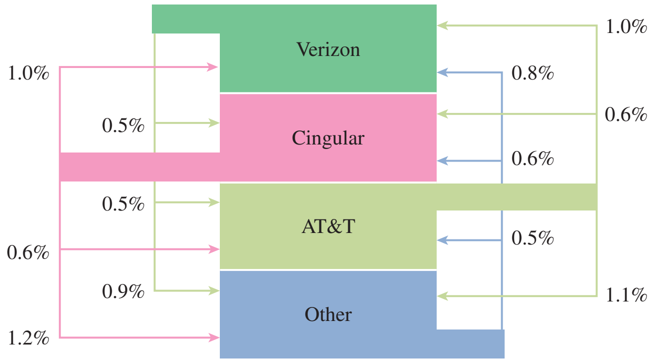
Father's Income	Eldest Son's Income: Bottom 10%	10–50%	50–90%	Top 10%
Bottom 10%	.30	.52	.17	.01
10–50%	.10	.48	.38	.04
50–90%	.04	.38	.48	.10
Top 10%	.01	.17	.52	.30

In the long term, what percentage of male earners would you expect to find in each category? Why are the long-range figures not necessarily 10% in the lowest 10% income bracket, 40% in the 10–50% range, 40% in the 50–90% range, and 10% in the top 10% range?

50. ***Income Distribution*** Repeat Exercise 49, using the following data:

Father's Income	Son's income: Bottom 10%	10–50%	50–90%	Top 10%
Bottom 10%	.50	.32	.17	.01
10–50%	.10	.48	.38	.04
50–90%	.04	.38	.48	.10
Top 10%	.01	.17	.32	.50

Market Share: Cell Phones *Three of the largest cellular phone companies in 2004 were* Verizon*,* Cingular*, and* AT&T Wireless*. Exercises 51 and 52 are based on the following figure, which shows percentages of subscribers who switched from one company to another during the third quarter of 2003:*[82]



51. a. Use the diagram to set up an associated transition matrix.

b. At the end of the third quarter of 2003, the market shares were: Verizon: 29.7%, Cingular: 19.3%, AT&T: 18.1%, and Other: 32.9%. Use your Markov system to estimate the percentage shares at the *beginning* of the third quarter of 2003.

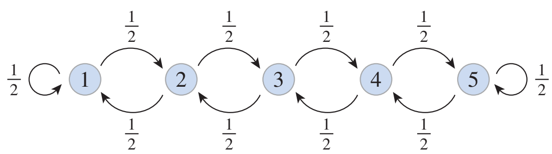
c. Using the information from part (b), estimate the market shares at the end of 2005. Which company is predicted to gain the most in market share?

52. a. Use the diagram to set up an associated transition matrix.

b. At the end of the third quarter of 2003, the market shares were: Verizon: 29.7%, Cingular: 19.3%, AT&T: 18.1%, Other: 32.9%. Use your Markov system to estimate the percentage shares one year earlier.

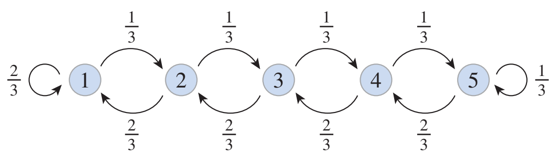
c. Using the information from part (b), estimate the market shares at the end of 2010. Which company is predicted to lose the most in market share?

53. ***Dissipation*** Consider the following five-state model of one-dimensional dissipation without drift:



Find the steady-state distribution vector.

54. ***Dissipation with Drift*** Consider the following five-state model of one-dimensional dissipation with drift:



Find the steady-state distribution vector.

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[81] Source: Gary Solon, University of Michigan/*New York Times*, May 18, 1992, p. D5. We have adjusted some of the figures so that the probabilities add to 1; they did not do so in the original table due to rounding.

[82] Published market shares and "churn rate" (percentage drops for each company) were used to estimate the individual transition percentages. "Other" consists of Sprint, Nextel, and T Mobile. No other cellular companies are included in this analysis. Source: The Yankee Group/*New York Times*, January 21, 2004, p. C1.

COMMUNICATION AND REASONING EXERCISES

55. ▼ Describe an interesting situation that can be modeled by the transition matrix

$$P = \begin{bmatrix} .2 & .8 & 0 \\ 0 & 1 & 0 \\ .4 & .6 & 0 \end{bmatrix}.$$

56. ▼ Describe an interesting situation that can be modeled by the transition matrix

$$P = \begin{bmatrix} .8 & .1 & .1 \\ 1 & 0 & 0 \\ .3 & .3 & .4 \end{bmatrix}.$$

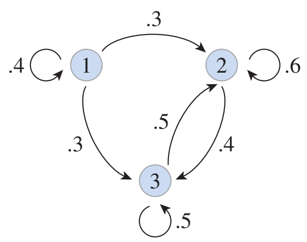
57. ▼ Describe some drawbacks to using Markov processes to model the behavior of the stock market with states (1) bull market (2) bear market.

