Part II

Cardinal polysplines in $\mathbb{R}^n$

The main purpose of Part II is to find and develop a proper polyspline analog to the notion of the cardinal splines. By definition, in the one-dimensional case the cardinal splines are those having knots at the integer points, or somewhat more generally, at the points $\alpha + \beta j$ where $j \in \mathbb{Z}$, for some fixed numbers α and β. There is a very beautiful theory, which was mainly developed by Schoenberg, the main results being summarized in his short monograph [18].[2] The results in this theory may be considered as a part of *harmonic analysis* due to the fact that the basic cardinal splines may be viewed as a Fourier transform of the function

$$\left(\frac{\sin \xi/2}{\xi/2}\right)^k .$$

Let us start with the *polysplines on strips*. Now trying to invent our *polyspline Ansatz*[1] let us imagine that we have polysplines on infinitely many strips, i.e. the *knot-surfaces* are infinitely many parallel hyperplanes. It seems very natural to term "cardinal" those polysplines which have equidistant hyperplanes. It is not very difficult to see that this is indeed a proper *Ansatz* and one may obtain many results by generalizing the one-dimensional case.[2]

For *polysplines on annuli*, i.e. when the knot-surfaces are infinitely many concentric spheres $S(0; r_j)$, finding the proper Ansatz is a real intellectual challenge. Its answer is far from evident but it is interesting that it is unique! The hint to the answer is hidden in the representation of polyharmonic functions in the annulus in Corollary 10.38, p. 173. By this corollary if $h(x)$ satisfies $\Delta^p h(x) = 0$ in the annulus $A_{r_j, r_{j+1}}$ and belongs to L_2 then $h(x)$ has the representation

$$h(x) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d_k} f_{k,\ell}(\log r) Y_{k,\ell}(\theta) \quad \text{for } r_j < r < r_{j+1},$$

where the one-dimensional function $f_{k,\ell}(v)$ is a solution to the equation

$$M_{k,p}\left(\frac{d}{dv}\right) f_{k,\ell}(v) = 0 \quad \text{for } r_j < e^v < r_{j+1}.$$

Recall that by formula (10.26), p. 169, the operator $M_{k,p}$ has constant coefficients. Furthermore, we have seen in Part I and more specially in Theorem 9.7, p. 124, that h is a polyspline if and only if for every two indexes k and ℓ the function $f_{k,\ell}(v)$ is an L-spline for the operator $L = M_{k,p}$! Now the question is whether we have a reasonable "cardinal" theory of such L-splines? Yes, we do! It has been developed by Micchelli [12, 13]. Some of the results have been given concise and elementary proofs by Schoenberg [19] in the same volume.

Eureka! We will call h a "cardinal polyspline on annuli" if all components $f_{k,\ell}$ are cardinal L-splines with knots at $j \in \mathbb{Z}$. Thus we see that the "break-radii" have to satisfy $r_j = e^j$, hence the break-surfaces for the polyspline h will be the spheres $S(0; e^j)$.

[1] The meaning of *Ansatz* was discussed in the footnote on p. 32.

[2] Due to the lack of space we omit the consideration of the cardinal polysplines on strips. We treat in detail only the case of the technically more complicated cardinal polysplines on annuli. The reader will be able to follow the same scheme and produce similar results for the cardinal polysplines on strips.

It now becomes clear to the reader what the motivation was for the compendium on representation of polyharmonic functions in the annulus, and further what the motivation is to have an exposition of the results of Micchelli on cardinal L-splines coming in the next chapter.

Last but not least the one-dimensional cardinal splines serve as a basic example for the wavelet analysis. We plan to mimic this construction by using *cardinal polysplines*. Thus a major motivation for the detailed study of the cardinal theory of polysplines in the present Part is their application to "polyharmonic wavelet analysis" in Part III.

Finally, we want to warn the reader that there will be some weak overlapping of the notations in some Chapters of the present Part. Following the tradition by $L(x)$ we will sometimes denote the fundamental spline function of Schoenberg and this may be mixed with the operator L for the L-spline. This overlapping is indeed very weak and we prefer to retain the original notations of Schoenberg. We will eventually repeat this warning at the proper place.

Chapter 13

Cardinal L-splines according to Micchelli

In the present chapter we provide an extended study of the cardinal L-splines following the approach of Ch. Micchelli, including results by I. Schoenberg, Dyn and Ron.[1]

13.1 Cardinal L-splines and the interpolation problem

The theory of cardinal splines and more specifically cardinal L-splines is a beautiful area of spline analysis which deserves much attention in view of its recent applications to wavelet analysis.

Within the general theory of splines the *theory of cardinal splines*, or the splines having only integers as knots, plays a very important and specific role. First, technically it may be considered more as a subset of harmonic analysis than of the general spline theory. Indeed, one may view the whole theory as study of Fourier inverse of functions of the type

$$\left(\frac{\sin \xi/2}{\xi/2}\right)^m = \left(\frac{e^{i\xi/2} - e^{-i\xi/2}}{i\xi}\right)^m = e^{i\xi m/2}\left(\frac{1 - e^{-i\xi}}{i\xi}\right)^m .$$

Let us denote by Q_m the usual polynomial B-spline of degree m with knots at the points $\{0, 1, \ldots, m\}$ and with support coinciding with the interval $[0, m]$, and let us introduce the "centralized" spline $M_m(x) = Q_m(x + m/2)$, having support $[-m/2, m/2]$ and knots at $\{-m/2, -m/2 + 1, \ldots, m/2\}$. Then by the properties of the Fourier transform

[1] In view of the terminology that has been established, see Chapter 11 and *Schumaker* [22], it would be more appropriate to use the name "cardinal Chebyshev splines" since the theory of Micchlli only concerns operators L having constant coefficients, and the corresponding splines are Chebyshev splines.

we obtain the following equality [18, pp. 11,12]:

$$\widehat{Q_m}(\xi) = \left(\frac{1 - e^{-i\xi}}{i\xi}\right)^m = \left(\widehat{Q_1}(\xi)\right)^m ,$$

$$\widehat{M_m}(\xi) = e^{i\xi m/2}\widehat{Q_m}(\xi) = \left(\frac{2\sin(\xi/2)}{\xi}\right)^m = \left(\widehat{M_1}(\xi)\right)^m .$$

By taking the inverse Fourier transform we see that

$$Q_m(x) = Q_{m-1}(x) * Q_1(x) = \int_0^1 Q_{m-1}(x - y)\, dy, \tag{13.1}$$

hence, we have a simple constructive and inductive definition of the compactly supported spline $Q_m(x)$.

The cardinal L-splines with compact support (TB-splines) which we will study may also be considered as a part of harmonic analysis since their Fourier transforms are given by

$$\widehat{Q_m}(x) = \frac{\prod_{j=1}^m (e^{-\lambda_j} - e^{-i\xi})}{\prod_{j=1}^m (i\xi - \lambda_j)},$$

where λ_j are real constants. Thus we have in a similar way

$$Q_{m-1}(x) = Q_{m-1}(x) * Q_1(x),$$

which provides a simple method to generate the most important function of the whole theory. So far this visual simplicity is only superficial.

The reader should be aware that one may start reading the present chapter from Section 13.10, where the compactly supported splines are introduced, since for the majority of standard numerical work one does not need much more. However, as will become clear in Chapter 14, p. 267, in order to understand the deeper properties of the functions Q_m, which are further necessary for the *wavelet analysis* in Part III, one really needs the whole theory developed in the present chapter. In particular, one needs the notions of Euler polynomials $A_m(x; \lambda)$ and the Euler–Frobenius polynomials $\Pi_m(\lambda) = A_m(0; \lambda)$, the location of their zeros etc.

The theory of polynomial cardinal splines, including the theory of the *Euler–Frobenius* and *Euler polynomials* related to them, was developed mainly by Schoenberg till the mid-1970s. He has summarized almost all the results in his fascinating book [18].[2]

During the last decade there has been renewed interest in cardinal splines in view of their applications to *wavelet theory*. One may even say that the cardinal splines were reborn in wavelet analysis in the works of Chui, [3], who generalized such a fundamental notion as the Euler–Frobenius polynomial for an arbitrary scaling function $\phi(x)$ generating a *multiresolution analysis*.

[2] One has to mention also the initiating work of Quade and Collatz [16], and that of Tchakaloff [35] of which Schoenberg was apparently not aware. More about the beautiful analytic work of Tchakaloff, which has been published in Bulgarian with a French summary, [2, p. 39].

As we have already said the theory of *cardinal L-splines* has been developed by Micchelli and Schoenberg.[3] The cardinal L-splines possess most of the advantageous properties of the usual (polynomial) cardinal splines. However, one important property which distinguishes the L-splines from the polynomial splines is that they are not scale-invariant, i.e. if $f(x)$ is an L-spline for some operator L then $f(\alpha x)$ is not an L-spline for the same operator. In the case of constant coefficient operators L the function $f(\alpha x)$ is still an L-spline, but for another operator. This is essentially used in the theory of *non-stationary wavelets* developed by de Boor *et al.* ([5] p. 150). The last is a construction which we will need for the wavelet analysis using *cardinal polysplines on annuli* to be treated in Part III of the present book, and for that reason all details of the one-dimensional construction will be studied here.

Cardinal L-splines are a basic tool for our study. For that reason we prefer to give an independent exposition of the theory, which does not refer to the fundamental theory of L-splines (and Chebyshev splines) with general knots developed by Schumaker [22], which we have already used in Part I. Such an exposition will give an opportunity for a reader who is mainly interested in wavelet analysis to have a complete and logically closed understanding of the subject. Let us note that the same results may be obtained [6, 7] by following the approach to L-splines of [22].

In the present section we will follow closely the approach and most of Micchelli's notations [12, 13]. Let us give some basic definitions and notations. Let the real numbers $\lambda_1, \ldots, \lambda_{Z+1}$ be given. We will consider the *nonordered* vector

$$\Lambda := \Lambda_{Z+1} := [\lambda_1, \lambda_2, \ldots, \lambda_{Z+1}], \tag{13.1a}$$

where some of the numbers λ_j may have repetitions. The number of repetitions of a number λ in Λ will be termed the **multiplicity** of λ.

There are different ways to give a good representation of such vectors Λ but they are all overburdened with indices. For example, we might write [22, p. 20]

$$t_1 \le t_2 \le \cdots \le t_m = \overbrace{\tau_1, \ldots, \tau_1}^{l_1}, \ldots, \overbrace{\tau_d, \ldots, \tau_d}^{l_d},$$

where $\sum_{i=1}^{d} l_i = m$. Another possibility is to put [2, p. 5]

$$(t_1, t_2, \ldots, t_m) = ((\tau_1, l_1), \ldots, (\tau_d, l_d)).$$

By using the notation $[\cdot]$ for such a vector we avoid having to describe the multiplicity of the entries every time. For almost all our purposes the representation by a nonordered vector Λ will be adequate.[4]

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[3] Micchelli constructs his theory in a way close to the meditative approach to cardinal splines developed by Schoenberg. This is based mainly on the Euler exponential spline.

[4] We note that a large part of the theory in the present chapter holds for complex numbers λ_j. In such a case the so-called W-property, associated with the name of Polya, holds only for intervals with bounded length, i.e. the set U_{Z+1} is not Chebyshev over arbitrary large intervals and one has to keep this in mind. See for examples, the comment of Schoenberg [19, p. 251]. The results which we need for the cardinal polysplines require no such generality, while the last would overburden some proofs.

