

Chapter 1

Introduction

The last decade of the twentieth century was marked by the penetration of partial differential equations (PDEs) and the methods used to study them into the multivariate constructive theory of functions, in particular in approximation theory and spline analysis. This trend differs from previous developments when standard objects such as polynomial and rational functions and splines were used to approximate solutions of partial differential equations.

The present book introduces and develops a new type of multivariate spline known as polysplines.¹ Although in the one-dimensional case there is a satisfactory theory of one-dimensional splines, which includes all kinds of generalizations such as Chebyshev splines and L -splines, in the multivariate case there are several alternatives which can be considered to be multivariate splines in their own right, such as box splines, simplex splines, radial basis functions etc. So far there are general principles which come from an intuitive understanding of what a multivariate spline should be and which we wish to discuss.

What is a multivariate spline? We will keep close to the following understanding of a spline: assume that a domain $D \subset \mathbb{R}^n$ be given and a disjoint family of subdomains D_j such that $\cup D_j = D$, and the boundaries $\text{bdry}(D_j)$ are smooth enough, so that the normal n exists almost everywhere on $\text{bdry}(D_j)$. Then a *spline* is a function u defined in D which is assembled of functions u_j defined on D_j . These pieces are of similar nature and match up to a certain degree d of smoothness on the joint boundaries. Imagine for simplicity that $D \subset \mathbb{R}^2$ and $D = D_1 \cup D_2$, and $\overline{D}_1 \cap \overline{D}_2 = \Gamma$ which is a curve (see Figure 1.1).

Throughout this book the joint boundary Γ where two pieces match will be called the **interface** or **break-surface**.

¹ The name which completely characterizes the polysplines is “piecewise polyharmonic splines”. So far the name “polyharmonic splines” has been used by W. Madych for radial basis functions made of the fundamental solution of the polyharmonic operator.

2 Multivariate polysplines

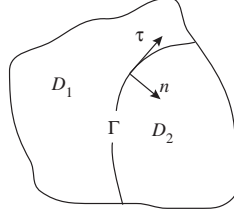


Figure 1.1.

Then we require that

$$\begin{aligned} u_1 &= u_2 \quad \text{on } \Gamma, \\ \frac{\partial}{\partial n} u_1 &= \frac{\partial}{\partial n} u_2 \quad \text{on } \Gamma, \\ &\dots \\ \frac{\partial^d}{\partial n^d} u_1 &= \frac{\partial^d}{\partial n^d} u_2 \quad \text{on } \Gamma, \end{aligned}$$

where $\partial/\partial n$ denotes the normal derivative (one of the two directions) on Γ . If we also require more smoothness of the functions u_1 and u_2 on the joint boundary Γ , say $u_1 \in C^{d_1}(\overline{D_1})$ and $u_2 \in C^{d_1}(\overline{D_2})$, if Γ is also smooth enough, we may differentiate the above equalities in the direction τ tangential to Γ and obtain the equalities of the mixed derivatives up to order d_1 , i.e.

$$\frac{\partial^l}{\partial \tau^l} \frac{\partial^k}{\partial n^k} u_1 = \frac{\partial^l}{\partial \tau^l} \frac{\partial^k}{\partial n^k} u_2,$$

where the indices l and k satisfy $l + k \leq d_1$ and $0 \leq k \leq d$. Let us fix a point y on Γ . To write the last equalities in a simpler way, let us introduce a local coordinate system on the surface Γ by putting $y = 0$ and by choosing the normal vector (one of the two directions) to Γ to coincide with the coordinate axis x_2 . Then the above equality at the point y will read as follows:

$$\frac{\partial^l}{\partial x_1^l} \frac{\partial^k}{\partial x_2^k} u_1(0) = \frac{\partial^l}{\partial x_1^l} \frac{\partial^k}{\partial x_2^k} u_2(0).$$

The really big questions arise if we are given a “data function” f on the set Γ which has to be *interpolated* by the spline u , i.e. if we would like to have

$$u_1 = u_2 = f \quad \text{on } \Gamma.$$

Then the problem is to find for every d a reasonable *class of functions* u_1 and u_2 for which this interpolation equality can be solved for a large class of data functions f . This problem is a real intellectual challenge. In the present book we provide a solution only for the integers $d = 2p - 2 \geq 0$, where $p \geq 1$ is an integer. The functions u_1 and u_2

then satisfy the equations

$$\begin{aligned}\Delta^p u_1 &= 0 && \text{in } D_1, \\ \Delta^p u_2 &= 0 && \text{in } D_2,\end{aligned}$$

where Δ^p is the polyharmonic operator.