58. ▼ Can the repeated toss of a fair coin be modeled by a Markov process? If so, describe a model; if not, explain the reason.

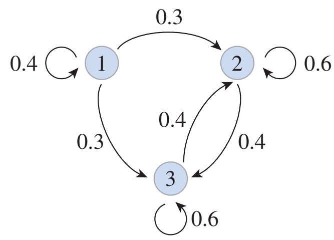
59. ▼ Explain: If Q is a matrix whose rows are steady-state distribution vectors, then $QP = Q$.

60. ▼ Construct a four-state Markov system so that both [.5 .5 0 0] and [0 0 .5 .5] are steady-state vectors. HINT [Try one in which no arrows link the first two states to the last two.]

61. ▼ Refer to the following state transition diagram and explain in words (without doing any calculation) why the steady-state vector has a zero in position 1.



62. ▼ Without doing any calculation, find the steady-state distribution of the following system and explain the reasoning behind your claim.



63. ◆ Construct a regular state transition diagram that possesses the steady-state vector [.3 .3 .4].

64. ◆ Construct a regular state transition diagram possessing a steady-state vector [.6 .3 0 .1].

65. ◆ Show that if a Markov system has two distinct steady-state distributions v and w, then $\dfrac{v+w}{2}$ is another steady-state distribution.

66. ◆ If higher and higher powers of P approach a fixed matrix Q, explain why the rows of Q must be steady-state distributions vectors.

CHAPTER 7 REVIEW

KEY CONCEPTS

Web Site www.FiniteMath.org

Go to the student Web site at www.FiniteMath.org to find a comprehensive and interactive Web-based summary of Chapter 7.

7.1 Sample Spaces and Events

Experiment, outcome, sample space *p. 446*

Event *p. 448*

The complement of an event *p. 452*

Unions of events *p. 452*

Intersections of events *p. 453*

Mutually exclusive events *p. 454*

7.2 Relative Frequency

Relative frequency or estimated probability *p. 460*

Relative frequency distribution *p. 460*

Properties of relative frequency distribution *p. 463*

7.3 Probability and Probability Models

Probability distribution, probability: $0 \leq P(s_i) \leq 1$ and $P(s_1) + \cdots + P(s_n) = 1$ *p. 469*

If $P(E) = 0$, we call E an impossible event *p. 469*

Probability models *p. 469*

Probability models for equally likely outcomes: $P(E) = n(E)/n(S)$ *p. 471*

Addition principle: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ *p. 475*

If A and B are mutually exclusive, then $P(A \cup B) = P(A) + P(B)$ *p. 476*

If S is the sample space, then $P(S) = 1$, $P(\emptyset) = 0$, and $P(A') = 1 - P(A)$ *p. 478*

7.4 Probability and Counting Techniques

Use counting techniques from Chapter 6 to calculate probability *p. 487*

7.5 Conditional Probability and Independence

Conditional probability: $P(A \mid B) = P(A \cap B)/P(B)$ *p. 497*

Multiplication principle for conditional probability: $P(A \cap B) = P(A \mid B)P(B)$ *p. 500*

Independent events: $P(A \cap B) = P(A)P(B)$ *p. 503*

7.6 Bayes' Theorem and Applications

Bayes' Theorem (short form):

$$P(A \mid T) = \frac{P(T \mid A)P(A)}{P(T \mid A)P(A) + P(T \mid A')P(A')} \quad p.\ 515$$

Bayes' Theorem (partition of sample space into three events):

$$P(A_1 \mid T) = \frac{P(T \mid A_1)P(A_1)}{P(T \mid A_1)P(A_1) + P(T \mid A_2)P(A_2) + P(T \mid A_3)P(A_3)}$$

p. 517

7.7 Markov Systems

Markov system, Markov process, states, transition probabilities *p. 522*

Transition matrix associated with a given Markov system *p. 522*

A vector having non-negative entries adding up to 1 is called a probability vector *p. 524*

A distribution vector is a probability vector giving the probability distribution for finding a Markov system in its various possible states *p. 524*

If v is a distribution vector, then the distribution vector one step later will be vP. The distribution after m steps will be vP^m *p. 524*

P^m is the m-step transition matrix. The *ij*th entry in P^m is the probability of a transition from state i to state j in m steps *p. 526*

A steady-state (probability) vector is a probability vector v such that $vP = v$ *p. 527*

Calculation of the steady-state distribution vector *p. 528*

A regular Markov system, long-term behavior of a regular Markov system *p. 527*

An absorbing state is one for which the probability of staying in the state is 1 (and the probability of leaving it for any other state is 0) *p. 530*

REVIEW EXERCISES

In each of Exercises 1–6, say how many elements are in the sample space S, list the elements of the given event E, and compute the probability of E.

1. Three coins are tossed; the result is one or more tails.

2. Four coins are tossed; the result is fewer heads then tails.

3. Two distinguishable dice are rolled; the numbers facing up add to 7.

4. Three distinguishable dice are rolled; the number facing up add to 5.

5. A die is weighted so that each of 2, 3, 4, and 5 is half as likely to come up as either 1 or 6; however, 2 comes up.

6. Two indistinguishable dice are rolled; the numbers facing up add to 7.

In each of Exercises 7–10, calculate the relative frequency $P(E)$.

