1° Classical problems

Consider a system of two differential equations with one unknown function

$$\begin{cases} p_1 u = f_1, \\ p_2 u = f_2 \end{cases}$$

in $\mathbb{R}^n$.

If the system has a solution u and p_1, p_2 commute, then

$$p_2 f_1 = p_1 f_2.$$

Hence, the latter equation is necessary for the local solvability of the genuine system (but not sufficient, for take $p_1 = p_2$!)

2° Compatibility operators

Let A be a differential operator of type $E \to F$ and order m on a C^∞ manifold X.

This operator is called overdetermined if there is a differential operator B with $BA = 0$ (and $B \neq 0$).

A differential operator A^1 is called a compatibility operator for A if $A^1 A = 0$ and from $BA = 0$ it follows that B factors through A^1, i.e., $B = CA^1$.

The "formal" theory says that any A has a compatibility operator

3° C^∞ Poincaré lemma

Pick a compatibility operator A^1 for A on X.

One says that the C^∞ Poincare lemma holds for A if for any open set U on X and each $f \in C^\infty(U, F)$ satisfying $A^1 f = 0$ in U there is an open set $V \subset U$ and $u \in C^\infty(V, E)$, such that $Au = f$ in V.

If A is not elliptic there is a counterexample of Hans Levi (1953).

In the elliptic case the problem is open if $n \neq 2$.

4° Reduction to selfadjoint operators

Consider a bounded operator $T: H_1 \to H_2$ in Hilbert spaces.

The equation $Tu = f$ has a solution only if $f \perp \mathrm{Nul}\, T^*$.

L. Let $f \perp \mathrm{Nul}\, T^*$. Then the equation $Tu = f$ has a solution if and only if so does $T^*Tu = T^*f$.

For the unbounded operator $T: D_T \to L^2(\mathcal{D}, F)$ on $L^2(\mathcal{D}, E)$ induced by a differential operator A the condition $f \perp \mathrm{Nul}\, T^*$ reduces to $f \perp \mathcal{H}^1(\mathcal{D}) := \{g \in D_{A^1} \cap D_{A^*} : A^1 g = A^* g = 0\}$.

5° Iterations

Suppose M is a selfadjoint operator on a Hilbert space H.

Th. If $\|M\| \leq 1$ then the limit $\lim\limits_{N\to\infty} M^N$ exists in the strong topology of $\mathcal{L}(H)$, and

$$1 = \pi_{\mathrm{Nul}(1-M)} + \sum_{n=0}^{\infty} M^n(1-M).$$

Proof Since

$$M = \int_{0-}^{1} \lambda \, dE_\lambda$$

it follows that

$$\lim_{N\to\infty} M^N = \lim_{N\to\infty} \int_{0-}^{1} \lambda^N \, dE_\lambda$$

$$= E_1 .$$

□

6° Green operators

Let $\mathcal{D}$ be a relatively compact domain with C^∞ boundary on X.

Using the Green function G for $A^* A$ on X yields an integral representation

$$u = Mu + T_{\mathcal{D}} A u$$

for all $u \in H^m(\mathcal{D}, E)$, where $T_{\mathcal{D}} = G A^* \chi_{\mathcal{D}}$.

Th. The Hermitian form $h_{\mathcal{D}}(u,v) := (Ae(u), Ae(v))_X$ is a scalar product on $H^m(\mathcal{D},E)$ inducing the same topology, and $h_{\mathcal{D}}(T_{\mathcal{D}} f, v) = (f, Av)_{L^2(\mathcal{D},F)}$

Main result: For each $f \in L^2(\mathcal{D},F)$, the series $Rf := \sum_{n=0}^{\infty} (1 - T_{\mathcal{D}} A)^n T_{\mathcal{D}} f$ converges in $H^m_{loc}(\mathcal{D},E)$.

7° Hodge decomposition

Th. For any $f \in L^2(\mathcal{D}, F)$ satisfying $A^1 f = 0$,

$$f = \pi_{\mathcal{H}^1(\mathcal{D})} f + A R f.$$

Denote by D_A the set of $u \in H^m_{loc}(\mathcal{D}, E)$ such that

1) $Au \in L^2(\mathcal{D}, F)$;
2) there is $\{u_j\} \subset H^m(\mathcal{D}, E)$ with $u_j \to u$ in $H^m_{loc}(\mathcal{D}, E)$ and $Au_j \to Au$ in $L^2(\mathcal{D}, F)$.

Col. Let $f \in L^2(\mathcal{D}, F)$. Then there is a $u \in D_A$ with $Au = f$ iff $A^1 f = 0$ and $f \perp \mathcal{H}^1(\mathcal{D})$.

8° Neumann problem

Write A for the maximal operator $L^2(\mathcal{D},E)\to L^2(\mathcal{D},F)$ induced by A.

The operator $\Delta = A A^* + A^{1*} A^1$ on $L^2(\mathcal{D},F)$ with domain

$$D_\Delta := \{u\in D_{A^1}\cap D_{A^*} : A^*u\in D_A,\ A^1u\in D_{A^{1*}}\}$$

is called the generalised Laplacian.

It is easy to verify that

$$\mathcal{H}^1(\mathcal{D}) = \mathrm{Nul}\ \Delta.$$

L. (weak orthogonal decomposition)

$$L^2(\mathcal{D},F) = \mathcal{H}^1(\mathcal{D}) \oplus \overline{\Delta D_\Delta}.$$

If $\mathcal{H}^1(\mathcal{D})=0$ for small $\mathcal{D}$ then the C^∞ Poincaré lemma holds