Further we introduce the polynomial

$$q_{Z+1}(z) := q_{Z+1}[\Lambda](z) := \prod_{j=1}^{Z+1}(z-\lambda_j) \tag{13.2}$$

and the operator $\mathcal{L}_{Z+1}$ defined by

$$\mathcal{L}_{Z+1}[\Lambda]f(x) := q_{Z+1}\left(\frac{d}{dx}\right)f(x) = \prod_{j=1}^{Z+1}\left(\frac{d}{dx}-\lambda_j\right)f(x) \tag{13.3}$$

where, if it is clear from the context, we will drop the dependence on the set Λ and simply write $\mathcal{L}_{Z+1}f$ or q_{Z+1}.[5]

Let us introduce the set of solutions, sometimes called L-**polynomials**, over the whole real axis:

$$U_{Z+1} := U_{Z+1}[\Lambda] = \{u \text{ in } C^{\infty}(\mathbb{R}):\quad \mathcal{L}_{Z+1}[\Lambda]u(x) = 0 \text{ for } x \text{ in } \mathbb{R}\}.$$

As will be discussed in the sections below the fact that λ_j are real constants provides the following important properties:

1. The set U_{Z+1} is Chebyshev over the whole real axis, i.e. every $\varphi \in U_{Z+1}$ has no more than Z real zeros.
2. The set U_{Z+1} is translation invariant, i.e. if $\varphi \in U_{Z+1}$ then for every real number α we have $\varphi(x-\alpha) \in U_{Z+1}$.
3. The classical polynomial case is obtained as a special case, when $\lambda_1 = \lambda_2 = \cdots = \lambda_{Z+1} = 0$. In this case we have the following:

$$q_{Z+1}(z) = z^{Z+1}, \quad \mathcal{L}_{Z+1}[\Lambda]f(x) = \frac{d^{Z+1}}{dx^{Z+1}}f(x),$$

and U_{Z+1} is the set of all polynomials of degree $\leq Z$.

We are using the notation $Z+1$ in order to make our notation consistent with the standard one-dimensional polynomial case. As is known, the dimension of the space U_{Z+1} is

$$\dim U_{Z+1} = Z+1,$$

but in the polynomial case the degree of the polynomials is $\leq Z$.

Definition 13.1 *The class of* ***cardinal*** L***-splines*** *for the operator* $\mathcal{L}_{Z+1}[\Lambda]$ *is defined as the set of those functions* $u(x) \in C^{Z-1}(\mathbb{R})$ *which on every interval* $(j, j+1)$ *is a solution of* $\mathcal{L}_{Z+1}[\Lambda]u(x) = 0$*, i.e.*

$$\mathcal{S}_{Z+1} := \mathcal{S}_{Z+1}[\Lambda] := \{u \text{ in } C^{Z-1}(R):\ u_{|(j,j+1)} \text{ in } U_{Z+1} \text{ for all } j \in \mathbb{Z}\}. \tag{13.4}$$

[5] As we will see below, the global C^{∞} solutions of $\mathcal{L}_{P+1}f(x) = 0$ are linear combinations of expressions $R_j(x)\cdot e^{\lambda_j x}$, where $R_j(x)$ is a polynomial with $\deg R_j \leq$ (multiplicity of t_j) -1. For that reason these splines are sometimes called *exponential*. This terminology should not be mixed with the so-called "exponential Euler spline" of Schoenberg which we will meet below.

In this definition by $g_{|(j,j+1)}$ we have, as usual, denoted the restriction of the function g to the interval $(j, j+1)$. In order to save notation, by $g_{|(j,j+1)} \in U_{Z+1}$ we mean that $g_{|(j,j+1)}$ is a restriction of an element of U_{Z+1}.

(We use q_{Z+1} for the polynomial instead of p_{Z+1} of Micchelli [12, 13]. We also write $\mathcal{S}_{Z+1}$ for the space of splines instead of $\mathcal{S}_Z$. Let us note again that his notation [12, 13] tends to preserve the tradition of the polynomial splines where the *degree* of the polynomials is Z and the dimension of the space is $Z+1$. We put as a central index $Z+1$ instead.)

The main problem solved by Schoenberg for polynomial splines and by Micchelli for the above introduced L-splines is the so-called *cardinal interpolation problem*. They have found the conditions within which the problem:

$$u(j+\alpha) = y_j \quad \text{for all } j \text{ in } \mathbb{Z}, \tag{13.5}$$

has a solution u in $\mathcal{S}_{Z+1}$. Here α is a constant such that $0 \le \alpha < 1$.[6] In order to formulate the complete solution for (13.5) we need the class of null L-splines, which is defined as

$$\mathcal{S}^0_{Z+1} := \{u \text{ in } \mathcal{S}_{Z+1} \colon u(j+\alpha) = 0 \text{ for all } j \text{ in } \mathbb{Z}\}.$$

We always assume that α is fixed. The following result is basic in Micchelli [12, 13, p. 204], and Schoenberg [19]. It generalizes the classical result of Schoenberg about cardinal interpolation through polynomial splines from his book [18].

Theorem 13.2 *1. The space $\mathcal{S}^0_{Z+1}$ has dimension*

$$\dim(\mathcal{S}^0_{Z+1}) = \begin{cases} m = Z-1 & \text{for } \alpha = 0, \\ m = Z & \text{for } 0 < \alpha < 1. \end{cases}$$

2. The space $\mathcal{S}^0_{Z+1}$ is spanned by m eigensplines $S_1, S_2, \ldots, S_m$ which satisfy the equation

$$S_i(x+1) = \tau_i S_i(x) \quad \text{for } i = 1, \ldots, m,$$

where the constants τ_i (called the eigenvalues of the problem) satisfy

$$\tau_1 < \tau_2 < \cdots < \tau_m < 0.$$

3. Let $\tau_i \neq -1$ for $i = 1, \ldots, m$. Then there exists a fundamental cardinal L-spline $L(x) \in \mathcal{S}_{Z+1}$, i.e. a spline such that[7]

$$L(j+\alpha) = \begin{cases} 0 & \text{for } j \neq 0, \\ 1 & \text{for } j = 0. \end{cases}$$

[6] The results about solubility of this problem are extended by Schoenberg [19] without any extra effort to the cardinal grid $\{jh + \alpha \colon \text{ for all } j \text{ in } \mathbb{Z}\}$, where $h > 0$ is an arbitrary constant.

[7] As we said already in the Introduction to this Part, the reader does not have to mix this L with the operator L. We have preserved Schoenberg's original notation which was also used by Micchelli and Chui.

There exist positive constants A, B such that

$$|L(x)| \leq Ae^{-B|x|} \quad \textit{for all } x \textit{ in } \mathbb{R}.$$

4. Let $\tau_i \neq -1$ for $i = 1, \ldots, m$. Let the sequence y_j be of power growth, i.e. for some $\gamma \geq 0$ it satisfies $y_j = O(|j|^\gamma)$. Then there exists a unique $u \in \mathcal{S}_{Z+1}$ which has a power growth, i.e.

$$|u(x)| = O(|x|^\gamma) \quad \textit{for all } x \textit{ in } \mathbb{R},$$

and which interpolates the data y_j, i.e.

$$u(j + \alpha) = y_j \quad \textit{for all } j \textit{ in } \mathbb{Z}.$$

We have the representation

$$u(x) = \sum_{i=-\infty}^{\infty} y_j L(x - j).$$

Later, in Theorem 13.33, p. 238, we will provide another criterion for solving the cardinal interpolation problem. One of our main purposes in Part II will be to find an analog to the above theorem for cardinal polysplines.

13.2 Differential operators and their solution sets U_{Z+1}

Let us introduce some operators decomposing the operator $\mathcal{L}_{Z+1}$ of Section 13.1.

When the nonordered vector $\Lambda = [\lambda_1, \lambda_2, \ldots, \lambda_{Z+1}]$ is given we define the following operators:

$$\mathcal{D}_j f(x) := \left(\frac{d}{dx} - \lambda_j\right) f(x) = e^{\lambda_j x} \frac{d}{dx} e^{-\lambda_j x} f(x) \quad \text{for } j = 1, \ldots, Z+1, \qquad (13.6)$$

$$\mathcal{D}_0 f(x) := f(x).$$

Evidently, for every integer $s \geq 1$ we have

$$[\mathcal{D}_j]^s f(x) = e^{\lambda_j x} \frac{d^s}{dx^s} e^{-\lambda_j x} f(x).$$

For every integer $s \geq 1$ we will define the following differential operators:

$$\mathcal{L}_s f(x) := \mathcal{D}_1 \cdots \mathcal{D}_s f(x),$$

$$\mathcal{L}_0 f(x) := f(x).$$

As was said above, the space of C^∞ solutions of the equation

$$\mathcal{L}_{Z+1} f(x) = 0 \quad \text{for } x \text{ in } \mathbb{R},$$

which we have denoted by $U_{Z+1}[\Lambda]$ will be important.

In order to develop some intuition in the reader who is not experienced in differential equations, we provide the following simple, standard facts from the theory of ODEs concerning the space $U_{Z+1}[\Lambda]$, see Pontryagin [15].

Example 13.3 $\dim U_{Z+1} = Z + 1$.

Example 13.4 *If*

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{Z+1} = 0$$

then

$$\mathcal{L}_{Z+1} f(x) = \frac{d^{Z+1}}{dx^{Z+1}} f(x)$$

and U_{Z+1} is the set of all algebraic polynomials of degree $\leq Z$, i.e.

$$U_{Z+1} = \{1, x, x^2, \ldots, x^Z\}_{\mathrm{lin}}.$$

Here $\{\cdot\}_{\mathrm{lin}}$ denotes the linear hull of the set of functions inside the brackets.