7. Two coins are tossed 50 times, and two heads come up 12 times. E is the event that at least one tail comes up.

8. Ten stocks are selected at random from a portfolio. Seven of them have increased in value since their purchase, and the rest have decreased. Eight of them are Internet stocks and two of those have decreased in value. E is the event that a stock has either increased in value or is an Internet stock.

9. You have read 150 of the 400 novels in your home, but your sister Roslyn has read 200, of which only 50 are novels you have read as well. E is the event that a novel has been read by neither you nor your sister.

10. You roll two dice 10 times. Both dice show the same number 3 times, and on 2 rolls, exactly one number is odd. E is the event that the sum of the numbers is even.

In each of Exercises 11–14, calculate the probability $P(E)$.

11. There are 32 students in categories A and B combined. Some are in both, 24 are in A, and 24 are in B. E is the event that a randomly selected student (among the 32) is in both categories.

12. You roll two dice, one red and one green. Losing combinations are doubles (both dice showing the same number) and outcomes in which the green die shows an odd number and the red die shows an even number. The other combinations are winning ones. E is the event that you roll a winning combination.

13. The *jPlay* portable music/photo/video player and bottle opener comes in three models: A, B, and C, each with five colors to choose from, and there are equal numbers of each combination. E is the event that a randomly selected *jPlay* is either orange (one of the available colors), a Model A, or both.

14. The Heavy Weather Service predicts that for tomorrow there is a 50% chance of tornadoes, a 20% chance of a monsoon, and a 10% chance of both. What is the probability that we will be lucky tomorrow and encounter neither tornadoes nor a monsoon?

A bag contains four red marbles, two green ones, one transparent one, three yellow ones, and two orange ones. You select five at random. In each of Exercises 15–20, compute the probability of the given event.

15. You have selected all the red ones.

16. You have selected all the green ones.

17. All are different colors.

18. At least one is not red.

19. At least two are yellow.

20. None are yellow and at most one is red.

In each of Exercises 21–26, find the probability of being dealt the given type of 5-card hand from a standard deck of 52 cards. (None of these is a recognized poker hand,) Express your answer in terms of combinations.

21. Kings and Queens: Each of the five cards is either a king or a queen.

22. Five Pictures: Each card is a picture card (J, Q, K).

23. Fives and Queens: Three fives, the queen of spades, and one other queen.

24. Prime Full House: A full house (three cards of one denomination, two of another) with the face value of each card a prime number (Ace = 1, J = 11, Q = 12, K = 13).

25. Full House of Commons: A full house (three cards of one denomination, two of another) with no royal cards (that is, no J, Q, K, or Ace).

26. Black Two Pair: Five black cards (spades or clubs), two with one denomination, two with another, and one with a third.

Two dice, one green and one yellow, are rolled. In each of Exercises 27–32, find the conditional probability, and also say whether the indicated pair of events is independent.

27. The sum is 5, given that the green one is not 1 and the yellow one is 1.

28. That the sum is 6, given that the green one is either 1 or 3 and the yellow one is 1.

29. The yellow one is 4, given that the green one is 4.

30. The yellow one is 5, given that the sum is 6.

31. The dice have the same parity, given that both of them are odd.

32. The sum is 7, given that the dice do not have the same parity.

A poll shows that half the consumers who use Brand A switched to Brand B the following year, while the other half stayed with Brand A. Three quarters of the Brand B users stayed with Brand B the following year, while the rest switched to Brand A. Use this information to answer Exercises 33–36.

33. Give the associated Markov state distribution matrix with state 1 representing using Brand A, and state 2 represented by using Brand B.

34. Compute the associated two- and three-step transition matrices. What is the probability that a Brand A user will be using Brand B three years later?

35. If two-thirds of consumers are presently using Brand A and one-third are using Brand B, how are these consumers distributed in three years' time?

36. In the long term, what fraction of the time will a user spend using each of the two brands?

APPLICATIONS

OHaganBooks.com currently operates three warehouses: one in Washington, one in California, and the new one in Texas. Book inventories are shown in the following table.

	Sci-Fi	Horror	Romance	Other	Total
Washington	10,000	12,000	12,000	30,000	64,000
California	8,000	12,000	6,000	16,000	42,000
Texas	15,000	15,000	20,000	44,000	94,000
Total	33,000	39,000	38,000	90,000	200,000

A book is selected at random. In each of Exercises 37–42, compute the probability of the given event.

37. That it is either a sci-fi book or stored in Texas (or both).

38. That it is a sci-fi book stored in Texas.

39. That it is a sci-fi book, given that it is stored in Texas.

40. That it was stored in Texas, given that it was a sci-fi book.

41. That it was stored in Texas, given that it was not a sci-fi book.

42. That it was not stored in Texas, given that it was a sci-fi book.

In order to gauge the effectiveness of the OHaganBooks.com site, you recently commissioned a survey of online shoppers. According to the results, 2% of online shoppers visited OHaganBooks.com during a one-week period, while 5% of them visited at least one of OHaganBooks.com's two main competitors: JungleBooks.com and FarmerBooks.com. Use this information to answer Exercises 43–50.

