Aggregation-based multilevel preconditioning of non-conforming FEM elasticity problems

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CONTENTS

- *Co-authors of some related joint papers:*

  O. Axelsson, R. Blaheta, Tz. Kolev, M. Neytcheva, and P.S. Vassilevski.

1. Introduction

The target problem in this paper is the Lamé system of elasticity:

$$\begin{aligned} \textstyle\sum_{j=1}^{2} \frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0 \qquad & x \in \Omega, \qquad & i = 1, 2 \\ u = 0 \qquad & x \in \Gamma_D & \\ \textstyle\sum_{j=1}^{2} \frac{\partial \sigma_{ij}}{\partial x_j} n_j = g_i \qquad & x \in \Gamma_N, \qquad & i = 1, 2, \end{aligned}$$

where Ω is a polygonal domain in $\mathbb{R}^2$ and (n_1, n_2) is the external unit normal to Ω.

The stresses σ_{ij} and the strains ε_{ij} are defined by the classical Hooke's law, i.e.

$$\sigma_{ij}(u) = \lambda(\sum_{k=1}^{2} \varepsilon_{kk})\delta_{ij} + 2\mu\varepsilon_{ij}(u),$$

$$\varepsilon_{ij}(u) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}).$$

The unknowns of the problem are the displacements $u^t = (u_1, u_2)$.

The Lamé coefficients are given by

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)},$$

where E stands for the modulus of elasticity, and $\nu \in (0, \frac{1}{2})$ is the Poisson ratio.

- The notion almost incompressible is used for the case $\nu = \frac{1}{2} - \delta$, $\delta > 0$ is a small parameter.
- Note that the boundary value problem becomes ill-posed when $\nu = \frac{1}{2}$.

Let $\mathcal{T}_\ell$ be obtained by ℓ regular refinement steps of the coarse triangulation $\mathcal{T}_0$ of Ω. The following weak formulation of the problem is considered:

$$a(u,v) = \int_\Omega f v dx + \int_{\Gamma_N} g v ds, \qquad \forall v \in \mathcal{V},$$

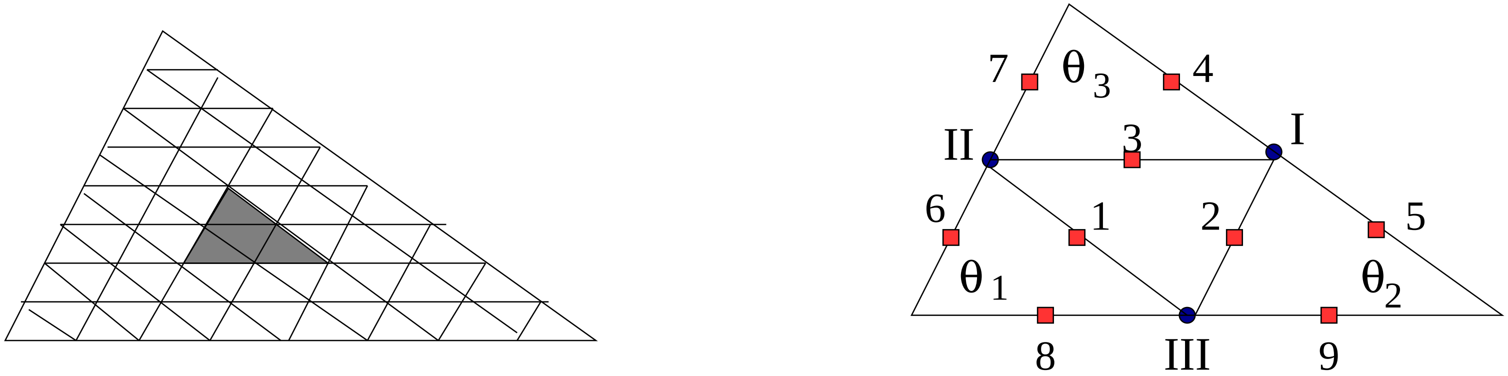
$$a(u,v) = \lambda a_1(u,v) + 2\mu a_2(u,v),$$

$$a_1(u,v) = \int_\Omega div(u) div(v) dx,\, a_2(u,v) = \sum_{i,j=1}^{2} \int_\Omega \varepsilon_{ij}(u) \varepsilon_{ij}(v) dx.$$