Our approach to this problem is widely based as will be seen in Chapters 3 and 4.

The theory of polysplines does not appear from nowhere. Even in its simplest cases it relies heavily upon (and even reduces to) Chebyshev's one-dimensional theory and L -splines.

The reader has to be clear about the theory of polysplines. It is a theory which is a genuine synthesis between two important areas in mathematics: approximation theory and elliptic partial differential equations. This has increased the volume of the present book by the inclusion of four appendixes: the *Compendium* on spherical harmonics and polyharmonic functions, and *appendixes* on elliptic boundary value problems (BVPs), Chebyshev splines, and Fourier analysis.

Let us discuss right at the beginning of this book the so-called “data concept” of the polysplines, addressing mainly the readers interested in practical applications: Where do such data come from on a whole curve Γ in \mathbb{R}^2 (or a surface Γ in \mathbb{R}^n)? Is not this rather restrictive and making the polysplines uninteresting for many applications, since the practical measurements are on a discrete set of points?

The answer to this argument is: In many practical applications (airborne, satellite data, CAGD data) one may find a natural set of curves in \mathbb{R}^2 (respectively surfaces in \mathbb{R}^n) which contain the discrete data points. Further, in order to obtain data defined on the whole curves one may apply the well-known one-dimensional methods for extending data – one-dimensional splines, etc. In a similar way, and inductively, one proceeds in the case of dimension $n \geq 3$.²

1.1 Organization of material

1.1.1 Part I: Introduction of polysplines

In Part I we provide a logical basis for the notion of polysplines. In Chapter 2 we cover some basic results in one-dimensional spline theory. In Chapters 3 and 4 we explain in detail the meaning of the data, smoothness and object concepts of the polyharmonic paradigm, and their implementation in spline analysis. In Chapter 5 we introduce polysplines in the plane \mathbb{R}^2 when the interface set is equal to a finite number of parallel straight lines (polysplines on strips). In Chapter 8 we consider the case when the interface set equals a finite number of concentric circles in \mathbb{R}^2 (polysplines on annuli). In the adjacent chapters we provide the necessary basics on harmonic and polyharmonic functions on the strip, annulus and ball. The advantage of the two-dimensional case is that the reader has absolutely no need to be familiar with PDEs. All that is necessary is some basic results on the Fourier series which are provided in Chapter 12. The main

² See also one of the statements in the last Chapter 24, “Afterword”.

4 Multivariate polysplines

result is that the computation of the interpolation polysplines in the two cases reduces to a computation of infinitely many proper one-dimensional L -splines, where L are operators with constant coefficients. In the case of polysplines on strips in \mathbb{R}^2 these operators are given by

$$L\left(\frac{d}{dt}\right) = \left(\frac{d^2}{dt^2} - \xi^2\right)^p,$$

where ξ is a real number, and in the case of the polysplines on annuli in \mathbb{R}^2 the operators are given by

$$L\left(\frac{d}{dt}\right) = \prod_{l=0}^{p-1} \left(\frac{d}{dt} - (k + 2l)\right) \prod_{l=1}^p \left(\frac{d}{dt} - (-2 - k + 2l)\right)$$

for all integers $k \geq 0$.

In Chapter 6 we provide experimental proofs for the superiority of the polysplines over well established methods such as kriging, minimum curvature, radial basis functions (RBFs) etc.³ This may be the most important reason for a reader whose main interest is in numerical analysis to study polysplines. To such a reader we have to say that almost all of Part I is simplified and algorithmic – the results are programmable.

Simple as they are, the polysplines in \mathbb{R}^2 have all the main features of the polysplines with arbitrary interfaces and the proof of their existence is not easier. For that reason we leave all the proofs of the existence of the interpolation polysplines to Part IV.

In Chapter 9 we define the polysplines on strips and on annuli in \mathbb{R}^n . For the polysplines on annuli we need the structure of a polyharmonic function on the annulus (Almansi-type theorem by I. Vekua in \mathbb{R}^2 and by S. L. Sobolev in \mathbb{R}^n). This is thoroughly studied in Chapter 10. Matters are definitely simpler for the polysplines on strips since we only need the Fourier transform of functions on strips. All necessary facts about the Fourier transform are provided in Chapter 12. The basic result of Chapter 9 is again as in the two-dimensional case – the polysplines on strips and annuli reduce to some special types of one-dimensional L -splines where L is an operator with constant coefficients (see Theorem 9.3, p. 119, and Theorem 9.7, p. 124).