43. What percentage of online shoppers never visited OHaganBooks.com?

44. What percentage of shoppers never visited either of OHaganBooks.com's main competitors?

45. Assuming that visiting OHaganBooks.com was independent of visiting a competitor, what percentage of online shoppers visited either OHaganBooks.com or a competitor?

46. Assuming that visiting OHaganBooks.com was independent of visiting a competitor, what percentage of online shoppers visited OHaganBooks.com but not a competitor?

47. Under the assumption of Exercise 45, what is the estimated probability that an online shopper will visit none of the three sites during a week?

48. If no one who visited OHaganBooks.com ever visited any of the competitors, what is the estimated probability that an online shopper will visit none of the three sites during a week?

49. Actually, the assumption in Exercise 45 is not what was found by the survey, because an online shopper visiting a competitor was in fact more likely to visit OHaganBooks.com than a randomly selected online shopper. Let H be the event that an online shopper visits OHaganBooks.com, and let C be the event that he visits a competitor. Which is greater: $P(H \cap C)$ or $P(H)P(C)$? Why?

50. What the survey found is that 25% of online shoppers who visited a competitor also visited OHaganBooks.com. Given this information, what percentage of online shoppers visited OHaganBooks.com and neither of its competitors?

Not all visitors to the OHaganBooks.com site actually purchase books, and not all OHaganBooks.com customers buy through the Web site (some phone in their orders and others use a mail-order catalog). According to statistics gathered at the Web site, 8% of online shoppers who visit the site during the course of a single week will purchase books. However, the survey mentioned prior to Exercise 43 revealed that 2% of online shoppers visited the site during the course of a week. Another survey estimated that 0.5% of online shoppers who did not visit the site during the course of a week nonetheless purchased books at OHaganBooks.com. Use this information to answer Exercises 51–54.

51. Complete the following sentence: Online shoppers who visit the OHaganBooks.com Web site are ___ times as likely to purchase books as shoppers who do not.

52. What is the probability that an online shopper will not visit the site during the course of a week but still purchase books?

53. What is the probability that an online shopper who purchased books during a given week also visited the site?

54. Repeat exercise 53 in the event that 1% of online shoppers visited OHaganBooks.com during the week.

As mentioned earlier, OHaganBooks.com has two main competitors, JungleBooks.com and FarmerBooks.com, and no other competitors of any significance. The following table shows the movement of customers during July.[83] *(Thus, for instance, the first row tells us that 80% of OHaganBooks.com's customers remained loyal, 10% of them went to JungleBooks.com and the remaining 10% to FarmerBooks.com.)*

			To	
		OHaganBooks	**JungleBooks**	**FarmerBooks**
From	**OHaganBooks**	80%	10%	10%
	JungleBooks	40%	60%	0%
	FarmerBooks	20%	0%	80%

At the beginning of July, OHaganBooks.com had an estimated market share of one-fifth of all customers, while its two competitors had two-fifths each. Use this information to answer Exercises 55–58.

55. Estimate the market shares each company had at the end of July.

56. Assuming the July trends continue in August, predict the market shares of each company at the end of August.

57. Name one or more important factors that the Markov model does not take into account.

58. Assuming the July trend were to continue indefinitely, predict the market share enjoyed by each of the three e-commerce sites.

[83] By a "customer" of one of the three e-commerce sites, we mean someone who purchases more at that site than at any of the two competitors' sites.

Case Study The Monty Hall Problem

Everett Collection

Here is a famous "paradox" that even mathematicians find counterintuitive. On the game show *Let's Make a Deal*, you are shown three doors, A, B, and C, and behind one of them is the Big Prize. After you select one of them—say, door A—to make things more interesting the host (Monty Hall), who knows what is behind each door, opens one of the other doors—say, door B—to reveal that the Big Prize is not there. He then offers you the opportunity to change your selection to the remaining door, door C. Should you switch or stick with your original guess? Does it make any difference?

Most people would say that the Big Prize is equally likely to be behind door A or door C, so there is no reason to switch.[84] In fact, this is wrong: The prize is more likely to be behind door C! There are several ways of seeing why this is so. Here is how you might work it out using Bayes' theorem.

