On the Convergence of a General Class of Finite Volume Methods

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Finite Volume Methods for Conservation Laws

Conservation Laws

$$\frac{\partial}{\partial t}u(x,t) + \sum_{\ell=1}^{d} \frac{\partial}{\partial x_\ell} f_\ell(u(x,t)) = 0$$

- $u : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}^n$ vector-valued solution to be conserved
- $f_\ell : \mathbb{R}^n \to \mathbb{R}^n$ denote the so-called flux functions
- + Initial conditions $u(x,0) = u_0(x)$
- + Boundary conditions

Weak formulation:

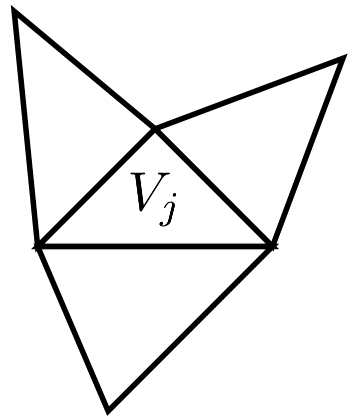
$$\frac{d}{dt}\int_V u(x,t)dx = -\int_{\partial V} \sum_{\ell=1}^{d} f_\ell(u(x,t))\eta_\ell(x)dS$$

- $V \subseteq \mathbb{R}^d$ test volume
- $\eta(x)$ outer normal unit vector

Finite Volumes

- Decompose Ω into small simplices V_j of size h

$$\Omega = \bigcup_{j=1}^{N} V_j.$$



- Introduce *cell averages*

$$\lambda_j(u)(t) := \overline{u}_j(t) = \frac{1}{|V_j|} \int_{V_j} u(x,t)dx, \qquad 1 \le j \le N.$$

- Rewrite weak formulation

$$\frac{d}{dt}\lambda_j(u)(t) = -\frac{1}{|V_j|} \sum_{V \in \mathcal{N}_j} \int_{\partial V \cap \partial V_j} \sum_{\ell=1}^{d} f_\ell(u)\eta_\ell^{(V)} dS,$$

$\eta^{(V)} =$ outer normal vector to the boundary face $\partial V \cap \partial V_j$ of V.

Numerical Flux Function

- Numerical flux function $H : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$

$$H(u,u;\eta) = \sum_{\ell=1}^{d} f_\ell(u)\eta_\ell.$$

- + Quadrature rule

leads to

$$\frac{d}{dt}\lambda_j(u)(t) = -\frac{1}{|V_j|}\sum_{V\in\mathcal{N}_j}\sum_{\nu=1}^{n_Q} w_\nu H(u(x_\nu(V),t),u(x_\nu(V),t);\eta_\ell^{(V)}) + \mathcal{O}(h^{m_Q}),$$

where m_Q denotes the order of the employed quadrature rule.

Recovery from Cell Averages

Problem: $u(x_\nu(V),t)$ is unknown.
Solution: Recover by a function s_u with

$$\begin{aligned} \lambda_j(s_u)(t) &= \lambda_j(u)(t), \qquad 1 \le j \le N, \\ \|u - s_u\|_{L_\infty(\Omega)} &= \mathcal{O}(h^p), \qquad h \to 0. \end{aligned}$$

Leads to:

$$\frac{d}{dt}\lambda_j(u)(t) = -\frac{1}{|V_j|} \sum_{V \in \mathcal{N}_j} \sum_{\nu=1}^{n_Q} w_\nu H(s_u(x_\nu(V),t), s_u(x_\nu(V),t); \eta_\ell^{(V)}) + \mathcal{O}(h^{\min\{p,m_Q\}}).$$

Error Estimates for Cell Average Reconstruction

Sobolev Norms

$$
\begin{aligned}
|u|^p_{W^k_p(\Omega)} &= \sum_{|\alpha|=k} \|D^\alpha u\|^p_{L^p(\Omega)} \\
\|u\|^p_{W^k_p(\Omega)} &= \sum_{|\alpha|\le k} \|D^\alpha u\|^p_{L_p(\Omega)} \\
|u|^p_{W^{k+s}_p(\Omega)} &= \sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{\|x-y\|_2^{d+ps}} dx dy \\
\|u\|^p_{W^{k+s}_p(\Omega)} &= \|u\|^p_{W^k_p(\Omega)} + |u|^p_{W^{k+s}_p(\Omega)} \\
\|u\|^2_{\ell_2(Y)} &= \frac{1}{M} \sum_{j=1}^{M} |u(y_j)|^2
\end{aligned}
$$

Reconstruction from Cell Averages

- Given $u \in W_p^\tau(\Omega)$ with $\lambda_j(u) = 0$, $1 \le j \le N$.
- $\tau > m + d/p$, $1 \le q \le \infty$.
- Every ball $B(x_c, h)$ contains one sub-domain V_j.