Part I is logically very self-contained and would meet the interests of a reader occupied with smoothing methods and computer-aided geometric design (CAGD) but not with wavelet analysis.

1.1.2 Part II: Cardinal polysplines

In Part II we concentrate on the polyspline analog to Schoenberg's one-dimensional cardinal splines. This theory will be very important for the wavelet analysis studied in Part III.

As became clear in Part I, we need the theory of the one-dimensional L -splines for the successful study of polysplines on strips and annuli. In Chapter 13 we start with the theory of cardinal L -splines (where L is an operator with constant coefficients) as

³ The comparison has been made with all methods which are implemented in the "Surfer" package (Golden Software, Golden, Colorado).

presented by Ch. Micchelli and I. Schoenberg, and include some results from N. Dyn and A. Ron. The classical polynomial case studied by Schoenberg is a simple special case when the operator $L = (d^{n+1})/(dt^{n+1})$.

The major discovery of Part II is that the *cardinal polysplines on annuli* are polysplines which have as interfaces (break-surfaces) all concentric spheres S_j with radii e^j (or more generally ab^j where the constants $a, b > 0$). They are studied in Chapter 15. We prove that if on every such sphere S_j a function f_j is prescribed such that $\|f_j\|_{L_2}$ has a “power growth” as $|j|^\gamma$ for some $\gamma > 0$ then the interpolation polyspline u exists, i.e.

$$u = f_j \quad \text{on } S_j \quad \text{for all } j \in \mathbb{Z},$$

and satisfies an estimate of the type

$$\|u(r\theta)\|_{L_2(\mathbb{S}_\theta^{n-1})} \leq C |\log r|^\gamma.$$

Here \mathbb{S}^{n-1} is the unit sphere in \mathbb{R}^n .

In Chapter 14 we prove that the shifts of the compactly supported L -spline Q (for a fixed operator L) form a Riesz basis. This result is a generalization of the polynomial case considered by Ch. Chui. This is a preparation for the wavelet analysis in Part III.

The case of cardinal polysplines on strips is definitely easier to consider. In that case the cardinal polysplines have as break-surfaces infinitely many parallel hyperplanes which are equidistant. There are again two cases – the periodic polysplines and the polysplines with fast decay. Unfortunately, lack of space prevents us from giving a detailed description of these results.

1.1.3 Part III: Wavelet analysis using polysplines

The *wavelet analysis* in Part III can be read without any preparation in the area, but it would be best if the reader were already familiar with Chui’s results on cardinal spline wavelet analysis.

In Chapter 16 we present briefly Chui’s results on cardinal spline wavelet analysis so that the reader is familiar with the material that will be generalized.

In Chapter 17 we provide a thorough study of the cardinal L -spline wavelet analysis, i.e. we use cardinal L -splines on refining grids $2^j\mathbb{Z}$ and study the structure of the spaces. The germs of this theory have been laid by C. de Boor, R. DeVore and A. Ron. The case of nonuniform L -spline wavelets (finite case) has been considered by T. Lyche and L. Schumaker. We will prove generalizations of all Chui’s basic results for cardinal spline wavelet analysis.

Chapter 18 may be considered as the apex of the pyramid built in the previous chapters. We use all results proved up to this point as bricks for the synthesis of the “polyspline wavelets” on annuli which represent a *spherical polyharmonic multiresolution analysis*. It should be noted that they are compactly supported. It seems that this will be a new framework for further studies in multivariate multiresolution analysis.

On the other hand the *cardinal polysplines on strips* generate a *parallel polyharmonic multiresolution analysis*. Technically its description is easier but no compactly supported “polyspline wavelets” exist!