Crouzeix-Raviart FEM

- The variational problem is next discretized using Crouzeix-Raviart non-conforming finite elements, i.e., the space $\mathcal{V}$ is replaced by a finite dimensional subspace $\mathcal{V}^{(\ell)}$.
- Here, and in what follows, $\{\mathcal{V}^{(k)}\}_{k=0}^{\ell}$ and $\{A^{(k)}\}_{k=0}^{\ell}$ stand for the FE spaces and for the stiffness matrices corresponding to $\{\mathcal{T}_k\}_{k=0}^{\ell}$.
- Let us recall that this FE approximation provides some attractive stability properties for parameter-dependent problems.

Hierarchical decomposition



Regular refinement (a), and macroelement local node numbering (b)

Let us consider two consecutive mesh refinements $\mathcal{T}_k$ and $\mathcal{T}_{k+1}$. As already mentioned, for Crouzeix-Raviart non-conforming linear elements, the FE spaces associated with consecutive mesh refinements are not nested.

The proposed hierarchical two-level decomposition of the Crouzeix-Raviart system is locally introduced by the splitting

$$\mathcal{V}\,(E) = \text{span}\ \{\phi_1, \ldots, \phi_9\}\ =\ \widetilde{\mathcal{V}}_1(E) \oplus \widetilde{\mathcal{V}}_2(E)$$

where

$$\widetilde{\mathcal{V}}_1\,(E) = \text{span}\{\phi_1,\ \phi_2, \phi_3, \phi_4 - \phi_5, \phi_6 - \phi_7, \phi_8 - \phi_9\},$$

$$\widetilde{\mathcal{V}}_2\,(E) = \text{span}\{\phi_1 + \phi_4 + \phi_5, \phi_2 + \phi_6 + \phi_7, \phi_3 + \phi_8 + \phi_9\}.$$

The transformation matrix J_{DA} corresponds to the introduced DA splitting. Accordingly, J_{DA} transforms the macroelement stiffness matrix into a hierarchical form

$$\widetilde{A}_E = J_{DA}^T A J_{DA} = \begin{bmatrix} \widetilde{A}_{E,11} & \widetilde{A}_{E,12} \\ \widetilde{A}_{E,21} & \widetilde{A}_{E,22} \end{bmatrix} \begin{matrix} \widetilde{\phi}_i \in \widetilde{\mathcal{V}}_1(E) \\ \widetilde{\phi}_i \in \widetilde{\mathcal{V}}_2(E) \end{matrix} .$$

Then, the global matrix is assembled as follows

$$\widetilde{A}^{(k+1)} = \sum_{E \in \mathcal{T}_{k+1}} \widetilde{A}_E .$$

CBS constant

The introduced decomposition is first used for construction of two-level preconditioners. The related PCG convergence rate is simply determined in terms of the strengthened CBS constant γ_{DA}.

Theorem 1. *For any element size and shape and for any Poisson ratio* $\nu \in (0, \frac{1}{2})$*, there holds*

$$\gamma_{DA} \leq \sqrt{\frac{3}{4}}.$$

Multilevel preconditioning

The standard multiplicative two-level preconditioner can be written in the form

$$\widetilde{C}^{(k+1)} = \begin{bmatrix} \widetilde{A}_{11}^{(k+1)} & 0 \\ \widetilde{A}_{21}^{(k+1)} & \widetilde{A}_{22}^{(k+1)} \end{bmatrix} \begin{bmatrix} I_1 & \widetilde{A}_{11}^{(k+1)-1} \widetilde{A}_{12}^{(k+1)} \\ 0 & I_2 \end{bmatrix}.$$

The convergence rate of the related PCG iterative method is based on the spectral condition number estimate

$$\kappa(\widetilde{C}^{(k+1)-1} \widetilde{A}^{(k+1)}) < \frac{1}{1-\gamma_{DA}^2} < 4.$$

The following theorem is useful for extending the two-level to multilevel preconditioners.