Example 13.5 *If all λ_j are pairwise different, i.e. $\lambda_i \neq \lambda_j$ for $i \neq j$, then*

$$U_{Z+1} = \{e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_{Z+1} x}\}_{\mathrm{lin}}.$$

Example 13.6 *The constants belong to the set $U_{Z+1}[\Lambda]$ if and only if there exists an index j for which $\lambda_j = 0$.*

Example 13.7 *If*

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{Z+1}$$

then the set $U_{Z+1}[\Lambda]$ coincides with all algebraic polynomials of degree $\leq Z$, times $e^{\lambda_1 x}$, i.e.

$$\begin{aligned} U_{Z+1}[\Lambda] &= \{e^{\lambda_1 x}, xe^{\lambda_1 x}, \ldots, x^Z e^{\lambda_1 x}\}_{\mathrm{lin}} \\ &= \{R(x)e^{\lambda_1 x} : R \textit{ is a polynomial of } \deg R \leq Z\}. \end{aligned}$$

Example 13.8 *More generally, let the set Λ be given by*

$$\Lambda = \left[\underbrace{\tau_1, \tau_1, \ldots, \tau_1}_{m_1}, \underbrace{\tau_2, \tau_2, \ldots, \tau_2}_{m_2}, \ldots, \ldots, \ldots, \underbrace{\tau_\ell, \tau_\ell, \ldots, \tau_\ell}_{m_\ell} \right],$$

where $m_1 + m_2 + \cdots + m_\ell = Z + 1$. Then

$$U_{Z+1}[\Lambda] = \{R_1(x)e^{\tau_1 x}, R_2(x)e^{\tau_2 x}, \ldots, R_\ell(x)e^{\tau_\ell x}\}_{\mathrm{lin}}$$

where the polynomials R_j satisfy

$$\deg R_1(x) \leq m_1 - 1, \ \deg R_2(x) \leq m_2 - 1, \ldots, \ \deg R_\ell(x) \leq m_\ell - 1.$$

13.3 Variation of the set $U_{Z+1}[\Lambda]$ with Λ and other properties

Let us see how the set $U_{Z+1}[\Lambda]$ changes with the variation of Λ. If all values of λ_j are pairwise different, we have seen in Example 13.5, p. 227, that $U_{Z+1}[\Lambda] = \{e^{\lambda_1 x}, e^{\lambda_2 x}, \dots, e^{\lambda_{Z+1} x}\}_{\text{lin}}$. Now let $\lambda_2 \to \lambda_1$. Obviously, for $\lambda_2 \neq \lambda_1$ we have

$$U_{Z+1}[\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_{Z+1}] = \left\{ e^{\lambda_1 x}, \frac{e^{\lambda_1 x} - e^{\lambda_2 x}}{\lambda_1 - \lambda_2}, e^{\lambda_3 x}, \dots, e^{\lambda_{Z+1} x} \right\}_{\text{lin}},$$

which in the limit $\lambda_2 \to \lambda_1$ gives

$$U_{Z+1}[\lambda_1, \lambda_1, \lambda_3, \dots, \lambda_{Z+1}] = \{e^{\lambda_1 x}, x e^{\lambda_1 x}, e^{\lambda_3 x}, \dots, e^{\lambda_{Z+1} x}\}_{\text{lin}}.$$

In this way we obtain the sets U_{Z+1} in Examples 13.7 and 13.8.

This kind of limiting process will often be used below – the reason is that several formulas are much simpler to write in the case of pairwise different λ_js. Then, using the above limiting argument, we will also obtain the result in the case of arbitrary λ_js.

Theorem 13.9 *The space $U_{Z+1}[\Lambda]$ is translation invariant, i.e. if $\varphi(x)$ belongs to $U_{Z+1}[\Lambda]$ then $\varphi(x-c)$ belongs to $U_{Z+1}[\Lambda]$ for every real number c. The space $U_{Z+1}[\Lambda]$ is not scaling invariant, i.e. if $\varphi(x)$ belongs to $U_{Z+1}[\Lambda]$ then in general it does not follow that $\varphi(hx)$ belongs to $U_{Z+1}[\Lambda]$ for arbitrary real number h.*

The first statement is due to the fact that

$$e^{\lambda_j (x-c)} = e^{-\lambda_j c} \cdot e^{\lambda_j x} \text{ belongs to } U_{Z+1}[\Lambda],$$

and the last is true since only for $h = \lambda_i / \lambda_j$, we have

$$e^{h \lambda_j x} \text{ belongs to } U_{Z+1}[\Lambda].$$

So far, if we consider another operator, namely

$$\mathcal{L}_{Z+1}[hT] = \prod_{j=1}^{Z+1} \left(\frac{d}{dx} - h t_j \right),$$

then evidently

$$e^{h t_j x} \text{ belongs to } U_{Z+1}[hT].$$

Here we have used the notation for the nonordered vector

$$hT := [h t_1, \dots, h t_{Z+1}].$$

This simple fact will be used further in the wavelet analysis.

Theorem 13.10 *The space $U_{Z+1}[\Lambda]$ is* Chebyshev *on the whole real line, i.e. if $\varphi(x)$ belongs to $U_{Z+1}[\Lambda]$ then φ has no more than Z real zeros.*

Theorem 13.10 is in fact a reformulation of Theorem 11.4, p. 188, and is important to us.

Exercise 13.11 *Prove Theorem 13.10 in the case of pairwise different λ_js.*

Hint: *If some $\varphi(x) \in U_{Z+1}[\Lambda]$ has $Z+1$ different zeros $x_1, \ldots, x_{Z+1}$, then on every interval (x_j, x_{j+1}) we have a point ξ_j where $\varphi'(\xi_j) = \lambda_1 \varphi(\xi_j)$. Indeed, on the interval (x_j, x_{j+1}) the continuous function $\varphi'(x)$ changes its sign. However, on the same interval the continuous function $\lambda_1 \varphi(x)$ is zero at both endpoints. It follows that the function $\varphi'(x) - \lambda_1 \varphi(x)$ changes sign on the interval (x_j, x_{j+1}), hence, by Rolle's theorem there exists a $\xi_j \in (x_j, x_{j+1})$ such that $\varphi'(\xi_j) = \lambda_1 \varphi(\xi_j)$. Now the function $\varphi'(x) - \lambda_1 \varphi(x)$ belongs to $U_{Z+1}[\lambda_2, \lambda_3, \ldots, \lambda_{Z+1}]$. Proceed further using inductive reasoning.*

13.4 The Green function $\phi_Z^+(x)$ of the operator $\mathcal{L}_{Z+1}$

Here we introduce the so-called Green function associated with the operator $\mathcal{L}_{Z+1}$. This function is the analog to the function $(x-t)_+^Z$ in the polynomial case. We put

$$\phi_Z(x) := [\lambda_1, \lambda_2, \ldots, \lambda_{Z+1}]_z e^{xz},$$

where the index of the *divided difference* $[\lambda_1, \lambda_2, \ldots, \lambda_{Z+1}]_z$ means that it is taken with respect to the variable z. Let us note that, using the equivalent definition of divided difference through residuum, see Chapter 11, formula (11.12), p. 193, we have

$$\phi_Z(x) = \int_\Gamma \frac{e^{xz}}{q_{Z+1}(z)}\, dz, \tag{13.7}$$

where the contour Γ in the complex plane surrounds the zeros of the polynomial $q_{Z+1}(z)$. In particular, in the case of pairwise different λ_js we obtain

$$\phi_Z(x) = \sum_{j=1}^{Z+1} \frac{e^{\lambda_j x}}{q'_{Z+1}(\lambda_j)}. \tag{13.8}$$

We define the function $\phi_Z^+(x)$ as follows:

$$\phi_Z^+(x) := \begin{cases} \phi_Z(x) & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases}$$

The following result shows that the function $\phi_Z^+(x)$ is the Green function for the operator $\mathcal{L}_{Z+1}$.

Proposition 13.12 *The function $\phi_Z^+(x)$ is the* Green function *for the operator $\mathcal{L}_{Z+1}$, i.e. it satisfies the following three equivalent properties.*

1. $\phi_Z^+(x)$ *belongs to* $C^{Z-1}(\mathbb{R})$.

2. The following equalities hold:

$$\begin{cases} \mathcal{D}_0\mathcal{D}_1\cdots\mathcal{D}_\ell\phi_Z(x)_{|x=0+} = 0 & \text{for } \ell = 0,\ldots,Z-1, \\ \mathcal{D}_0\mathcal{D}_1\cdots\mathcal{D}_Z\phi_Z(x)_{|x=0+} = 1, \end{cases} \tag{13.9}$$

where the operators $\mathcal{D}_j$ *were defined in (13.6), p. 226.*

3. The equalities in (13.9) are equivalent to the following:

$$\frac{d^\ell}{dx^\ell}\phi_Z(x)_{|x=0+} = 0 \quad \text{for } \ell = 0,\ldots,Z-1,$$
$$\frac{d^Z}{dx^Z}\phi_Z(x)_{|x=0+} = 1.$$

The function $\phi_Z(x)$ *is also the unique element in* U_{Z+1} *which satisfies these equalities.*

Proof By the residuum representation (13.7), p. 229, we obtain, for $\ell = 0,\ldots,Z-1$, the equalities

$$\mathcal{D}_0\mathcal{D}_1\cdots\mathcal{D}_\ell\phi_Z(x)_{|x=0+} = \left[\int_\Gamma \frac{\mathcal{D}_0\mathcal{D}_1\cdots\mathcal{D}_\ell e^{xz}}{q_{Z+1}(z)}\,dz\right]_{|x=0_+}$$
$$= \int_\Gamma \prod_{j=1}^{\ell}(z-\lambda_j)\frac{1}{q_{Z+1}(z)}\,dz.$$

Taking for Γ the large circle $\Gamma_R = \{R\cdot e^{i\theta} : 0 \le \theta < \theta\}$, we obtain the estimate

$$|\mathcal{D}_0\mathcal{D}_1\cdots\mathcal{D}_\ell\phi_Z(x)_{|x=0+}| \le \int_0^{2\pi}\left|\prod_{j=1}^{\ell}(z-\lambda_j)\frac{1}{q_{Z+1}(z)}\right| R\,d\theta$$
$$\le \int_0^{2\pi}\frac{C}{R^{Z+1-\ell}}R\,d\theta = \frac{2\pi C}{R^{Z-\ell}}$$

which after letting $R \longrightarrow \infty$ proves the first part of (2).

Since

$$\left[\frac{\mathcal{D}_0\mathcal{D}_1\cdots\mathcal{D}_Z e^{xz}}{q_{Z+1}(z)}\right]_{|x=0+} = \frac{1}{z-\lambda_{Z+1}},$$

and by the Cauchy residuum theorem

$$\int_\Gamma \frac{1}{z-\lambda_{Z+1}}\,dz = 2\pi i,$$

we obtain the second equality in (2).

Point (3) follows easily by induction in s since

$$\prod_{j=1}^{s}\left(\frac{d}{dx}-\lambda_j\right) = \frac{d^s}{dx^s} + \sum_{j=0}^{s-1} c_j\frac{d^j}{dx^j}.$$

The uniqueness as stated follows since the dimension of the space $U_{Z+1}(\Lambda)$ is $Z+1$. ■

Let us denote by $\phi_Z^+[\lambda_1+\gamma,\ldots,\lambda_{Z+1}+\gamma](x)$ the Green function corresponding to the nonordered vector

$$\Lambda+\gamma := [\lambda_1+\gamma,\lambda_2+\gamma,\ldots,\lambda_{Z+1}+\gamma].$$

Proposition 13.13 *The Green function ϕ_Z^+ satisfies the following identity:*

$$\phi_Z^+[\Lambda+\gamma](x) = e^{\gamma x}\phi_Z^+[\Lambda](x).$$

Proof Since the function $e^{\gamma x}$ is C^∞ it follows that $e^{\gamma x}\phi_Z^+[\lambda_1,\lambda_2,\ldots,\lambda_{Z+1}](x) \in C^{Z-1}(R)$, hence, the function $e^{\gamma x}\phi_Z^+[\lambda_1,\lambda_2,\ldots,\lambda_{Z+1}](x)$ satisfies the first row of conditions in (13.9), p. 230. The last condition in (13.9) is satisfied owing to the Leibnitz formula for differentiation of a product

$$\begin{aligned}
&\left\{\prod_{j=1}^{Z}\left(\frac{d}{dx}-\lambda_j\right)\right\}[e^{\gamma x}\phi_Z^+[\lambda_1,\lambda_2,\ldots,\lambda_{Z+1}](x)]_{|x=0+}\\
&=\left\{\frac{d^Z}{dx^Z}+\sum_{j=0}^{Z-1}c_j\frac{d^j}{dx^j}\right\}[e^{\gamma x}\phi_Z^+[\lambda_1,\lambda_2,\ldots,\lambda_{Z+1}](x)]_{|x=0+}\\
&=\frac{d^Z}{dx^Z}[e^{\gamma x}\phi_Z^+[\lambda_1,\lambda_2,\ldots,\lambda_{Z+1}](x)]_{|x=0+}\\
&=\left[e^{\gamma x}\frac{d^Z}{dx^Z}\phi_Z^+[\lambda_1,\lambda_2,\ldots,\lambda_{Z+1}](x)\right]_{|x=0+}\\
&=1,
\end{aligned}$$

which completes the proof. ■

Exercise 13.14 *Prove the above theorem without residuum, assuming for simplicity that all λ_j are pairwise different.*

