A posteriori error estimates for some finite volume methods

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Plan

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- The numerical scheme
- The interpolant
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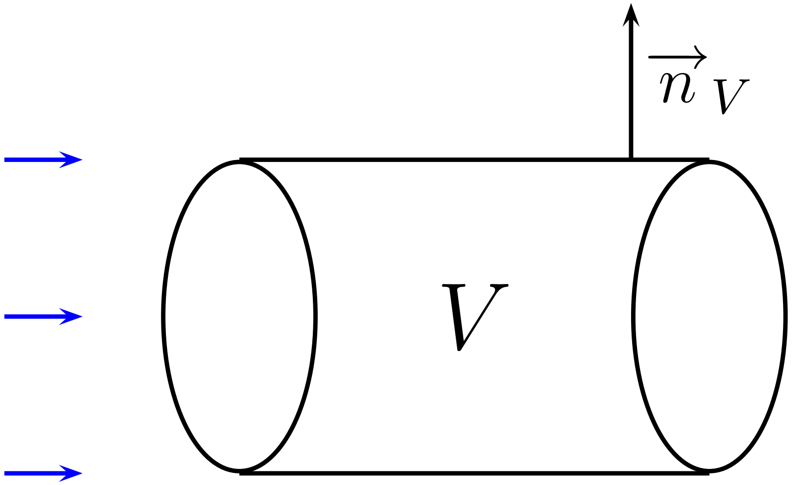
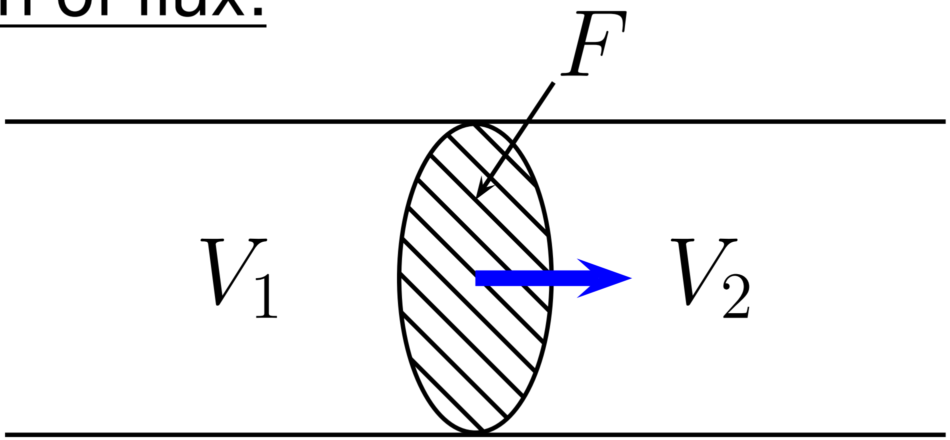
Introduction to Finite Volume methods

Green's Formula:

$$\int_V \text{div}\,\overrightarrow{T}\ dV = \int_{\partial V} \overrightarrow{T}.\overrightarrow{n}_V\ ds$$



Conservation of flux:



Introduction to adaptive methods

The different steps:

1. Discretization (by a finite element method or a finite volume method) on a mesh $\tau \Rightarrow$ approximated solution u_h
2. Estimator $\eta := \{\eta_K\}_{K \in \tau}$ such that $\eta_K(u_h) \sim |||u - u_h|||_K$
3. Subdivision of the elements K of τ where the estimator η_K is too large. Back to point 1.

The problem

Let Ω be a bounded domain of $\mathbb{R}^2$ with a polygonal boundary.
Let u be the solution of

$$\left\{\begin{array}{l} -\varepsilon \,\text{div}(\overrightarrow{\nabla}\, u) + \text{div}(\overrightarrow{b}\, u) + c\, u = f \text{ dans } \Omega\ , \\ \\ u = 0 \text{ sur } \partial\Omega\ , \end{array}\right. \tag{1}$$

where $\varepsilon \in \mathbb{R}_+^*$ and $\overrightarrow{b},\ c,\ f$ are regular.

$$\frac{\text{div}(\overrightarrow{b}\,)}{2} + c \ \geqslant\ 0 \text{ in } \ \Omega \qquad \Rightarrow \qquad \exists!\ u \in H^1(\Omega) \text{ solution of } \ (1$$