1.1.4 Part IV: Polysplines on general interfaces

Part IV has a quite different flavour. It is not concrete analysis as the previous parts where we have considered the two special cases of polysplines. It considers the polysplines for general interfaces (break-surfaces). Even more generally, we introduce a very general class of polysplines, which are piecewise solutions of a large class of higher-order elliptic equations. We prove a generalization of the Holladay extremal property as well as the existence of interpolation polysplines for the so-called “even-order polysplines”. Unfortunately, there is no simpler setting than Sobolev spaces or Hölder spaces if one wants to obtain solutions. For that reason Part IV may be read without problem only by somebody who is already familiar with the classical theory of elliptic BVPs. Almost all the necessary references are given in Chapter 23. We also provide an extensive introduction to explain the leading ideas of polyspline theory, and references which are classified according to their level and accessibility. Although this is the most abstract part of the book the most technical and deep remains Part III (wavelet analysis).

The following are the surveys and appendixes which we have provided as necessary for our study:

- *Compendium* on spherical harmonics and polyharmonic functions in annular domains following Stein and Weiss, and Sobolev (Chapter 10, p. 129). Not available elsewhere.
- *Survey* of cardinal L -splines (or Chebyshev splines) following Micchelli (Chapter 13, p. 221). Not available elsewhere.
- *Survey* of the sharp estimates and properties of the fundamental function L of the cardinal polynomial splines following several papers by Schoenberg (Section 15.5, p. 294). Not available elsewhere.
- *Survey* of the cardinal polynomial wavelets following Chui (Chapter 16, p. 313).
- *Appendix* on Chebyshev splines following Schumaker (Chapter 11, p. 187).
- *Appendix* on elliptic BVPs in Sobolev and Hölder spaces (Chapter 23, p. 461).
- *Appendix* on Fourier analysis (Chapter 12, p. 209).

1.2 Audience

Part I (except for the existence theorems in Chapter 9), and Parts II and III could be read by anyone who has a good knowledge of classical mathematical analysis, while the *Compendium* on spherical harmonics and polyharmonic functions makes the exposition self-contained. One may say that technically this is nineteenth-century mathematics, wavelets also being included. Part IV is somewhat different. The application of elliptic BVPs is essential and is unavoidable. It would be best if the reader was already familiar with this material. However, we have supplied Part IV with an extended and comprehensive introduction, to at least enable the reader who is not competent in this area to understand that the leading ideas stem from one-dimensional spline theory. This accords with the author’s conviction that ideas are more important to the development of mathematics than formulas.

The author has misgivings about not providing elementary proofs to the existence theorems in Chapter 9, but that might have been a lengthy process. For that reason the elementary proofs are left to the reader; we have only supplied the main hints.

The following parts of the book may be used as graduate course texts:

1. Introduction to polysplines (Chapters 2–8). Knowledge of PDEs is not required. Chapter 9, devoted to the polysplines in \mathbb{R}^n , may be included with the exception of the existence theorems.
2. Introduction to spherical harmonics and polyharmonic functions (Chapter 10).
3. Introduction to cardinal L -splines (Micchelli's approach) (Chapter 13). Introduction to cardinal polysplines (Chapter 15).
4. Introduction to cardinal L -spline wavelet analysis (approach of Chui and de Boor *et al.*) (Chapter 17). Introduction to cardinal polyharmonic wavelet analysis (Chapter 18).

1.3 Statements

- The author is convinced that, in the near future, polyharmonic functions will be studied in *multivariate mathematical analysis* in the same way that polynomials are studied in one-dimensional mathematical analysis. From this point of view the constructive theory of functions (including approximation theory) is a qualitative theory of PDEs, whereas the solutions to the elliptic equations form the case of Chebyshev systems.
- Why polyharmonic functions? That is the most natural question which would be asked by any practically oriented mind. "Polynomials are so simple! And *radial basis functions* are simpler to compute! Why should we learn this complicated theory?" First, the answer to this question is far from obvious. There are two ways to answer the above questions. The first is by supplying sufficiently beautiful theorems and other notional arguments which show that the theory of polysplines is a natural multivariate spline theory. The second way is simply by providing more experimental material which shows that the polysplines are better than the usual splines, kriging, minimum curvature, RBFs, and other known methods.

This book is devoted mainly to the first way, the notional one. We show the flexibility of the polysplines concept and provide sufficient argumentation to show that it is a very natural generalization of the notion of one-dimensional splines. We also devote a chapter to showing what the polysplines can do in practice.