Hint: *Then*

$$\phi_Z(x) = \sum_{j=1}^{Z+1}\frac{1}{q_Z'(\lambda_j)}e^{\lambda_j x}.$$

and it follows that

$$\begin{aligned}
\mathcal{D}_0\mathcal{D}_1\cdots\mathcal{D}_Z\phi_Z(x)_{|x=0_+} &= \prod_{j=1}^{Z}(\lambda_{Z+1}-\lambda_j)\cdot\frac{1}{q_Z'(\lambda_{Z+1})}\cdot[e^{\lambda_{Z+1}x}]_{|x=0+}\\
&=\prod_{j=1}^{Z}(\lambda_{Z+1}-\lambda_j)\cdot\frac{1}{q_Z'(\lambda_{Z+1})}\\
&=1.
\end{aligned}$$

In Theorem 11.25, p. 200, we provided the basic result about the one-sided basis generated by the Green function for general Chebyshev splines. Here we specify that general result for the case of the cardinal splines $\mathcal{S}_{Z+1}$ defined in (13.4), p. 224.

Theorem 13.15 *Let us denote by $\mathcal{S}_{Z+1}[a, b]$ the L-splines in $\mathcal{S}_{Z+1}$ having break-points in the interval $[a, b]$ where a, b are integers. Then the set of shifts $\{\phi_Z^+[\Lambda](x - j) : j = a - 1, \ldots, b\}$ is a linear basis for the space $\mathcal{S}_{Z+1}[a, b]$.*

This is a classic result in spline theory and is proved simply by counting dimensions.

Corollary 13.16 *The function $\phi_Z^+(x) \in L_2(\mathbb{R})$ if and only if $\lambda_j < 0$ for all $j = 1, \ldots, Z + 1$.*

This will be used later in wavelet analysis using L-splines.

Exercise 13.17 *Prove Corollary 13.16.* Hint*: Use representation (13.8), p. 229, of $\phi_Z(x)$ in the case of pairwise different λ_j. Another possibility is to use representation (13.7).*

Remark 13.18 *1. Let us note that the function $\phi_Z^+(x)$ may be considered as an L-spline with the only knot the point 0. Since the dimension of U_{Z+1} is $Z + 1$ (as we have seen in Proposition 13.12, p. 229) this is the only L-spline with a knot at 0. This corollary means, roughly speaking, that the space of splines in $L_2(\mathbb{R})$ with the only knot 0 has dimension zero or one, and both cases are described.*

2. Let us note that due to the translation invariance of the space U_{Z+1} it follows that the function $\phi_Z^+[\Lambda](x - y)$ is the Green function associated with the operator $\mathcal{L}_{Z+1}$ *in the most general sense of this notion, see Section 11.3.1, especially formula (11.18) on p. 198.*

13.5 The dictionary: L-polynomial case

In order to make the transition from the classical polynomial case to the case of solutions of the operator $\mathcal{L}_{Z+1}$ called L-polynomials we provide the following dictionary of notions:

$$\frac{d^{Z+1}}{dx^{Z+1}} \longrightarrow \prod_{j=1}^{Z+1} \left(\frac{d}{dx} - \lambda_j \right),$$

$$\pi_Z \longrightarrow U_{Z+1},$$

$$(x - t)_+^Z \longrightarrow \phi_Z^+(x - t),$$

where, as usual, π_Z denotes the set of polynomials of degree $\leq Z$.

13.6 The generalized Euler polynomials $A_Z(x; \lambda)$

In the classical theory of cardinal splines the so-called *Euler* and *Euler–Frobenius* polynomials, see Schoenberg [18, p. 21], play a major role. These two notions may be generalized to the case of the differential operators $\mathcal{L}_{Z+1}$, and they also play an analogous role in the theory of cardinal L-splines.

Lemma 13.19 *Let $\lambda \neq e^{\lambda_i}$ for $i = 1, \dots, Z+1$. If for some function u in U_{Z+1} we have $u^{(j)}(1) = \lambda u^{(j)}(0)$ for $j = 0, 1, \dots, Z$, then $u \equiv 0$.*

Proof For $Z = 0$ we have $\Lambda = [\lambda_1]$ and the space $U_1[\Lambda]$ is one-dimensional with elements $Ce^{\lambda_1 x}$ for an arbitrary constant C. Hence, $u(1) = u(0)$ implies $\lambda = e^{\lambda_1}$ if $C \neq 0$.

For $Z \geq 1$ we obtain from the first two conditions

$$u(1) = u(0), \quad u'(1) = u'(0),$$

that

$$\left(\frac{d}{dx} - \lambda_{Z+1}\right) u(1) = \left(\frac{d}{dx} - \lambda_{Z+1}\right) u(0).$$

However, $((d/dx) - \lambda_{Z+1})u(x)$ belongs to $U_Z[\lambda_1, \dots, \lambda_Z]$, which shows that we may proceed inductively. ■

Let us note that the above proof is similar to the proof of the equivalence of points (2) and (3) in Proposition 13.12, p. 229, and we can see that conditions $u^{(j)}(1) = \lambda u^{(j)}(0)$ are equivalent to conditions $\mathcal{D}_0\mathcal{D}_1 \cdots \mathcal{D}_\ell u(1) = \lambda \mathcal{D}_0\mathcal{D}_1 \cdots \mathcal{D}_\ell u(0)$.

Thanks to the above lemma we may introduce the very important function $A(x; \lambda)$ of the theory developed by Micchelli which is a generalization of the classical *Euler polynomial* considered by Schoenberg [18, p. 21].[8] Let us first put

$$G(z; x, \lambda) := \frac{e^{xz}}{e^z - \lambda}.$$

Definition 13.20 *Let $\lambda \neq e^{\lambda_i}$ for $i = 1, \dots, Z+1$. Then the function $A_Z(x; \lambda) = A_Z[\Lambda](x; \lambda)$ is defined as the unique element in $U_{Z+1} = U_{Z+1}[\Lambda]$ which is a solution of the boundary value problem*

$$A_Z^{(j)}(1; \lambda) = \lambda A_Z^{(j)}(0; \lambda) \quad \textit{for } j = 0, \dots, Z-1,$$

$$A_Z^{(Z)}(1; \lambda) = \lambda A_Z^{(Z)}(0; \lambda) + 1.$$

It will be called the ***Euler polynomial****.*

In fact $A_Z(x; \lambda)$ is L-polynomial. Where necessary we will write $A_Z[\Lambda](x; \lambda)$ or $A_Z[\lambda_1, \lambda_2, \dots, \lambda_{Z+1}](x; \lambda)$ in order to stress the dependence in the nonordered vector Λ.

Recalling the operators $\mathcal{D}_j$ defined in (13.6), p. 226, we can state the most important properties of the function $A(x; \lambda)$.

[8] We have taken the notation $A(x; \lambda)$ from Schoenberg.

Theorem 13.21 *The function $A_Z[\Lambda](x;\lambda)$ satisfies the following properties:*

1. $\begin{cases} \mathcal{L}_i A_Z(1;\lambda) = \lambda \mathcal{L}_i A_Z(0;\lambda) & \textit{for } i = 0, 1, \ldots, Z-1, \textit{ and} \\ \mathcal{L}_Z A_Z(1;\lambda) = \lambda \mathcal{L}_Z A_Z(0;\lambda) + 1. \end{cases}$
2. $\mathcal{D}_{Z+1} A_Z[\lambda_1, \lambda_2, \ldots, \lambda_{Z+1}](x;\lambda) = A_{Z-1}[\lambda_1, \lambda_2, \ldots, \lambda_Z](x;\lambda)$.
3. $A_Z(x;\lambda) = [\lambda_1, \lambda_2, \ldots, \lambda_{Z+1}]_z G(z; x, \lambda)$, *the last being the divided difference with respect to the variable z.*
4. $A_Z(x+1;\lambda) - \lambda A_Z(x;\lambda) = \phi_Z(x)$.

Exercise 13.22 *Prove properties (1) and (2).*
Hint: *Use induction as in the proof of Lemma 13.19, p. 233.*

Proof Let us prove properties (3) and (4).

Assuming for simplicity that $\lambda_j \neq \lambda_i$ for $j \neq i$, we see that the space U_{Z+1} is spanned by the exponentials $e^{\lambda_j x}$, hence, for some constants σ_j we have

$$A_Z(x;\lambda) = \sum_{j=1}^{Z+1} \sigma_j e^{\lambda_j x}.$$

Now the conditions in Definition 13.20 give

$$\sum_{j=1}^{Z+1} \sigma_j \lambda_j^i (e^{\lambda_j} - \lambda) = 0 \quad \text{for } i = 0, \ldots, Z-1,$$
$$\sum_{j=1}^{Z+1} \sigma_j \lambda_j^Z (e^{\lambda_j} - \lambda) = 1,$$

which is a linear system with respect to σ_j which has a determinant, multiple of the Vandermonde

$$\prod_{j=1}^{Z+1} (e^{\lambda_j} - \lambda) \cdot \det[\lambda_j^i]_{i=0,j=1}^{Z,Z+1} = \prod_{j=1}^{Z+1} (e^{\lambda_j} - \lambda) \cdot \prod_{i<j} (\lambda_i - \lambda_j).$$

The solution is given by

$$\sigma_j = \frac{1}{(e^{\lambda_j} - \lambda)} \cdot \frac{1}{q'_{Z+1}(\lambda_j)},$$

which proves

$$A_Z(x;\lambda) = \sum_{j=1}^{Z+1} \frac{1}{q'_{Z+1}(\lambda_j)} \cdot \frac{e^{\lambda_j x}}{(e^{\lambda_j} - \lambda)} \tag{13.10}$$

which is exactly (3). From this formula property (4) follows directly. ∎

Exercise 13.23 *Prove (3) by checking directly that all conditions of Definition 13.20, p. 233, are satisfied by the function $[\lambda_1, \lambda_2, \ldots, \lambda_{Z+1}]_z G(z; x, \lambda)$.*

We have the following useful corollary.

Corollary 13.24 *For every* λ *such that* $\lambda \neq e^{\lambda_j}$ *for all* $j = 1, 2, \ldots, Z+1$, *the function* $A_Z(x; \lambda)$ *permits the residuum representation*

$$A_Z(x; \lambda) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{q_{Z+1}(z)} \frac{e^{xz}}{e^z - \lambda} dz, \tag{13.11}$$

where the closed contour Γ *surrounds the zeros of* $q_{Z+1}(z)$, *i.e. all elements of* $\Lambda = [\lambda_1, \lambda_2, \ldots, \lambda_{Z+1}]$, *and excludes the zeros of the function* $\exp(xz)/(e^z - \lambda)$.