Let A be the event that the Big Prize is behind door A, B the event that it is behind door B, and C the event that it is behind door C. Let F be the event that Monty has opened door B and revealed that the prize is not there. You wish to find $P(C \mid F)$ using Bayes' theorem. To use that formula you need to find $P(F \mid A)$ and $P(A)$ and similarly for B and C. Now, $P(A) = P(B) = P(C) = 1/3$ because at the outset, the prize is equally likely to be behind any of the doors. $P(F \mid A)$ is the probability that Monty will open door B if the prize is actually behind door A, and this is $1/2$ because we assume that he will choose either B or C randomly in this case. On the other hand, $P(F \mid B) = 0$, because he will never open the door that hides the prize. Also, $P(F \mid C) = 1$ because if the prize is behind door C, he must open door B to keep from revealing that the prize is behind door C. Therefore,

$$\begin{aligned} P(C \mid F) &= \frac{P(F \mid C)P(C)}{P(F \mid A)P(A) + P(F \mid B)P(B) + P(F \mid C)P(C)} \\ &= \frac{1 \cdot \frac{1}{3}}{\frac{1}{2} \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3}} = \frac{2}{3}. \end{aligned}$$

You conclude from this that you *should* switch to door C because it is more likely than door A to be hiding the Prize.

Here is a more elementary way you might work it out. Consider the tree diagram of possibilities shown in Figure 22. The top two branches of the tree give the cases in which the prize is behind door A, and there is a total probability of $1/3$ for that case. The remaining two branches with nonzero probabilities give the cases in which the prize is behind the door that you did not choose, and there is a total probability of $2/3$ for that case. Again, you conclude that you should switch your choice of doors because the one you did not choose is twice as likely as door A to be hiding the Big Prize.

[84] This problem caused quite a stir in late 1991 when this problem was discussed in Marilyn vos Savant's column in *Parade* magazine. Vos Savant gave the answer that you should switch. She received about 10,000 letters in response, most of them disagreeing with her, including several from mathematicians.

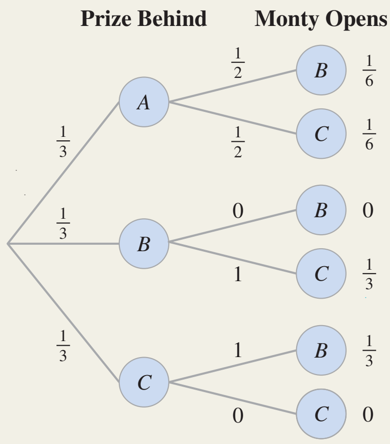


Figure 22

EXERCISES

1. The answer you came up with, to switch to the other door, depends on the strategy Monty Hall uses in picking the door to open. Suppose that he actually picks one of doors B and C at random, so that there is a chance that he will reveal the Big Prize. If he opens door B and it happens that the Prize is not there, should you switch or not?
2. What if you know that Monty's strategy is always to open door B if possible (i.e., it does not hide the Big Prize) after you choose A?

   a. If he opens door B, should you switch?
   b. If he opens door C, should you switch?
3. Repeat the analysis of the original game, but suppose that the game uses four doors instead of three (and still only one prize).
4. Repeat the analysis of the original game, but suppose that the game uses 1,000 doors instead of 3 (and still only one prize).

TI-83/84 Plus Technology Guide

Section 7.2

The TI-83/84 Plus has a random number generator that we can use to simulate experiments. For the following example, recall that a fair coin has probability $1/2$ of coming up heads and $1/2$ of coming up tails.

Example Use a simulated experiment to check the following.

a. The estimated probability of heads coming up in a toss of a fair coin approaches $1/2$ as the number of trials gets large.

b. The estimated probability of heads coming up in two consecutive tosses of a fair coin approaches $1/4$ as the number of trials gets large.[85]

Solution with Technology

a. Let us use 1 to represent heads and 0 to represent tails. We need to generate a list of **random binary digits** (0 or 1). One way to do this—and a method that works for most forms of technology—is to generate a random number between 0 and 1 and then round it to the nearest whole number, which will be either 0 or 1.

We generate random numbers on the TI-83/84 Plus using the "`rand`" function. To round the number X to the nearest whole number on the TI-83/84 Plus, follow MATH → NUM, select "`round`," and enter `round(X,0)`. This instruction rounds X to zero decimal places—that is, to the nearest whole number. Since we wish to round a random number we need to enter

`round(rand,0)` To obtain `rand`, follow MATH → PRB.

The result will be either 0 or 1. Each time you press ENTER you will now get another 0 or 1. The TI-83/84 Plus can also generate a random integer directly (without the need for rounding) through the instruction

`randInt(0,1)` To obtain `randInt`, follow MATH → PRB.

In general, the command `randInt(`m, n`)` generates a random integer in the range $[m, n]$. The following sequence of 100 random binary digits was produced using technology.[86]

0	1	0	0	1	1	0	1	0	0
0	1	0	0	0	0	0	0	1	0
1	1	0	0	0	1	0	0	1	1
1	1	1	0	1	0	0	0	1	0
1	1	1	1	1	1	1	0	0	1
1	0	1	1	1	0	0	1	1	0
0	1	0	1	1	1	0	1	1	1
1	0	0	0	0	0	0	1	1	1
1	1	1	1	0	0	1	1	1	0
1	1	1	0	1	1	0	1	0	0