Then

$$|u|_{W_q^m(\Omega)} \le ch^{\tau - m - d(1/p - 1/q)_+} |u|_{W_p^\tau(\Omega)},$$

In particular,

$$\|u\|_{L_\infty(\Omega)} \le ch^{\tau - d/2} |u|_{W_2^\tau(\Omega)}.$$

Idea of Proof

- Cover Ω by local, "nice" patches $\mathcal{D} = \cup V_j$ of size $\mathcal{O}(h)$
- On each patch:
  - $\|D^\alpha u\|_{L_\infty(\mathcal{D})} \le \underbrace{\|D^\alpha(u-p)\|_{L_\infty(\mathcal{D})}}_{\le C h^{\tau-|\alpha|-d/2}|u|_{W_2^\tau(\mathcal{D})}} + \|D^\alpha p\|_{L_\infty(\mathcal{D})}$, $p \in \pi_k(\mathbb{R}^d)$
  - $$\begin{aligned} D^\alpha p(x) &= \sum_j a_j(x)\lambda_j(p) = \sum_j a_j(x)[\lambda_j(p) - \lambda_j(u)] \\ &= \sum_j a_j(x)\frac{1}{|V_j|}\int_{V_j}[p(y)-u(y)]dy \end{aligned}$$
  - $|D^\alpha p(x)| \le \underbrace{\sum_j |a_j(x)|}_{Ch^{-|\alpha|}} \underbrace{\|p-u\|_{L_\infty(\mathcal{D})}}_{\le Ch^{\tau-d/2}|u|_{W_2^s(\mathcal{D})}}$
- Piece things together

Norming Sets

Def: $\mathcal{P}$ finite dimensional. Then, $\Lambda = \{\lambda_1, \ldots, \lambda_N\} \subseteq \mathcal{P}^*$ is a norming set for $\mathcal{P}$ if the mapping

$$T : \mathcal{P} \to \mathbb{R}^N, \qquad p \mapsto (\lambda_j(p))_{1 \le j \le N},$$

is injective.

Theorem: If Λ is a norming set for $\mathcal{P}$ and $\eta \in \mathcal{P}^*$ then there exists a vector $c = c(\eta) \in \mathbb{R}^N$ such that

$$\begin{aligned} \eta|\mathcal{P} &= \sum_{j=1}^{N} c_j \lambda_j | P, \\ \|c\|_1 &\le \|\eta|\mathcal{P}\| \, \|T^{-1}\|. \end{aligned}$$

Polynomials

- Here: $\mathcal{P} = \pi_k(\mathbb{R}^d)$, $\quad \lambda_j(p) = \frac{1}{|V_j|}\int_{V_j} p(y)dy$, $\quad \eta(p) = D^\alpha p(x)$.
- Requires:

$$\gamma \|p\|_{L_\infty(\Omega)} \le \max_{1\le j\le N} \frac{1}{|V_j|}\left|\int_{V_j} p(y)dy\right| \quad \Rightarrow \quad \|T^{-1}\| \le 1/\gamma.$$

$$\|D^\alpha p\|_{L_\infty(\mathcal{D})} \le \left(\frac{2k^2}{r\sin(\theta)}\right)^{|\alpha|} \|p\|_{L_\infty(\mathcal{D})} \quad \Rightarrow \quad \|\eta\| \le \left(\frac{2k^2}{r\sin(\theta)}\right)^{|\alpha|}.$$

- Result:

Theorem: If $h \le c/k^2$ then there exists $a_j^\alpha(x)$ such that

$$D^\alpha p(x) = \sum a_j^\alpha(x)\lambda_j(p), \qquad x \in \mathcal{D},\ p \in \pi_k(\mathbb{R}^d)$$

with

$$\sum |a_j^\alpha(x)| \le \frac{1}{\gamma}\left(\frac{2k^2}{r\sin(\theta)}\right)^{|\alpha|} = Ch^{-\alpha}$$

Optimal Recovery from Cell Averages

Optimal Recovery

Goal: For $u \in W_2^\tau(\Omega)$ find $s_u \in W_2^\tau(\Omega)$ such that

- $\lambda_j(u) = \lambda_j(s_u)$, $1 \le j \le N$.
- $\|u - s_u\|_{W_2^\tau(\Omega)} \le \|u\|_{W_2^\tau(\Omega)}$.

Possible approach: s_u as solution of

$$\min\{\|s\|_{W_2^\tau(\mathbb{R}^n)} : \lambda_j(s) = \lambda_j(u)\}$$

Solution is computable: Let $\Phi(\cdot,\cdot)$ be the reproducing kernel of $W_2^\tau(\mathbb{R}^n)$. Then,

$$s_u(x) = \sum_{j=1}^N \alpha_j \lambda_j^y \Phi(x,y),$$

where $\alpha \in \mathbb{R}^N$ is determined by interpolation conditions

$$\lambda_j(s_u) = \lambda_j(u), \qquad 1 \le j \le N.$$

Reproducing Kernels

- $\mathcal{H} = W_2^\tau(\mathbb{R}^d)$:
$$\widehat{\Phi}(\omega) \sim (1 + \|\omega\|_2^2)^{-\tau}.$$
Examples: Sobolev-splines, CS-RBF.

- $\mathcal{H} = \mathrm{BL}_\tau(\mathbb{R}^d)$:
$$\widehat{\Phi}(\omega) \sim \|\omega\|_2^{-2\tau}.$$
Examples: Thin-plate splines

In both cases: the reproducing kernels are radial.