The proof is due to the Frobenius representation of divided difference as residuum, see formula (11.12), p. 193.

Let us put

$$r(\lambda) := \prod_{j=1}^{Z+1} (e^{\lambda_j} - \lambda) = \sum_{j=0}^{Z+1} r_j \lambda^j, \tag{13.12}$$

$$s(\lambda) := \prod_{j=1}^{Z+1} (e^{-\lambda_j} - \lambda) = \sum_{j=0}^{Z+1} s_j \lambda^j. \tag{13.13}$$

We have the following important representation of the function $A_Z(x; \lambda)$.

Corollary 13.25 *The expression*

$$\Pi_Z(\lambda; x) := r(\lambda) A_Z(x; \lambda) \tag{13.14}$$

is a polynomial of degree $\leq Z$ *in* λ. *The polynomial* $\Pi_Z(\lambda; 0)$ *has degree* $\leq Z - 1$.

The polynomial $\Pi_Z(\lambda := \Pi_Z(\lambda; 0)$ is known as the **Euler–Frobenius polynomial**.

Corollary 13.25 follows directly from formula (13.10). We obtain the representation

$$\Pi_Z(\lambda; x) = \phi_Z(x)\lambda^Z + \cdots + e^{\lambda_1 + \cdots + \lambda_{Z+1}} \phi_Z(x - 1).$$

We see that the points $\lambda = e^{\lambda_j}$ are singular for the function $A_Z(x; \lambda)$. However, the polynomials $\Pi_Z(\lambda; x)$ also make sense for such values of the parameter λ.

Proposition 13.26 *Assume that all* $\lambda_s \in \Lambda$ *are pairwise different. Then for every* $x \in \mathbb{R}$ *and for every* $\lambda_s \in \Lambda$ *we have the following equality:*

$$\Pi_Z(e^{\lambda_s}; x) = -r'(e^{\lambda_s}) \cdot \frac{e^{\lambda_s x}}{q'_{Z+1}(\lambda_s)}. \tag{13.15}$$

Proof By the definition of the function in formula (13.14) we obtain

$$\begin{aligned} \Pi_Z(\lambda; x) = r(\lambda) A_Z(x; \lambda) &= \prod_{j=1}^{Z+1} (e^{\lambda_j} - \lambda) \cdot \sum_{j=1}^{Z+1} \frac{1}{q'_{Z+1}(\lambda_j)} \cdot \frac{e^{\lambda_j x}}{e^{\lambda_j} - \lambda} \\ &= \sum_{j=1}^{Z+1} \frac{e^{\lambda_j x}}{q'_{Z+1}(\lambda_j)} \cdot \prod_{\ell=1, \ell \neq j}^{Z+1} (e^{\lambda_\ell} - \lambda). \end{aligned} \tag{13.16}$$

Hence,

$$\Pi_Z(e^{\lambda_s}; x) = -r'(e^{\lambda_s}) \cdot \frac{e^{\lambda_s x}}{q'_{Z+1}(\lambda_s)},$$

which completes the proof. ∎

Exercise 13.27 *Find an expression for $\Pi_Z(\lambda; x)$ when the values of λ_j are not pairwise different as in equality (13.16). Consider the case when $Z+1 = 2p$ with $\lambda_1 = \cdots = \lambda_p$ and $\lambda_{p+1} = \cdots = \lambda_{2p}$.*

13.7 Generalized divided difference operator

In the general theory of Chebyshev splines presented in Section 11.2, p. 191, we have a divided difference operator which is not uniquely determined. It is important for the cardinal L-splines that the coefficients of the polynomials $r(\lambda)$ and $s(\lambda)$ determine an elegant expression for a *divided difference operator*.

Let us consider the polynomial

$$q^*_{Z+1}[\Lambda](z) := q^*_{Z+1}(z) := q_{Z+1}[-\Lambda](z) = \prod_{j=1}^{Z+1} (z + \lambda_j),$$

which evidently satisfies $q^*_{Z+1}(z) = (-1)^{Z+1} q_{Z+1}(-z)$ by the definition of $q^*_{Z+1}(z)$ in (13.2), p. 224.

Naturally, we will define by $U^*_{Z+1}[\Lambda]$ the space of C^∞-solutions of the equation

$$\mathcal{L}^*_{Z+1}[\Lambda] f := q^*_{Z+1}[\Lambda]\left(\frac{d}{dx}\right) f(x) = 0 \quad \text{for } x \text{ in } \mathbb{R}.$$

We have the following *divided difference operators* for cardinal L-splines.

Proposition 13.28 *If the coefficients r_j and s_j are those defined, respectively, in (13.12) and (13.13), p. 235, then for every function $f(x)$ in $U_{Z+1}[\Lambda]$*

$$\sum_{j=1}^{Z+1} r_j f(j) = 0. \tag{13.17}$$

Also, for every f in $U^*_{Z+1}[\Lambda]$

$$\sum_{j=1}^{Z+1} s_j f(j) = 0. \tag{13.18}$$

Proof For simplicity, we first assume that all λ_j are pairwise different. In such a case every solution to $\mathcal{L}_{Z+1} f(x) = 0$ is a linear combination of simple exponents

$$f(x) = \sum_{l=1}^{Z+1} \sigma_l e^{\lambda_l x}.$$

But for every $l = 1, \ldots, Z+1$

$$\sum_{j=1}^{Z+1} r_j e^{\lambda_l j} = \sum_{j=1}^{Z+1} r_j (e^{\lambda_l})^j = r(e^{\lambda_l}) = 0$$

holds, hence $\sum_{j=1}^{Z+1} r_j f(j) = 0$.

For arbitrary values of λ_j let us note that all the equalities above depend continuously on the parameters $\lambda_j,\ j = 1, \ldots, Z+1$. We proceed by perturbing the coinciding λ_js so that the perturbed values do not coincide and then we apply a limiting argument in all the above equalities. In a similar way we prove the second part, since the solutions of $\mathcal{L}^*_{Z+1}(d/dx) f(x) = 0$ are linear combinations of exponents of the type $e^{-\lambda_j x}$. ■

13.8 Zeros of the Euler–Frobenius polynomial $\Pi_Z(\lambda)$

Recall that we have termed the polynomial

$$\Pi_Z(\lambda) := \Pi_Z(\lambda; 0)$$

Euler–Frobenius polynomial

From formula (13.10), p. 234, we see that the values $\lambda = e^{\lambda_i}$ are generally speaking singular for the function $A_Z(x; \lambda)$. Lemma 13.29 gives an answer to what happens if $\lambda = e^{\lambda_i}$ for some i. It plays a central role in solving the *cardinal interpolation problem* (13.5), p. 225.

Lemma 13.29 *Let* $U_{Z+1}[\Lambda] = \{u_1, \ldots, u_{Z+1}\}_{\text{lin}}$.

For any α *with* $0 \le \alpha < 1$ *the system of equations*

$$y^{(i)}(1) = \lambda y^{(i)}(0) \quad \text{for } i = 0, \ldots, Z-1,$$
$$y(\alpha) = 0,$$

has a nontrivial solution y *in* U_{Z+1} *if and only if* $\lambda \ne e^{\lambda_i}$ *for* $i = 1, \ldots, Z+1$ *and* $A_Z(\alpha; \lambda) = 0$.

More precisely, the determinant of the above linear system with respect to the variables c_j*, where* $y(x) = \sum_{j=1}^{Z+1} u_j(x)$*, is proportional to* $A_Z(\alpha; \lambda)$.

Exercise 13.30 *Prove the above lemma.* Hint: *Follow a way similar to the one we used to obtain formula (13.10), p. 234.*

We will not prove the following fundamental theorem since its proof will not be necessary later in our study.

Theorem 13.31 *1. If* $\lambda \ge 0$ *and* $\lambda \ne e^{\lambda_i}$ *for* $i = 1, \ldots, Z+1$*, then as a function of* x*,* $A_Z(x; \lambda)$ *has no zeros in the interval* $(0, 1)$*. If* $\lambda < 0$ *then* $A_Z(x; \lambda)$ *has exactly one simple zero in the interval* $[0, 1)$.

2. Let us fix α *with* $0 < \alpha < 1$*. Then as a function of* λ*,* $A_Z(\alpha; \lambda)$ *has exactly* Z *different zeros*

$$\tau_1(\alpha) < \cdots < \tau_Z(\alpha) < 0$$

which interlace the zeros of $A_{Z-1}(\alpha, \lambda) = A_{Z-1}[\lambda_1, \ldots, \lambda_Z](\alpha, \lambda)$.

3. For $Z \geq 2$ the polynomial $\Pi_Z(\lambda) = r(\lambda)A_Z(0;\lambda)$ has exactly $Z-1$ negative zeros which interlace the $Z-2$ zeros

$$\tau_1(0) < \cdots < \tau_{Z-1}(0) < 0$$

of $A_{Z-1}(0;\lambda) = A_{Z-1}[\lambda_1,\ldots,\lambda_Z](0,\lambda)$.

Micchelli proves this theorem by applying a generalized Budan–Fourier-type result for the zeros of L-polynomials [13, pp. 210–211]. Schoenberg has provided a more elementary proof of the above result [19, p. 256, Theorems 1 and 2, pp. 258, Lemma 1].

13.9 The cardinal interpolation problem for L-splines

In view of the above results we see that for every α with $0 < \alpha < 1$ there exist precisely Z solutions of the zero interpolation problem (13.5), p. 225, i.e. elements of the space $\mathcal{S}_Z^0$. They correspond to the different solutions of equation $A_Z(\alpha,\lambda) = 0$.

Proposition 13.32 *For every α satisfying $0 < \alpha < 1$ the dimension of $\mathcal{S}_Z^0$ is exactly Z, while for $\alpha = 0$ it is $Z-1$.*

For the proof see the illuminating explanation by Schoenberg either in his book [18, Lecture 4, pp. 35, 36], or in his paper [19, p. 269].

We put

$$S_j(x) := A_Z(x,\tau_j(\alpha)) \quad \text{for } 0 \leq x \leq 1$$

and extend it for every x in $\mathbb{R}$ by means of the functional equation

$$S_j(x+1) = \tau_j(\alpha)S_j(x).$$

We proceed in a similar way for $\alpha = 0$ but there we use the $Z-1$ zeros of $A_Z(0;\lambda)$.

Let us note that all these elements have an exponential growth. Indeed, if $-1 < \tau_j(\alpha) < 0$ then due to

$$S_j(m) = \tau_j^m(\alpha)S_j(0) \quad \text{for all } m \text{ in } \mathbb{Z},$$

for all $m < 0$ we have an exponential growth for $m \to \infty$. If $\tau_j(\alpha) < -1$ then we obtain exponential growth for all $m > 0$ for $m \to -\infty$.

Let us denote by ξ the unique simple zero of $A_Z(x;-1)$ satisfying $0 \leq \xi < 1$. Thus, if $\alpha \neq \xi$ an element $\mathcal{S}_Z^0$ of power growth does not exist.

This obtains the main result of the cardinal interpolation.