The numerical scheme

- $\mathcal{T}$: A mesh of Ω of size $h > 0$
- $\mathcal{P} := \{x_K\}_{K\in\mathcal{T}}$

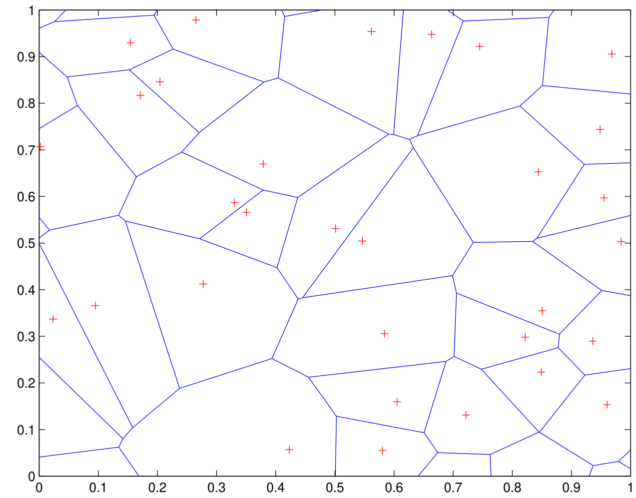


Figure 1: Example of a mesh $\mathcal{T}$

- $\{u_K\}_{K\in\mathcal{T}}$:= unknowns (u_K approximated value of u at x_K)

The numerical scheme

Integrate the equation (1) on $K \in \tau$

and use Green's formula

$$\Downarrow$$

$$-\varepsilon \sum_{\sigma \in F_K} \int_\sigma \overrightarrow{\nabla} u . \overrightarrow{n}_{K,\sigma} + \sum_{\sigma \in F_K} \int_\sigma \overrightarrow{b} . \overrightarrow{n}_{K,\sigma} u + \int_K c\, u = \int_K f \ , \forall\, K \in \tau$$

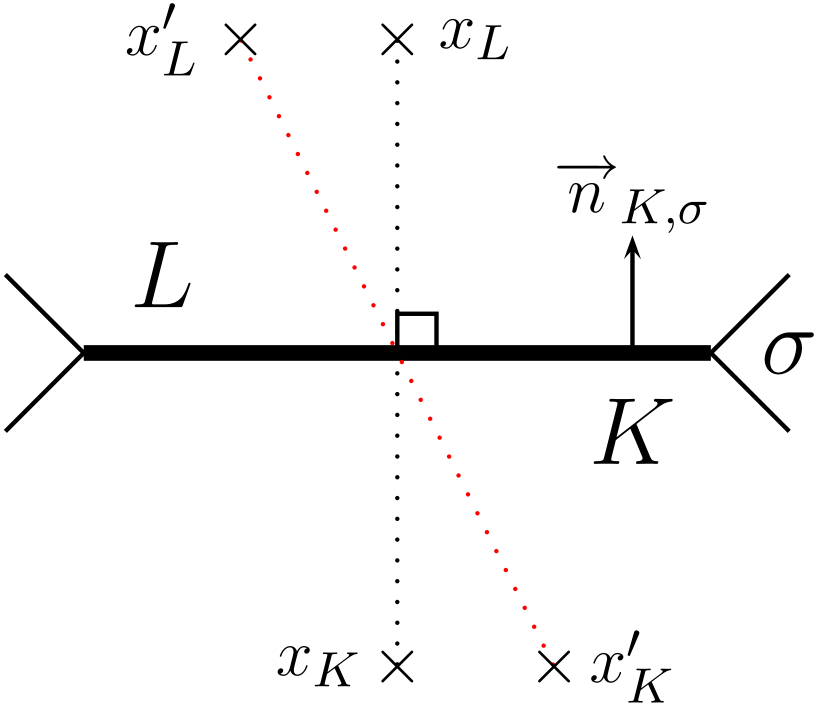
F_K : set of faces of $K, \ K \in \tau$,
$\overrightarrow{n}_{K,\sigma}$: normal outward unit vector to K along σ ,
$\sigma \in F_K \ , \ K \in \tau.$