- It was the initial intention of the author to scatter the results of this research in several papers. In fact, the most important fundamental results have been published in journals or conference proceedings. But the brevity of such publications does not allow for an extended presentation, explaining the ideas, making comparisons and finishing with a complete description of the illustrative examples.
- One may consider as a proof that the polysplines are a genuine generalization of the one-dimensional splines the fact that all cases of symmetric interfaces are reduced to studying one-dimensional Chebyshev splines. We prove this in Chapter 9 for the case of polysplines on strips, and for the case of polysplines on annuli, all in \mathbb{R}^n .

8 Multivariate polysplines

- The Chebyshev systems (see work by Karlin and Studden, or Schumaker) play an important role in univariate approximation theory. In fact, all the beautiful results in the classical and trigonometric moment problem and best approximation theory are also available when the Chebyshev systems are used instead.

The very deep reason for using pieces of polyharmonic functions as bricks for constructing splines comes from *approximation theory*. It is based on the observation (thus far not rigorously proved) that the polyharmonic functions play the role of Chebyshev systems in several dimensions. We would refer to a series of papers by the author and others establishing theorems in approximation theory through polyharmonic functions which are analogs to one-dimensional approximation theory through polynomials. This area is called by the author the *polyharmonic paradigm*.

- Finally, we have to say that a major motivation to write “yet another theory in analysis” was the experimental success of the polysplines. Experiments with data from magnetism have shown that polysplines give much better results than the Briggs algorithm (minimum curvature), kriging, and RBFs functions which are very popular in that area. The experiments with data in CAGD have shown the superiority of the polysplines even over the usual spline methods, for which CAGD is a priority area.
- The exposition is by no means uniform. Some places are written at an extremely elementary level, things are oversimplified, especially where algorithmic questions are concerned, in order to allow a constructively thinking reader to be able to make the algorithm. Other places, in particular those devoted to the existence theory and applications of Sobolev spaces etc., are intended for the more advanced reader and may be rather tough.
- Many of the results in this book are appearing for the first time. If it is not indicated that the result belongs to someone else, then the result is new.
- We wish to help the reader to find references to a formula, theorem, or chapter–section–subsection, by providing a *page number* to most of these references.
- The name *spline* is associated with smoothness, or smoothly joining pieces of analytic entity. This will be our point of view for the generalization which we plan to do. For that reason it is clear that we have to employ the notions of multivariate smoothness as Sobolev and Hölder spaces. This makes things a lot more complicated than the one-dimensional case but it could not be otherwise if one wants to obtain a genuine multivariate spline theory.
- Many of the books or papers cited here have Russian, French, or other versions. For that reason we quote the number of the theorem, proposition, lemma, section etc., which is independent of the translation.

1.4 Acknowledgements

Many results in the present book were obtained by the author during his stay as a Humboldt Research Fellow at the University of Duisburg in 1992–1994, and later by a

grant from the VW-Foundation in 1996–1999. The author wishes to express his gratitude to the Department of Mathematics, University of Duisburg, and especially to Professor Werner Haussmann, for the scientific collaboration and hospitality. The author has lectured over parts of the book as a Visiting Professor at the University of Hamburg in 1997 and 1999, for which he is very grateful to Professor Wolf Hoffmann. The Working Group on Elliptic Boundary Value Problems (formerly at the Max Planck Society) of Professor Bert-Wolfgang Schulze at the University of Potsdam has often played host to the author since 1992, and many solutions and new ideas for the future have taken shape near Sanssouci.

