MIC(0) Preconditioning of Rotated Bilinear FEM Elliptic Systems

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Introduction

$$a_h(u,v) = \sum_{e \in \Omega_h} \int_e a(e) \sum_{i=1}^{2} u_{x_i} v_{x_i} de$$

$\Omega_h = \omega_1 \times \omega_2,\ \Omega \subset \mathbf{R}^2$
$\psi_e : \widehat{e} \to e,\ \widehat{e} = [-1,1]^2$

$$\{\phi_i\}_{i=1}^4 = \{\widehat{\phi}_i \circ \psi_e^{-1}\}_{i=1}^4,$$

$$\widehat{\phi}_i \in span\{1, \xi_j, \xi_j^2 - \xi_{j+1}^2, \quad j = 1,2\}$$

$$\widehat{\phi}_i(b_\Gamma^j) = \delta_{ij},$$

$$\{\widehat{\phi}_i\}_{i=1}^4 = \left\{ \left(1 \pm 2\xi_j + \xi_j^2 - \xi_{j+1}^2 \right) /4, \quad j = 1,2 \right\}.$$

MIC(0) preconditioning

Theorem 1 *Let $A = (a_{ij})$ be a symmetric real $N \times N$ matrix and let $A = D - L - L^t$ be the splitting of A. Let us assume that*

$$\begin{array}{rcl} L & \geq & 0 \\ A\underline{e} & \geq & 0 \\ A\underline{e} + L^t\underline{e} & > & 0 \qquad \underline{e} = (1, \cdots, 1)^t \in \mathcal{R}^{\mathcal{N}}, \end{array}$$

i.e. that A is a weakly diagonally dominant matrix with nonpositive offdiagonal entries and that $A + L^t = D - L$ is strictly diagonally dominant.

Then the relation

$$x_i = a_{ii} - \sum_{k=1}^{i-1} \frac{a_{ik}}{x_k} \sum_{j=k+1}^{N} a_{kj} > 0$$

and the diagonal matrix $X = diag(x_1, \cdots, x_N)$ defines stable $MIC(0)$ factorization of A,

$$MIC(0)(A) = (X - L)X^{-1}(X - L)^t$$

M-matrices

$$M_n = \left\{ A \in \mathbf{R}^{n\times n} : a_{ii} > 0, a_{ij} \leq 0, i \neq j, \sum_{j=1}^{n} a_{ij} \geq 0 \right\}$$

$$h_x < h_y$$

$$p = \frac{h_y}{h_x}, \quad 0 < p < 1$$

$$A_e = \frac{1}{3p} \begin{pmatrix} 1+4p^2 & -(1+p^2) & -(1+p^2) & -(2p^2-1) \\ -(1+p^2) & 4+p^2 & -(2-p^2) & -(1+p^2) \\ -(1+p^2) & -(2-p^2) & 4+p^2 & -(1+p^2) \\ -(2p^2-1) & -(1+p^2) & -(1+p^2) & 1+4p^2 \end{pmatrix}$$

$$p \in (0, \sqrt{2}/2), \quad -(2p^2-1) > 0$$

$$B_1 = \tfrac{1}{3p} \begin{pmatrix} 2+2p^2 & -(1+p^2) & -(1+p^2) & 0 \\ -(1+p^2) & 4+p^2 & -(2-p^2) & -(1+p^2) \\ -(1+p^2) & -(2-p^2) & 4+p^2 & -(1+p^2) \\ 0 & -(1+p^2) & -(1+p^2) & 2+2p^2 \end{pmatrix}$$

$$B_2 = \frac{1}{3} \begin{pmatrix} 5 & -2 & -2 & -1 \\ -2 & 5 & -1 & -2 \\ -2 & -1 & 5 & -2 \\ -1 & -2 & -2 & 5 \end{pmatrix}$$

$$A \to B \to C = MIC(0)(B)$$

Remark 1 *The general conclusion is that the proposed algorithm is suitable for problems with moderate mesh anisotropy.*

Optimizaton problem

For given SPD matrix K, find SPD M-matrix B such that the condition number of the generalized eigenvalue problem

$$Ku = \lambda Bu$$

is small as possible.

U. Langer, S. Reitzinger, J. Schicho, *Symbolic Methods for the Element Preconditioning Technique*

$$B = \frac{1}{3p} \begin{pmatrix} 2+2p^2 & -(1+p^2) & -(1+p^2) & 0 \\ -(1+p^2) & 4+p^2 & -(2-p^2) & -(1+p^2) \\ -(1+p^2) & -(2-p^2) & 4+p^2 & -(1+p^2) \\ 0 & -(1+p^2) & -(1+p^2) & 2+2p^2 \end{pmatrix}$$

$$\kappa = \frac{1+p^2}{3p^2}$$

Numerical tests

$$\Omega = [0, 1]^2$$

$$-\Delta u = 0, \quad in \quad \Omega$$

$$u = 0, \quad on \quad \partial\Omega$$

PCG iterations

n \ k	0	1	2	3	4
3	4	8	12	26	44
7	7	11	20	40	78
15	11	18	37	75	148
31	28	28	70	149	302

Where, h_x and h_y, are mesh parametars and

$$h_x = 1/n, \quad h_y = h_x 2^k$$

Conclusions

$3D \rightarrow$ non-conforming rotated trilinear elements

Construction of efficient preconditioners for rotated (bi)trilinear nonconforming FEM elasticity systems