Theorem 13.33 *Let ξ be, as above, the unique zero of $A_Z(x;-1)$ in the interval $[0,1)$. Then for every α with $0 \leq \alpha < 1$ such that $\alpha \neq \xi$ and any bi-infinite sequence of power growth $\{y_j\}_{j=-\infty}^{\infty}$ there exists a unique spline $u(x)$ in $\mathcal{S}_Z$ of power growth for which*

$$u(\alpha+j) = y_j \quad \text{for all } j \text{ in } \mathbb{Z},$$

and $u(x)$ is given by the cardinal series

$$u(x) = \sum_{j=-\infty}^{\infty} y_j L(x - j).$$

Here $L(\cdot)$ is the fundamental cardinal L-spline of Theorem 13.2, p. 225.

13.10 The cardinal compactly supported L-splines Q_{Z+1}

One of the important features of Micchelli's approach to cardinal L-splines is the simple and natural way in which the cardinal TB-spline[9] functions are obtained, compared with the general construction of the TB-splines in Section 11.4.1, p. 201.

We will consider a more general situation by taking the mesh $h\mathbb{Z}$ instead of $\mathbb{Z}$. This will be particularly important when we study L-spline wavelets.

We assume as usual that the nonordered vector $\Lambda = [\lambda_1, \lambda_2, \ldots, \lambda_{Z+1}]$ of the real numbers is fixed. We will consider the *cardinal mesh*

$$h\mathbb{Z} := \{jh : \text{ all } j \text{ in } \mathbb{Z}\}$$

where we have taken some fixed number $h > 0$.[10] The reader may simplify the results below by putting $h = 1$.

Definition 13.34 *The (**forward**) TB-**spline** for the cardinal L-spline space $\mathcal{S}_Z(\Lambda)$ is defined by*

$$Q_{Z+1}(x) := Q_{Z+1}[\Lambda; h](x) := \sum_{j=0}^{Z+1} \phi_Z^+(x - jh) s_{j,h} \tag{13.19}$$

where

$$s_h(x) := s_h[\Lambda](x) := \prod_{j=1}^{Z+1} (e^{-\lambda_j h} - x) = \sum_{j=0}^{Z+1} s_{j,h} x^j. \tag{13.20}$$

The notation $Q_{Z+1} = Q_{Z+1}[\Lambda; h] = Q[\Lambda; h]$ will be used on equal rights depending on what we want to emphasize. Obviously the notation $Q_{Z+1}[\Lambda; h]$ is redundant since Λ will normally have $Z + 1$ elements, but if this is not the case we will use this notation.

In the case $Z = 0$ we obtain

$$s_h(x) = e^{-\lambda_1 h} - x = s_0 + s_1 x, \quad \text{with}$$
$$s_0 = e^{-\lambda_1 h}, \quad s_1 = -1.$$

[9] The notion TB-spline is the generalization of the polynomial B-spline. It means an L-spline with a minimal compact support.

[10] Such a cardinal mesh is considered by Schoenberg [19]. The case considered by Micchelli [12, 13] is that of $h = 1$.

By the properties of $\phi_Z(x)$ in Proposition 13.12, p. 229, we have

$$\phi_0(x) = e^{\lambda_1 x},$$

hence,

$$Q_1(x) = e^{-\lambda_1 h} e^{\lambda_1 x} \chi_{[0,h]}(x), \tag{13.21}$$

where $\chi_{[0,h]}(x)$ is the characteristic function of the interval $[0, h]$, i.e. by definition

$$\chi_{[0,h]}(x) := \begin{cases} 1 & \text{for } 0 \le x \le h, \\ 0 & \text{elsewhere.} \end{cases} \tag{13.22}$$

Here is the most central result of classical spline theory but formulated in our L-spline setting (see the case of general Chebyshev splines in Theorem 11.29, p. 201):

Proposition 13.35 *1. The spline* $Q_{Z+1}(x) = Q_{Z+1}[\Lambda; h](x)$ *defined by formula (13.19) is a* TB*-spline for the operator* $\mathcal{L}_{Z+1}[\Lambda]$ *on the mesh* $h\mathbb{Z}$*, i.e. it is a nonnegative function, has a minimal compact support in the sense that no (nonzero)* L*-spline with smaller support exists, and it is the unique* L*-spline up to a multiplicative constant with support* $[0, Ph + h]$*. The above means that*

$$\begin{cases} Q_{Z+1}(x) > 0 & \textit{for } 0 < x < Ph + h, \\ Q_{Z+1}(x) = 0 & \textit{for } x \le 0 \quad \textit{or} \quad x \ge Ph + h. \end{cases}$$

Proof We prove only that the support of $Q_{Z+1}(x)$ is contained in the interval $[0, Ph + h]$.

Assuming for simplicity that all values of λ_j are pairwise different, by the definition of the function $\phi_Z(x)$ in formula (13.8), p. 229, we obtain

$$\phi_Z(x - jh) = \sum_{l=1}^{Z+1} \frac{1}{q'_{Z+1}(\lambda_l)} e^{\lambda_l (x - jh)},$$

hence, for $x \ge Z + 1$ we have

$$\begin{aligned} Q_{Z+1}(x) &= \sum_{j=0}^{Z+1} \phi_Z(x - jh) s_j \\ &= \sum_{j=0}^{Z+1} \left(\sum_{l=1}^{Z+1} \frac{1}{q'_{Z+1}(\lambda_l)} e^{\lambda_l (x - jh)} \right) s_j \\ &= \sum_{l=1}^{Z+1} \frac{1}{q'_{Z+1}(\lambda_l)} e^{\lambda_l x} \left(\sum_{j=0}^{Z+1} e^{-\lambda_l jh} s_j \right) \\ &= \sum_{l=1}^{Z+1} \frac{1}{q'_{Z+1}(\lambda_l)} e^{\lambda_l x} \cdot s_h(e^{-\lambda_l h}) = 0. \end{aligned}$$

Thus for $x \le 0$, $Q_{Z+1}(x) = 0$ follows immediately from the definition of the function $\phi_Z^+(x)$. ■

Exercise 13.36 *Prove the above result by using the residuum representation of* $\phi_Z^+(x)$ *in formula (13.7), p. 229.*

Exercise 13.37 *This is another classic result in spline theory. Prove the minimality of the support for* $Q_{Z+1}(x)$ *stated in Proposition 13.35.*

Hint: *Recall that the dimension of* U_{Z+1} *is* $Z+1$ *and check that the number of smoothness conditions on* $Q_{Z+1}(x)$ *(note* $Q_{Z+1}(x)$ *belongs to* $C^{Z-1}(\mathbb{R})$*) is* $(Z+2)Z$*. Count the dimensions.*

Theorem 13.38 is the most basic of all results about compactly supported spline, (for the general Chebyshev splines see Theorem 11.30, p. 202).

Theorem 13.38 *Take for simplicity* $h=1$.

1. No element of the set of shifts

$$\{Q_{Z+1}(x-j): \text{ for all } j \text{ in } \mathbb{Z}\}$$

is a finite linear combination of the others.

2. Denote by $\mathcal{S}_Z(\Lambda)[a,b]$ *the space of cardinal L-splines in* $\mathcal{S}_Z(\Lambda)$ *which have their support only in the interval* $[a,b]$ *where* a,b *are two integers. Then the set of shifts*

$$\{Q_{Z+1}(x-j): \text{ for } j=a-Z,\ldots,b+Z\}$$

forms a linear basis of $\mathcal{S}_Z(\Lambda)[a,b]$*. All elements in this set of shifts are linearly independent.*

13.11 Laplace and Fourier transform of the cardinal *TB*-spline Q_{Z+1}

Since the function $\phi_Z^+(x)$ has no compact support we may not consider its Fourier transform in the classical sense. On the other hand the function $Q_{Z+1}(x)=Q_{Z+1}[\Lambda;h](x)$ is a linear combination of shifts (integer translates) of $\phi_Z^+(x)$ but has a compact support and for that reason its Fourier transform is defined in a classical sense. Because of this we first compute the *Laplace transform* $\mathfrak{L}[\phi_Z^+](z)$ which makes sense for some subdomain of the complex plane and after that we extend by analytical argument the formula obtained for $\mathfrak{L}[Q_{Z+1}](z)$. Then we use the fact that the *Fourier* transform is obtained through the *Laplace transform* at the point $z=i\xi$.

Proposition 13.39 *The Laplace transform of the function* Q_{Z+1} *is given by*

$$\mathfrak{L}[Q_{Z+1}[\Lambda;h]](z)=\int_{\infty}^{\infty}Q_{Z+1}(x)e^{-xz}dx=\frac{s_h(e^{-zh})}{q_{Z+1}(z)}=\frac{\prod_{j=1}^{Z+1}(e^{-\lambda_j h}-e^{-zh})}{\prod_{j=1}^{Z+1}(z-\lambda_j)}$$

for every complex number $z\in C$*. The Fourier transform is, respectively,*

$$\widehat{Q_{Z+1}[\Lambda;h]}(\xi)=\mathfrak{L}[Q_{Z+1}](i\xi)=\frac{s_h(e^{-i\xi h})}{q_{Z+1}(i\xi)}=\frac{\prod_{j=1}^{Z+1}(e^{-\lambda_j h}-e^{-i\xi h})}{\prod_{j=1}^{Z+1}(i\xi-\lambda_j)}. \quad (13.23)$$

Proof Assuming for simplicity that all λ_js are pairwise different, we can easily see that

$$\mathfrak{L}[\phi_Z^+(x)](z) = \int_\infty^\infty \phi_Z^+(x)e^{-xz}\,dx \tag{13.24}$$
$$= \frac{1}{q_{Z+1}(z)} \quad \text{for Re}\, z > \max_{j=1,\dots,Z+1} \lambda_j,$$

which follows directly from the representation of the function $\phi_Z^+(x)$, formula (13.8), p. 229. Indeed, we have

$$\int e^{\lambda_j x}e^{-zx}\,dx = \frac{-1}{\lambda_j - z} \quad \text{for Re}\, z > \max_{j=1,\dots,Z+1} \lambda_j$$

and

$$\int_\infty^\infty \phi_Z^+(x)e^{-xz}\,dx = \sum_{j=1}^{Z+1} \frac{1}{q'_{Z+1}(\lambda_j)}\frac{1}{z-\lambda_j} = \frac{1}{q_{Z+1}(z)}.$$

The last equality is a standard result in the representation of a rational polynomial through simple fractions, easily checked by multiplying with $(z-\lambda_j)$ and substituting $z = \lambda_j$ thereafter.

Hence, by the standard properties of the *Laplace* transform we obtain

$$\int_\infty^\infty Q_{Z+1}(x)e^{-xz}\,dx = \left(\sum_{j=0}^{Z+1} s_j e^{-zjh}\right) \cdot \int_\infty^\infty \phi_Z^+(x)e^{-xz}dx = \frac{s_h(e^{-zh})}{q_{Z+1}(z)}$$

which is now true for every complex number $z \in C$ since $Q_{Z+1}(x)$ has a compact support and we extend the right-hand side analytically. For coinciding λ_js the result follows by a continuity argument. ∎

From the above there immediately follows a relationship between the TB-spline for different $h\mathbb{Z}$, which shows that we may reduce the study of $Q_{Z+1}[\Lambda; h]$ to the case of $h = 1$. For that reason it makes sense to introduce a simplified notation for $h = 1$, namely

$$Q_{Z+1}[\Lambda](x) := Q_{Z+1}[\Lambda; 1](x)\,. \tag{13.24a}$$

By formula (13.23) we obtain, through simple transformations,

$$Q_{\widehat{Z+1[\Lambda}; h]}(\xi) = \frac{\prod_{j=1}^{Z+1}(e^{-\lambda_j h} - e^{-i\xi h})}{\prod_{j=1}^{Z+1}(i\xi - \lambda_j)} \tag{13.25}$$
$$= h^{Z+1}\frac{\prod_{j=1}^{Z+1}(e^{-\lambda_j h} - e^{-i\xi h})}{\prod_{j=1}^{Z+1}(ih\xi - h\lambda_j)}$$
$$= h^{Z+1} \cdot Q_{\widehat{Z+1[h}\Lambda]}(h\xi).$$

Taking the inverse Fourier transform we obtain Proposition 13.40.