- Special thanks are due to Dr. Hermann Render (Department of Mathematics, University of Duisburg), for many useful remarks on the whole text. Also some basic Lemmata in the Chapter on interpolation cardinal polysplines are due to him, as well as proofs of missing argumentation at several places in Part III on wavelet analysis. Dr. Vladimir Nikiforov has checked carefully the text of Part I and made some useful linguistic remarks.
- At the early stages of the polysplines when they needed to prove practical efficiency, I got a good encouragement from Professor Erik Grafarend (Institute of Geodesy, University of Stuttgart). The first data set, that of airborne data from the Cobb Offset, has been provided to me by Dr. Richard O. Hansen (formerly Professor at Colorado School of Mines, now at Pearson & de Ridder & Johnson in Denver, Colorado), as well as the first “hurrah” confirming that polysplines perform definitely better than Kriging, Minimum Curvature, Radial Basis Functions, etc. I enjoyed also the enthusiasm and the support in working with practical data from Dr. Dimiter Ouzounov (currently at NASA, Goddard Space Flight Research Center), by the time he was at the Institute of Geophysics, Bulgarian Academy of Sciences. I learned from him about the “Surfer” and “Grapher” – both Golden Software products.
- In 1992 Kurt Jetter (then Professor at University of Duisburg, now at University of Hohenheim) invited me to give the first lectures on polysplines at the Seminar of Approximation Theory. In 1993 Kurt was really willing to see the *Polywavelets*. So far the expectation was that there exists a *refinement equation*. Who knew that no such exists? A rather more complicated structure as the *refinement operator* is the reality.
- During the galley-proof period of the present project the author enjoyed the spline-wavelet atmosphere at the University of Wisconsin at Madison thanks to the hospitality of Professors Carl de Boor and Amos Ron. Some improvements in the text are resulting from the experience gained at lecturing about polyharmonic functions and polysplines at the seminar on Approximation theory.
- I used the wonder of Internet by e-mailing from time to time Carl de Boor, Larry Schumaker and Charles Micchelli, and got fast and competent advice on cardinal splines and L -splines.
- I profited many times from the TEX and computer experience of Dr. Hermann Hoch (University of Duisburg) and Dr. Pencho Marinov (Institute of Parallel Computations, Bulgarian Academy of Sciences).

1.5 The polyharmonic paradigm

The polysplines of the present book are a result of the application of the so-called polyharmonic paradigm to spline analysis.

Until recently the *polynomial paradigm* has been a dominant concept in the multivariate constructive theory of functions. The polynomial paradigm comprises all theories which rely upon multivariate polynomials or “tensor product” constructs.

Let us briefly explain what will be understood by the term *polyharmonic paradigm*. Its aim is to distinguish the polyharmonic functions as a proper multivariate generalization of the *one-dimensional polynomials*. The main thesis is that the polyharmonic functions have to be used as a main building block in the multivariate constructive theory of functions. The polyharmonic paradigm has to be understood as an alternative to the polynomial paradigm.

The polyharmonic paradigm consists of *operator*, *object* and *data concepts* which are related, and which we briefly describe below.

1.5.1 The operator, object and data concepts of the polyharmonic paradigm

The *operator concept* consists of replacing the operator d/dt and its powers in one dimension through the Laplace operator Δ and its powers (called polyharmonic operators), or symbolically written

$$\begin{aligned} \frac{d}{dt} &\longrightarrow \Delta, \\ \frac{d^p}{dt^p} &\longrightarrow \Delta^p. \end{aligned} \tag{1.1}$$

The *objects concept* consists of replacing the one-dimensional polynomials which are solutions to the equation

$$\frac{d^p}{dt^p} f(t) = 0$$

with the solution of the polyharmonic equation

$$\Delta^p f(x) = 0.$$

The function t_+^{p-1} defined by

$$t_+^{p-1} := \begin{cases} t^{p-1} & \text{for } t \geq 0, \\ 0 & \text{for } t \leq 0, \end{cases}$$

satisfies

$$\frac{d^p}{dt^p} t_+^{p-1} = \delta(t)$$

where δ is the *Dirac* delta function in \mathbb{R} . We will replace it with the function $R_p(x)$ which satisfies

$$\Delta^p R_p(x) = \delta(x)$$

where δ is the Dirac delta function in \mathbb{R}^n . One look at the explicit formulas for R_p (see [3], or the solutions to the operator $L_{(0)}^p$ in Proposition 10.32, p. 168) shows how similar they are to the functions t_+^{p-1} .

The *data concept* consists of replacing the discrete set of data points $\{t_1, t_2, \dots, t_N\}$ in the one-dimensional case with surfaces $\{\Gamma_1, \Gamma_2, \dots, \Gamma_N\}$ of dimension $n - 1$ in \mathbb{R}^n , or

$$t_j \longrightarrow \Gamma_j.$$

In view of the *polyharmonicity paradigm* one may reconsider the basic notions of mathematical analysis in several dimensions, like *finite* (or *divided*) *differences*, *multivariate Taylor formula* etc.

We will illustrate the richness of the above formal side of the polyharmonic paradigm by considering the multivariate analogs of the Taylor formula.