If we use only the first row of data (corresponding to the first ten tosses), we find

$$P(\mathrm{H}) = \frac{fr(1)}{N} = \frac{4}{10} = .4.$$

Using the first two rows ($N = 20$) gives

$$P(\mathrm{H}) = \frac{fr(1)}{N} = \frac{6}{20} = .3.$$

Using all ten rows ($N = 100$) gives

$$P(\mathrm{H}) = \frac{fr(1)}{N} = \frac{54}{100} = .54.$$

This is somewhat closer to the theoretical probability of $1/2$ and supports our intuitive notion that the larger the number of trials, the more closely the estimated probability should approximate the theoretical value.[87]

b. We need to generate pairs of random binary digits and then check whether they are both 1s. Although the TI-83/84 Plus will generate a pair of random digits if you enter `round(rand(2),0)`, it would be a lot more convenient if the calculator could tell you right away whether both digits are 1s (corresponding to two consecutive heads in a coin toss). Here is a simple way of accomplishing this. Notice that if we *add* the two random binary digits, we obtain either 0, 1, or 2, telling us the number of heads that result from the two consecutive throws. Therefore, all we need to do is add the pairs of random digits and then count the number of times 2 comes up. A formula we can use is

```
randInt(0,1)+randInt(0,1)
```

What would be even *more* convenient is if the result of the calculation would be either 0 or 1, with 1 signifying

[85] Since the set of outcomes of a pair of coin tosses is {HH, HT, TH, TT}, we expect HH to come up once in every four trials, on average.

[86] The instruction `randInt(0,1,100)`→L_1 will generate a list of 100 random 0s and 1s and store it in L_1, where it can be summed with Sum(L_1) (under 2ND LIST → MATH).

[87] Do not expect this to happen every time. Compare, for example, $P(\mathrm{H})$ for the first five rows and for all ten rows.

TECHNOLOGY GUIDE

success (two consecutive heads) and 0 signifying failure. Then, we could simply add up all the results to obtain the number of times two heads occurred. To do this, we first divide the result of the previous calculation above by 2 (obtaining 0, .5, or 1, where now 1 signifies success) and then round *down* to an integer using a function called "int":

```
int(0.5*(randInt(0,1)+randInt(0,1)))
```

Following is the result of 100 such pairs of coin tosses, with 1 signifying success (two heads) and 0 signifying failure (all other outcomes). The last column records the number of successes in each row and the total number at the end.

1	1	0	0	0	0	0	0	0	0	2
0	1	0	0	0	0	0	1	0	1	3
0	1	0	0	1	1	0	0	0	1	4
0	0	0	0	0	0	0	0	1	0	1
0	1	0	0	1	0	0	1	0	0	3
1	0	1	0	0	0	0	0	0	0	2
0	0	0	0	0	0	0	0	0	1	1
0	1	1	1	1	0	0	0	0	1	5
1	1	0	1	0	0	1	1	0	0	5
0	0	0	0	0	0	0	0	1	0	1
										27

Now, as in part (a), we can compute estimated probabilities, with D standing for the outcome "two heads":

First 10 trials: $$P(D) = \frac{fr(1)}{N} = \frac{2}{10} = .2$$

First 20 trials: $$P(D) = \frac{fr(1)}{N} = \frac{5}{20} = .25$$

First 50 trials: $$P(D) = \frac{fr(1)}{N} = \frac{13}{50} = .26$$

100 trials: $$P(D) = \frac{fr(1)}{N} = \frac{27}{100} = .27.$$

Q: *What is happening with the data? The probabilities seem to be getting* less *accurate as N increases!*

A: Quite by chance, exactly 5 of the first 20 trials resulted in success, which matches the theoretical probability. Figure 23 shows an Excel plot of estimated probability versus N (for N a multiple of 10). Notice that, as N increases, the graph seems to meander within smaller distances of .25.

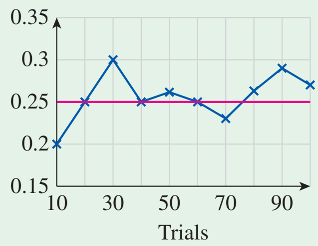


Figure 23

Q: *The previous techniques work fine for simulating coin tosses. What about rolls of a fair die, where we want outcomes between 1 and 6?*

A: We can simulate a roll of a die by generating a random integer in the range 1 through 6. The following formula accomplishes this:

```
1 + int(5.99999*rand).
```

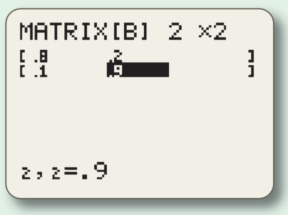
(We used 5.99999 instead of 6 to avoid the outcome 7.)

Section 7.7

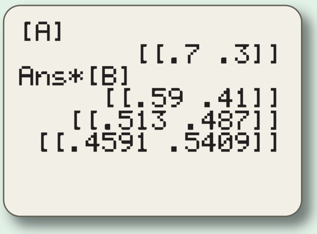
Example 1 (page 523) Consider the Markov system found by Gamble Detergents at the beginning of this section. Suppose that 70% of consumers are now using powdered detergent while the other 30% are using liquid. What will be the distribution one year from now? Two years from now? Three years from now?