Proposition 13.40 *The TB-spline $Q_{Z+1}[h\Lambda]$ on the mesh $\mathbb{Z}$ and the TB-spline $Q_{Z+1}[\Lambda; h](x)$ on the mesh $h\mathbb{Z}$ are related by the equality:*

$$Q_{Z+1}[\Lambda; h](x) = h^Z \cdot Q_{Z+1}[h\Lambda]\left(\frac{x}{h}\right). \tag{13.26}$$

It should be noted that the last function has singularities at hj, $j \in \mathbb{Z}$, since the function $Q_{Z+1}[h\Lambda](y)$ has singularities at $j \in \mathbb{Z}$.

Exercise 13.41 *Prove Proposition 13.40 by using the residuum representation of $\phi_Z(x)$ in formula (13.7), p. 229.*

13.12 Convolution formula for cardinal *TB*-splines

Here we will prove an important generalization of the inductive convolution formula known for the polynomial cardinal splines, see Schoenberg [18, p. 12, formula (1.9)], which we have mentioned in the introduction as formula (13.1), p. 220.

Assume that we are given the nonordered vector $\Lambda = [\lambda_1, \ldots, \lambda_N]$ and let us denote by m_j the number of entries in Λ for the number λ_j, i.e. m_j is the multiplicity of λ_j.[11] As above we denote by $Q[\Lambda](x) = Q_{N+1}[\Lambda](x)$ the $L-$spline on $\mathbb{Z}$ which corresponds, according to formula (13.19), p. 239, to the set Λ. ". We denote by $Q[\Lambda](x) = Q_{N+1}[\Lambda](x)$ the L-spline which corresponds according to formula (13.19), p. 239, to the set Λ. Here we drop the subindex $N+1$ of Q as inessential for the present consideration.

The Fourier transform of $Q[\Lambda](x)$ which we have by formula (13.23), p. 241, is equal to

$$\widehat{Q[\Lambda]}(\xi) = \frac{\prod_{j=1}^{N}(e^{-\lambda_j} - e^{-i\xi})}{\prod_{j=1}^{N}(i\xi - \lambda_j)}.$$

Assume that a *subdivision* of the nonordered vector Λ be given, i.e. two other nonordered vectors Λ_1 and Λ_2 are determined by $\Lambda_1 = [u_1, \ldots, u_{N_1}]$ and $\Lambda_2 = [v_1, \ldots, v_{N_2}]$ with $N = N_1 + N_2$, and the number of entries of λ_j in Λ_1 plus the number of entries of λ_j in Λ_2 is equal to m_j. Evidently, we have

$$\widehat{Q[\Lambda]}(\xi) = \widehat{Q[\Lambda_1]}(\xi) \cdot \widehat{Q[\Lambda_2]}(\xi).$$

By taking the inverse Fourier transform and using a basic property of the Fourier transform, namely to convert the convolution between two functions into their product, see (12.7), p. 212, we obtain

$$Q[\Lambda_1]\widehat{\ast\, Q}[\Lambda_2](\xi) = \widehat{Q[\Lambda_1]}(\xi) \cdot \widehat{Q[\Lambda_2]}(\xi).$$

This completes the proof.

[11] See the conventions about non–ordered vectors Λ on p. 223.

Proposition 13.42 *If the sets Λ_1, Λ_2 and Λ are defined as above then the corresponding TB-splines satisfy the following equality:*

$$Q[\Lambda](x) = Q[\Lambda_1](x) * Q[\Lambda_2](x).$$

In particular,

$$Q[\mu, \lambda_1, \ldots, \lambda_s](x) = e^{-\mu h} \cdot \int_0^h Q[\lambda_1, \ldots, \lambda_s](x - y) \cdot e^{\mu y} dy, \tag{13.27}$$

and

$$Q[\lambda_1, \ldots, \lambda_s](x) = Q[\lambda_1](x) * Q[\lambda_2](x) * \cdots * Q[\lambda_N](x). \tag{13.28}$$

Let us recall that in the last equality $Q[\lambda_j](x)$ is the TB-spline corresponding to the vector $\Lambda = [\lambda_j]$ which has a unique element and which by formula (13.21), p. 240, is given by

$$Q[\lambda_j](x) = e^{-\lambda_j h} e^{\lambda_j x} \chi_{[0,h]}(x)$$

or in the case of the mesh $\mathbb{Z}$ is given by $Q[\lambda_j](x) = e^{-\lambda_j} e^{\lambda_j x} \chi_{[0,1]}(x)$. It has the Fourier transform

$$\widehat{Q[\lambda_j]}(\xi) = e^{-\lambda_j h} \int_0^h e^{\lambda_j x} e^{-i\xi x}\, dx = e^{-\lambda_j h} \cdot \frac{e^{(\lambda_j - i\xi)h} - 1}{\lambda_j - i\xi}$$

which coincides with the general formula (13.23), p. 241.

13.13 Differentiation of cardinal *TB*-splines

We now prove Theorem 13.43 by means of the convolution formula for different order TB-splines.

Theorem 13.43 *If we use the notation for the TB-spline as in (13.19), p. 239, for the mesh $h\mathbb{Z}$, then the following formula holds:*

$$\begin{aligned}
&\left(\frac{d}{dx} - \mu\right) Q[\mu, \lambda_1, \ldots, \lambda_s](x) \\
&\qquad = -e^{-\mu h}(Q[\lambda_1, \ldots, \lambda_s](x - h) \cdot e^{\mu h} - Q[\lambda_1, \ldots, \lambda_s](x)) \\
&\qquad = e^{-\mu h} Q[\lambda_1, \ldots, \lambda_s](x) + Q[\lambda_1, \ldots, \lambda_s](x - h).
\end{aligned}$$

Proof In formula (13.27), p. 244, we have proved that

$$Q[\mu, \lambda_1, \ldots, \lambda_s](x) = e^{-\mu h} \cdot \int_0^h Q[\lambda_1, \ldots, \lambda_s](x - y) \cdot e^{\mu y}\, dy.$$

Let us differentiate it. We obtain after integration by parts, and using the fact that $(d/dx)g(x-y) = -(d/dy)g(x-y)$, the following equality:

$$\begin{aligned}
&\frac{d}{dx}Q[\mu,\lambda_1,\dots,\lambda_s](x)\\
&\quad= -e^{-\mu h}\cdot\int_0^h \frac{d}{dy}Q[\lambda_1,\dots,\lambda_s](x-y)\cdot e^{\mu y}\,dy\\
&\quad= -e^{-\mu h}\left(Q[\lambda_1,\dots,\lambda_s](x-y)\cdot e^{\mu y}\Big|_{y=0}^{y=h} - \mu\int_0^h Q[\lambda_1,\dots,\lambda_s](x-y)\cdot e^{\mu y}\,dy\right)\\
&\quad= -e^{-\mu h}(Q[\lambda_1,\dots,\lambda_s](x-h)\cdot e^{\mu h} - Q[\lambda_1,\dots,\lambda_s](x))\\
&\qquad+\mu\int_0^h Q[\mu,\lambda_1,\dots,\lambda_s](x-y)\cdot e^{\mu y}dy,
\end{aligned}$$

which completes the proof. ■

13.14 Hermite–Gennocchi-type formula

We may easily derive an analog to the classical Hermite–Gennocchi formula [2, p. 9].

In order to be able to apply the Fourier transform we have to work at least temporarily with functions in $L_2(\mathbb{R})$. For an arbitrary function $f \in L_{1,\mathrm{loc}}(\mathbb{R}) \cap L_2(\mathbb{R})$ let us consider

$$I := \int_{-\infty}^{\infty} f(x)Q[\Lambda](x)\,dx.$$

Using the Parseval identity (12.5), p. 212, and the convolution formula (13.28), p. 244, we obtain

$$\begin{aligned}
I &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f}(\xi)\cdot\overline{\widehat{Q[\Lambda]}(\xi)}\,d\xi\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f}(\xi)\cdot\overline{\widehat{Q[\lambda_1]}(\xi)}\cdot\overline{\widehat{Q[\lambda_2]}(\xi)}\cdot\ldots\cdot\overline{\widehat{Q[\lambda_N]}(\xi)}\,d\xi\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f}(\xi)\cdot\left\{\prod_{j=1}^{N}e^{-\lambda_j h}\right\}\cdot\left\{\prod_{j=1}^{N}\int_0^h e^{\lambda_j x}e^{i\xi x}dx\right\}\,d\xi\\
&= \frac{1}{2\pi}\left\{\prod_{j=1}^{N}e^{-\lambda_j h}\right\}\cdot\underbrace{\int_0^h\cdots\int_0^h}_{N}\int_{-\infty}^{\infty}\widehat{f}(\xi)\cdot\prod_{j=1}^{N}e^{\lambda_j x_j}e^{i\xi x_j}dx_1\cdots dx_N\,d\xi\\
&= \frac{1}{2\pi}\left\{\prod_{j=1}^{N}e^{-\lambda_j h}\right\}\cdot\underbrace{\int_0^h\cdots\int_0^h}_{N}\prod_{j=1}^{N}e^{\lambda_j x_j}\cdot\int_{-\infty}^{\infty}\widehat{f}(\xi)\cdot e^{i\xi(x_1+\cdots+x_N)}d\xi\,dx_1\cdots dx_N.
\end{aligned}$$

Further we apply the inverse Fourier transform $\mathcal{F}^{-1}$, see (12.4), p. 212. By formula (12.6), p. 212, it has the property that $\mathcal{F}^{-1}\mathcal{F} = id$, which implies the equality:

$$I = \left\{ \prod_{j=1}^{N} e^{-\lambda_j h} \right\} \cdot \underbrace{\int_0^h \cdots \int_0^h}_{N} \prod_{j=1}^{N} e^{\lambda_j x_j} \cdot f(x_1 + \cdots + x_N)\, dx_1 \cdots dx_N.$$

Let us note that both sides also make sense for functions in $L_{1,\mathrm{loc}}(\mathbb{R})$, and by the approximation argument we may prove it for all functions in $L_{1,\mathrm{loc}}(\mathbb{R})$.

Thus we have proved the *generalized Hermite–Gennocchi formula*[12].