1.5.2 The Taylor formula

In the case of the one-dimensional Taylor formula

$$\begin{aligned} f(t) = & f(t_0) + \frac{t-t_0}{1!} f'(t_0) + \frac{(t-t_0)^2}{2!} f''(t_0) + \dots + \frac{(t-t_0)^{p-1}}{(p-1)!} f^{(p-1)}(t_0) \\ & + \frac{1}{(p-1)!} \int_{t_0}^t f^{(p)}(\tau) (t-\tau)_+^{p-1} d\tau \end{aligned}$$

we find that the *Almansi*, *Pizzetti* and *Green* formulas are also *genuine analogs* to it.

The Almansi formula

Let $f(x)$ be a polyharmonic function in the ball $B = B(0; R)$. Then the Almansi formula reads as

$$f(x) = f_0(x) + |x|^2 f_1(x) + |x|^4 f_2(x) + \dots + |x|^{2p-2} f_{p-1}(x) \quad \text{for } x \in \mathbb{R}^n,$$

where $f_k(x)$ are harmonic functions in the ball B . Compared with the Taylor formula, it is obtained by the scheme

$$\frac{d}{dt} \longrightarrow \Delta, \quad \frac{d^p}{dt^p} \longrightarrow \Delta^p,$$

and since the constants $f^{(k)}(t_0)$ are solutions of

$$\frac{d}{dt} (f^{(k)}(t_0)) = 0,$$

they are replaced by solutions of the Laplace equation,

$$\Delta f_k(x) = 0.$$

The Pizzetti formula

This reads as

$$\int_{S(x_0; R)} f d\sigma(y) = f(x_0) + a_{n1} |x - x_0|^2 \Delta f(x_0) + a_{n2} |x - x_0|^4 \Delta^2 f(x_0) + \dots$$

for all $x \in S(x_0; R)$, where a_{nj} are some coefficients depending only on the dimension of the space n . Due to $|x - x_0| = R$ the replacement scheme is

$$\begin{aligned} \text{data point } t_0 &\longrightarrow \text{data surface } S(x_0; R) \\ \text{point value at } t_0 &\longrightarrow \int_{S(x_0; R)} f d\sigma(y), \\ \text{value } \frac{d^k}{dt^k} f(t_0) &\longrightarrow \Delta^k f(x_0). \end{aligned}$$

The Green formula

Let $f(x)$ be a polyharmonic function of order p in the domain D having sufficiently smooth boundary. Assume that f belongs to $C^{2p}(\bar{D})$. Then the Green formula reads as

$$f(x) = \Omega_n \sum_{j=0}^{p-1} \int_{\partial D} \left\{ \Delta^j f(z) \frac{\partial R_{j+1}(z-x)}{\partial v_z} - \frac{\partial \Delta^j f(z)}{\partial v_z} R_{j+1}(z-x) \right\} d\sigma(z),$$

see Aronszajn–Creese–Lipkin [3, p. 9], or formula (23.6) on p. 479. Here R_j is the properly normalized fundamental solution to the equation $\Delta^j R_j(x) = \delta(x)$, and Ω_n is a constant (in (10.3), p. 130); we have denoted by $\frac{\partial}{\partial v_z}$ the inward normal derivative.

The above formula is obtained through the rather unusual scheme

$$\begin{aligned} \text{data point } t &\longrightarrow \text{data point } x, \\ \text{value } f(t) &\longrightarrow \text{value } f(x), \\ \text{data point } t_0 &\longrightarrow \text{data surface } \partial D, \\ \text{value } \frac{d^{2j}}{dt^{2j}} f(t_0) &\longrightarrow \int_{\partial D} \Delta^j f(z) \frac{\partial R_{j+1}(z-x)}{\partial v_z} d\sigma(z), \\ \text{value } \frac{d^{2j+1}}{dt^{2j+1}} f(t_0) &\longrightarrow - \int_{\partial D} \frac{\partial \Delta^j f(z)}{\partial v_z} R_{j+1}(z-x) d\sigma(z). \end{aligned}$$

Note that the k 'th term

$$u_k(x) = \Omega_n \sum_{j=0}^k \int_{\partial D} \left\{ \Delta^j f(z) \frac{\partial R_{j+1}(z-x)}{\partial v_z} - \frac{\partial \Delta^j f(z)}{\partial v_z} R_{j+1}(z-x) \right\} d\sigma(z)$$

satisfies

$$\Delta^{k+1}u_k(x) = 0 \quad \text{in } D.$$

Thus we see that we have different ways to make a multivariate analog to the Taylor formula. When we generalize univariate results we have to choose a genuine generalization by finding the proper scheme.