Solution with Technology

In Chapter 3 we saw how to set up and multiply matrices. For this example, we can use the matrix editor to define `[A]` as the initial distribution and `[B]` as the transition matrix (remember that the only names we can use are `[A]` through `[J]`).



Entering `[A]` (obtained by pressing MATRX 1 ENTER) will show you the initial distribution. To obtain the distribution after 1 step, press X MATRX 2 ENTER, which has the effect of multiplying the previous answer by the transition matrix `[B]`. Now, just press ENTER repeatedly to continue multiplying by the transition matrix and obtain the distribution after any number of steps. The screenshot shows the initial distribution `[A]` and the distributions after 1, 2, and 3 steps.



Example 4 (page 527) Calculate the steady-state probability vector for the transition matrix in the preceding examples.

Solution with Technology

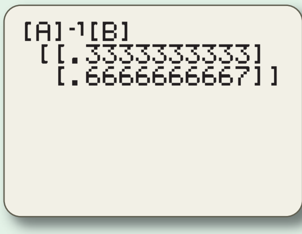
Finding the steady-state probability vector comes down to solving a system of equations. As discussed in Chapters 2 and 3, there are several ways to use a calculator to help. The most straightforward is to use matrix inversion to solve the matrix form of the system. In this case, as in the text, the system of equations we need to solve is

$$\begin{aligned} x + y &= 1 \\ -.2x + .1y &= 0. \end{aligned}$$

We write this as the matrix equation $AX = B$ with

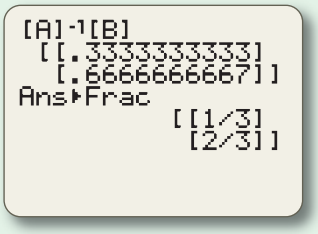
$$A = \begin{bmatrix} 1 & 1 \\ -.2 & .1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

To find $X = A^{-1}B$ using the TI-83/84 Plus, we first use the matrix editor to enter these matrices as `[A]` and `[B]`, then compute `[A]`$^{-1}$`[B]` on the home screen.



To convert the entries to fractions, we can follow this by the command

➤ Frac   MATH ENTER ENTER



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EXCEL Technology Guide

Section 7.2

Excel has a random number generator that we can use to simulate experiments. For the following example, recall that a fair coin has probability 1/2 of coming up heads and 1/2 of coming up tails.

Example Use a simulated experiment to check the following.

a. The estimated probability of heads coming up in a toss of a fair coin approaches 1/2 as the number of trials gets large.

b. The estimated probability of heads coming up in two consecutive tosses of a fair coin approaches 1/4 as the number of trials gets large.[88]

Solution with Technology

a. Let us use 1 to represent heads and 0 to represent tails. We need to generate a list of **random binary digits** (0 or 1). One way to do this—and a method that works for most forms of technology—is to generate a random number between 0 and 1 and then round it to the nearest whole number, which will be either 0 or 1.

In Excel, the formula `RAND()` gives a random number between 0 and 1.[89] The function `ROUND(X,0)` rounds `X` to zero decimal places—that is, to the nearest integer. Therefore, to obtain a random binary digit in any cell just enter the following formula:

```
=ROUND(RAND(),0)
```

[88] Because the set of outcomes of a pair of coin tosses is {HH, HT, TH, TT}, we expect HH to come up once in every four trials, on average.

[89] The parentheses after RAND are necessary even though the function takes no arguments.

Excel can also generate a random integer directly (without the need for rounding) through the formula

```
=RANDBETWEEN(0,1)
```

To obtain a whole array of random numbers, just drag this formula into the cells you wish to use.

b. We need to generate pairs of random binary digits and then check whether they are both 1s. It would be convenient if the spreadsheet could tell you right away whether both digits are 1s (corresponding to two consecutive heads in a coin toss). Here is a simple way of accomplishing this. Notice that if we *add* two random binary digits, we obtain either 0, 1, or 2, telling us the number of heads that result from the two consecutive throws. Therefore, all we need to do is add pairs of random digits and then count the number of times 2 comes up. Formulas we can use are

```
=RANDBETWEEN(0,1)+RANDBETWEEN(0,1)
```

What would be even *more* convenient is if the result of the calculation would be either 0 or 1, with 1 signifying success (two consecutive heads) and 0 signifying failure. Then, we could simply add up all the results to obtain the number of times two heads occurred. To do this, we first divide the result of the calculation above by 2 (obtaining 0, .5, or 1, where now 1 signifies success) and then round *down* to an integer using a function called "int":

```
=INT(0.5*(RANDBETWEEN(0,1)+
 RANDBETWEEN(0,1)))
```

Following is the result of 100 such pairs of coin tosses, with 1 signifying success (two heads) and 0 signifying failure (all other outcomes). The last column records the number of successes in each row and the total number at the end.