Theorem 2. *Let* $\tilde{A}_{22}^{(k+1)}$ *be the stiffness matrix corresponding to the space* $\widetilde{\mathcal{V}}_2^{(k+1)}$ *from the introduced DA splitting, and let* $A^{(k)}$ *be the stiffness matrix, corresponding to the coarser triangulation* $\mathcal{T}_k$*, equipped with the standard nodal finite element basis. Then*

$$\tilde{A}_{22}^{(k+1)} = 4\,A^{(k)}\,.$$

Coarse grid blocks preconditioning

- One possible approach is first to use a DD block-diagonal approximation of $\widetilde{A}_{11}^{(k+1)}$, and then to precondition the related elliptic blocks. This construction is optimal with respect to the mesh size and anisotropy for moderate values of ν.
- In the almost incompressible case, it is better firs to apply a macroelement level static condensation of $\widetilde{A}_{11}^{(k+1)}$. Then, the obtained Schur compliment has a condition number which is uniformly bounded for any $\nu \in (0, \frac{1}{2})$.

Numerical tests

- It is well known, that when the Poisson ratio ν tends to the incompressible limit, the so called locking phenomenon appears, if low order conforming finite elements are used in the construction of the approximation space.
- We use the Crouzeix-Raviart linear finite elements to get a locking-free FEM solution of the problem.
- Note that such a straightforward NC FEM discretization works well for the pure displacement problem only.

Relative error stability for $\nu \to 1/2$

Table 1: $\ell = 4$, $N = 1472$, and $\varepsilon = 10^{-9}$

ν	$\|u - u_h\|_{[L_2]^2} / \|f\|_{[L_2]^2}$
0.4	.31082491065035
0.49	.36959437474055
0.499	.37648796437736
0.4999	.37718890770387
0.49999	.37725911956136
0.499999	.37726614194014

Locking-free AMLI preconditioning

Table 2: $\varepsilon = 10^{-3}$, $\beta = 2$,$\nu \in (0.3, 0.499999)$

ℓ	N	0.3	0.49	0.4999	0.499999
4	1472	13	12	13	13
5	6016	12	12	13	13
6	24320	12	12	13	13
7	97792	11	11	13	13
8	196096	11	11	12	13

Remark 1. *$\beta = 2$ corresponds to the derived uniform estimate of γ, providing a total optimality of the algorithm.*

Concluding remarks

- This study is strongly motivated by the expanding interest in non-conforming finite elements, which are very useful for solving problems, where the standard FEM may suffer from the *locking effects*.
- The presented multilevel algorithm has some well expressed inherently parallel features. The key point here is that the considered approximations of the coarse grid complement blocks $\widetilde{A}_{11}$ are of either diagonal or generalized tridiagonal form.

Related publications

1. O. Axelsson, S. Margenov. On multilevel preconditioners which are optimal with respect to both problem and discretization parameters, *Comp. Meth. Appl. Math., Vol. 3, No. 1: 6-22, 2003.*
2. *R. Blaheta, S. Margenov, M. Neytcheva. Uniform estimate of the constant in the strengthened CBS inequality for anisotropic non-conforming FEM systems,* Numer. Lin. Algebra Appl.*, 2004.*
3. *Tz. Kolev, S. Margenov. Two-level preconditioning of pure displacement non-conforming FEM systems,* Numer. Lin. Algebra Appl., 6: 533-555, 1999.
4. Tz. Kolev, S. Margenov. AMLI preconditioning of pure displacement non-conforming elasticity FEM systems, *Springer LNCS, 1988: 482-489, 2001.*
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6. *S. Margenov, P.S. Vassilevski. Algebraic multilevel preconditioning of anisotropic elliptic problems,* SIAM J. Sci. Comp., V.15(5): 1026-1037, 1994.