The numerical scheme/ The diffusive term

$$\int_\sigma \overrightarrow{\nabla} u . \overrightarrow{n}_{K,\sigma}$$

$\wr$ finite differences

$$|\sigma| \; \frac{u_L - u_K}{d(x_K, x_L)}$$



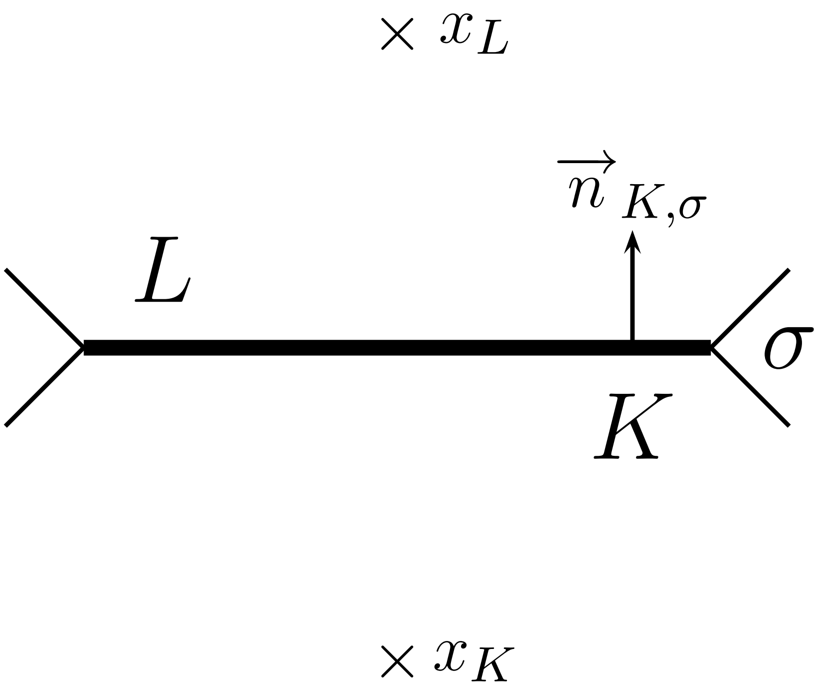
orthogonality conditions $\Rightarrow$ consistency of the approximation

The numerical scheme/ The convective term

$$\int_\sigma \overrightarrow{b}\,.\,\overrightarrow{n}_{K,\sigma}\; u$$

$\approx$ upstream scheme

$$u_\sigma \int_\sigma \overrightarrow{b}\,.\,\overrightarrow{n}_{K,\sigma}$$



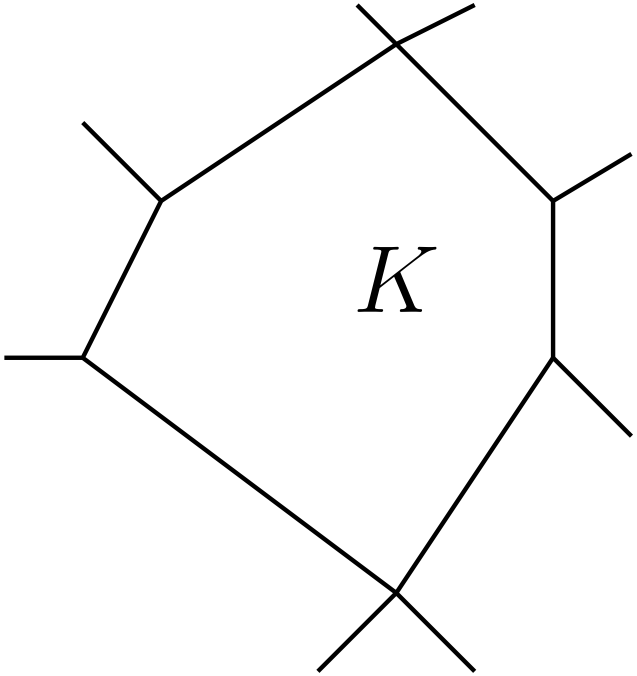
where $$u_\sigma := \begin{cases} u_K \text{ if } \displaystyle\int_\sigma \overrightarrow{b}\,.\,\overrightarrow{n}_{K,\sigma} \geqslant 0 \\ u_L \text{ if } \displaystyle\int_\sigma \overrightarrow{b}\,.\,\overrightarrow{n}_{K,\sigma} < 0 \end{cases}$$

The numerical scheme/ The reaction term

$$\int_K c\, u$$

$\wr$ quadrature rule

$$u_K \int_K c$$



The numerical scheme