1	1	0	0	0	0	0	0	0	0	2
0	1	0	0	0	0	0	1	0	1	3
0	1	0	0	1	1	0	0	0	1	4
0	0	0	0	0	0	0	0	1	0	1
0	1	0	0	1	0	0	1	0	0	3
1	0	1	0	0	0	0	0	0	0	2
0	0	0	0	0	0	0	0	0	1	1
0	1	1	1	1	0	0	0	0	1	5
1	1	0	1	0	0	1	1	0	0	5
0	0	0	0	0	0	0	0	1	0	1
										27

Now, as in part (a), we can compute estimated probabilities, with D standing for the outcome "two heads":

First 10 trials: $P(D) = \dfrac{fr(1)}{N} = \dfrac{2}{10} = .2$

First 20 trials: $P(D) = \dfrac{fr(1)}{N} = \dfrac{5}{20} = .25$

First 50 trials: $P(D) = \dfrac{fr(1)}{N} = \dfrac{13}{50} = .26$

100 trials: $P(D) = \dfrac{fr(1)}{N} = \dfrac{27}{100} = .27.$

Q: *What is happening with the data? The probabilities seem to be getting* less accurate *as N increases!*

A: Quite by chance, exactly 5 of the first 20 trials resulted in success, which matches the theoretical probability. Figure 24 shows an Excel plot of estimated probability versus N (for N a multiple of 10). Notice that, as N increases, the graph seems to meander within smaller distances of .25.

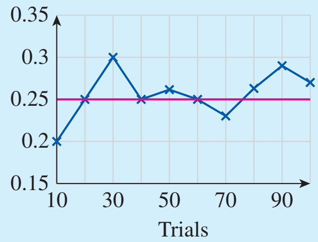

Figure 24

Q: *The previous techniques work fine for simulating coin tosses. What about rolls of a fair die, where we want outcomes between 1 and 6?*

A: We can simulate a roll of a die by generating a random integer in the range 1 through 6. The following formula accomplishes this:

```
=1 + INT(5.99999*RAND())
```

(We used 5.99999 instead of 6 to avoid the outcome 7.)

Section 7.7

Example 1 (page 523) Consider the Markov system found by Gamble Detergents at the beginning of this section. Suppose that 70% of consumers are now using powdered detergent while the other 30% are using liquid. What will be the distribution one year from now? Two years from now? Three years from now?

Solution with Technology

In Excel, enter the initial distribution vector in cells A1 and B1 and the transition matrix to the right of that, as shown.

	A	B	C	D	E
1	0.7	0.3		0.8	0.2
2				0.1	0.9

To calculate the distribution after one step, use the array formula

```
=MMULT(A1:B1,$D$1:$E$2)
```

The absolute cell references (dollar signs) ensure that the formula always refers to the same transition matrix, even if we copy it into other cells. To use the array formula, select cells A2 and B2, where the distribution vector will go, enter this formula, and then press Control+Shift+Enter.[90]

	A	B	C	D	E
1	0.7	0.3		0.8	0.2
2	=MMULT(A1:B1,$D$1:$E$2)			0.1	0.9

The result is the following, with the distribution after one step highlighted.

	A	B	C	D	E
1	0.7	0.3		0.8	0.2
2	0.59	0.41		0.1	0.9

To calculate the distribution after two steps, select cells A2 and B2 and drag the fill handle down to copy the formula to cells A3 and B3. Note that the formula now takes the vector in A2:B2 and multiplies it by the transition matrix to get the vector in A3:B3. To calculate several more steps, drag down as far as desired.

	A	B	C	D	E
1	0.7	0.3		0.8	0.2
2	0.59	0.41		0.1	0.9
3					
4					

	A	B	C	D	E
1	0.7	0.3		0.8	0.2
2	0.59	0.41		0.1	0.9
3	0.513	0.487			
4	0.4591	0.5409			

[90] On a Macintosh, you can also use Command+Enter.

Example 4 (page 527) Calculate the steady-state probability vector for the transition matrix in the preceding examples.

Solution with Technology

Finding the steady-state probability vector comes down to solving a system of equations. As discussed in Chapters 2 and 3, there are several ways to use Excel to help. The most straightforward is to use matrix inversion to solve the matrix form of the system. In this case, as in the text, the system of equations we need to solve is

$$\begin{aligned} x + \quad y &= 1 \\ -.2x + .1y &= 0. \end{aligned}$$

We write this as the matrix equation $AX = B$ with

$$A = \begin{bmatrix} 1 & 1 \\ -.2 & .1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We enter A in cells A1:B2, B in cells D1:D2, and the formula for $X = A^{-1}B$ in a convenient location, say B4:B5.

	A	B	C	D
1	1	1		1
2	-0.2	0.1		0
3				
4		=MMULT(MINVERSE(A1:B2),D1:D2)		
5				

When we press Control+Shift+Enter we see the result:

	A	B	C	D
1	1	1		1
2	-0.2	0.1		0
3				
4		0.3333333		
5		0.6666667		

If we want to see the answer in fraction rather than decimal form, we format the cells as fractions.

	A	B	C	D
1	1	1		1
2	-0.2	0.1		0
3				
4		1/3		
5		2/3		