Theorem 13.44 *If the nonordered vector $\Lambda = [\lambda_1, \ldots, \lambda_N]$ is given and the function f belongs to $L_{1,\mathrm{loc}}(\mathbb{R})$ (i.e. f belongs to $L_1(a, b)$ for every finite interval (a, b)). Then the corresponding TB-spline $Q[\Lambda](x)$ defined on the mesh $h\mathbb{Z}$ satisfies the identity*

$$\int_{-\infty}^{\infty} f(x) Q[\Lambda](x)\, dx$$

$$= \left\{ \prod_{j=1}^{N} e^{-\lambda_j h} \right\} \cdot \underbrace{\int_0^h \cdots \int_0^h}_{N} \prod_{j=1}^{N} e^{\lambda_j x_j} \cdot f(x_1 + \cdots + x_N)\, dx_1 \cdots dx_N. \quad (13.29)$$

Exercise 13.45 *Recall that the left-hand side of equality (13.29) is equal to the divided difference of a function g for which $\mathcal{L}^*_{P+1} g = f$, which will be proved in Theorem 13.59, p. 258. Combining both results we obtain the equality which is usually known as the* Hermite–Gennocchi *formula. Prove the above result for noncardinal L-splines when $Q[\Lambda](x)$ is the corresponding compactly supported TB-spline without using the Fourier transform.*

13.15 Recurrence relation for the *TB*-spline

As an application of the Fourier transform of the TB-spline $Q[\Lambda](x)$ we may easily prove a recurrence relation which expresses the values of $Q[\Lambda]$ through values of lower order TB-splines.[14][13]

[12] This result has been proved by Dyn and Ron [7].

[13] This result was first proved by Dyn and Ron [6, 7].

Theorem 13.46 *If $\lambda_1 \neq \lambda_{Z+1}$ then the following recurrence relation holds:*

$$\begin{aligned} Q[\lambda_1, \lambda_2, \dots, \lambda_{Z+1}](x) = \frac{e^{-\lambda_{Z+1}}}{\lambda_1 - \lambda_{Z+1}} Q[\lambda_2, \dots, \lambda_{Z+1}](x) \\ + \frac{-e^{-\lambda_1}}{\lambda_1 - \lambda_{Z+1}} Q[\lambda_1, \dots, \lambda_Z](x) \\ + \frac{-1}{\lambda_1 - \lambda_{Z+1}} Q[\lambda_2, \dots, \lambda_{Z+1}](x-1) \\ + \frac{1}{\lambda_1 - \lambda_{Z+1}} Q[\lambda_1, \dots, \lambda_Z](x-1). \end{aligned} \tag{13.30}$$

Proof By assumption $\lambda_1 \neq \lambda_{Z+1}$. We will be looking for the constants in the equality

$$\begin{aligned} Q[\lambda_1, \lambda_2, \dots, \lambda_{Z+1}](x) &= C_1 Q[\lambda_2, \dots, \lambda_{Z+1}](x) + C_2 Q[\lambda_1, \dots, \lambda_Z](x) \\ &\quad + C_3 Q[\lambda_2, \dots, \lambda_{Z+1}](x-1) + C_4 Q[\lambda_1, \dots, \lambda_Z](x-1). \end{aligned}$$

We carry out some algebraic operations. First, we take the Fourier transform on both sides and obtain

$$\begin{aligned} &\frac{\prod_{j=1}^{Z+1}(e^{-\lambda_j} - e^{-i\xi})}{\prod_{j=1}^{Z+1}(i\xi - \lambda_j)} \\ &\quad = C_1 \frac{\prod_{j=2}^{Z+1}(e^{-\lambda_j} - e^{-i\xi})}{\prod_{j=2}^{Z+1}(i\xi - \lambda_j)} + C_2 \frac{\prod_{j=1}^{Z}(e^{-\lambda_j} - e^{-i\xi})}{\prod_{j=1}^{Z}(i\xi - \lambda_j)} \\ &\qquad + C_3 e^{-i\xi} \frac{\prod_{j=2}^{Z+1}(e^{-\lambda_j} - e^{-i\xi})}{\prod_{j=2}^{Z+1}(i\xi - \lambda_j)} + C_4 e^{-i\xi} \frac{\prod_{j=1}^{Z}(e^{-\lambda_j} - e^{-i\xi})}{\prod_{j=1}^{Z}(i\xi - \lambda_j)}. \end{aligned}$$

Then we divide the last by $\left(\prod_{j=2}^{Z}(e^{-\lambda_j} - e^{-i\xi})\right) \Big/ \left(\prod_{j=2}^{Z}(i\xi - \lambda_j)\right)$, after putting

$$z = e^{-i\xi}, \quad T_1 = e^{-\lambda_1}, \quad T_2 = e^{-\lambda_{Z+1}},$$

we obtain

$$\begin{aligned} &\frac{(T_1 - z)(T_2 - z)}{(i\xi - \lambda_1)(i\xi - \lambda_{Z+1})} \\ &\quad = \frac{(C_1 + C_3 z)(i\xi - \lambda_{Z+1})(T_1 - z) + (C_2 + C_4 z)(i\xi - \lambda_1)(T_2 - z)}{(i\xi - \lambda_1)(i\xi - \lambda_{Z+1})}. \end{aligned}$$

By comparing both sides as polynomials of z we obtain the system:

$$\begin{aligned} C_1(i\xi - \lambda_{Z+1})T_1 + C_2(i\xi - \lambda_1)T_2 &= T_1 T_2, \\ -C_1(i\xi - \lambda_{Z+1}) + C_3(i\xi - \lambda_{Z+1})T_1 - C_2(i\xi - \lambda_1) + C_4(i\xi - \lambda_1)T_2 &= -(T_1 + T_2), \\ -C_3(i\xi - \lambda_{Z+1}) - C_4(i\xi - \lambda_1) &= 1. \end{aligned}$$

Now we compare the coefficients in front of the variable ξ in the first and the third equations which gives

$$C_1T_1 + C_2T_2 = 0,$$
$$C_3 + C_4 = 0,$$

which again gives, using the first and the third equations, the solution

$$C_1 = \frac{T_2}{\lambda_1 - \lambda_{Z+1}} = \frac{e^{-\lambda_{Z+1}}}{\lambda_1 - \lambda_{Z+1}},$$
$$C_2 = -\frac{C_1T_1}{T_2} = -\frac{-T_1}{\lambda_1 - \lambda_{Z+1}} = \frac{-e^{-\lambda_1}}{\lambda_1 - \lambda_{Z+1}},$$
$$C_3 = \frac{-1}{\lambda_1 - \lambda_{Z+1}},$$
$$C_4 = \frac{1}{\lambda_1 - \lambda_{Z+1}}.$$

One checks directly that these constants also satisfy the second equation, hence they solve the above system. ■

13.16 The adjoint operator $\mathcal{L}^*_{Z+1}$ and the *TB*-spline $Q^*_{Z+1}(x)$

In Section 13.7, p. 236, we introduced the adjoint polynomial q^*_{Z+1} and the adjoint operator $\mathcal{L}^*_{Z+1}$ for the purposes of the generalized divided difference operators. Here we will need them again for defining the adjoint *TB*-spline. It is helpful to work with the formally adjoint operator times $(-1)^{Z+1}$

$$\mathcal{L}^*_{Z+1}[\Lambda]\left(\frac{d}{dx}\right) := \mathcal{L}_{Z+1}[-\Lambda]\left(\frac{d}{dx}\right) := \prod_{j=1}^{Z+1}\left(\frac{d}{dx} + \lambda_j\right) \tag{13.31}$$

with the polynomial

$$q^*_{Z+1}(\lambda) := \prod_{j=1}^{Z+1}(\lambda + \lambda_j) = (-1)^{Z+1} q_{Z+1}(-\lambda). \tag{13.32}$$

The corresponding *TB*-spline on the mesh $h\mathbb{Z}$ is given by

$$Q^*_{Z+1}(x) := (-1)^{Z+1} \sum_{j=0}^{Z} \phi^+_Z(jh - x) r_j \tag{13.33}$$

where we have put, as in (13.12), p. 235,

$$r_h(x) := \sum_{j=0}^{Z+1} r_{j,h} x^j = \prod_{j=1}^{Z+1} (e^{\lambda_j h} - x).$$

We will drop the second index and write r_j instead of $r_{j,h}$ if the context allows.

Proposition 13.47 *The polynomials $r_h(x)$ and $s_h(x)$ are related through the equality*

$$x^{Z+1} r_h\left(\frac{1}{x}\right) = (-1)^{Z+1} e^{h(\lambda_1 + \cdots + \lambda_{Z+1})} \cdot s_h(x). \tag{13.34}$$

Proof We have evidently

$$\begin{aligned} x^{Z+1} r_h\left(\frac{1}{x}\right) &= x^{Z+1} \prod_{j=1}^{Z+1} \left(e^{\lambda_j h} - 1/x\right) = \prod_{j=1}^{Z+1} (x e^{\lambda_j h} - 1) \\ &= \exp\left(h\left(\sum_{j=1}^{Z+1} \lambda_j\right)\right) \cdot \prod_{j=1}^{Z+1} (x - e^{-\lambda_j h}) \end{aligned}$$

which proves the statement. ■

Due to the properties of ϕ_Z^+ proved in Proposition 13.12, p. 229, one may prove Proposition 13.48.

Proposition 13.48 *The following equality holds:*

$$Q^*_{Z+1}(Zh + h - x) = e^{(\lambda_1 + \cdots + \lambda_{Z+1})h} \cdot Q_{Z+1}(x). \tag{13.35}$$

Proof It is clear from

$$Q^*_{Z+1}(Zh + h - x) = (-1)^{Z+1} \sum_{j=0}^{Z+1} \phi_Z^+(x - (Z + 1 - j)h) r_j$$

that $Q^*_{Z+1}(Zh + h - x)$ is an L-spline for the operator $\mathcal{L}_{Z+1}$ with a support in the half-axis $x \geq 0$. By the definition of the polynomial $r(x)$, and by Proposition 13.35, p. 240, applied to the operator $\mathcal{L}^*_{Z+1}[\Lambda] = \mathcal{L}_{Z+1}[-\Lambda]$, it follows that the support of $Q^*_{Z+1}(Zh + h - x)$ coincides with the interval $[0, Zh + h]$. By the uniqueness of such a TB-spline, which we proved in Proposition 13.35 it follows that $Q^*_{Z+1}(Zh + h - x)$ and $Q_{Z+1}(x)$ are proportional, i.e. for some constant

$$Q^*_{Z+1}(Zh + h - x) = C Q_{Z+1}(x).$$

In order to obtain this constant it suffices to check this equality for $x = h$.

By the definition of the functions Q_{Z+1} and ϕ_Z^+ we have

$$Q_{Z+1}(h) = \phi_Z^+(h) s_{0,h},$$

$$Q^*_{Z+1} = (-1)^{Z+1} \phi_Z^+(h) r_{Z+1,h}.$$

It follows that

$$C = \frac{(-1)^{Z+1} r_{Z+1,h}}{s_{0,h}}.$$

From the definition of the polynomials $s_h(\lambda)$ and $r_h(\lambda)$ we see directly that

$$s_{0,h} = e^{-(\lambda_1 + \cdots + \lambda_{Z+1})h},$$
$$r_{Z+1,h} = (-1)^{Z+1},$$

which completes the proof. ■

13.17 The Euler polynomial $A_Z(x;\lambda)$ and the $\mathcal{TB}$-spline $Q_{Z+1}(x)$

For simplicity we consider only the case $h = 1$. The function $A_Z(x;\lambda)$ and the $\mathcal{TB}$-splines Q^*_{Z+1} are related by Proposition 13.49.

Proposition 13.49 *The following equality holds:*

$$A_Z(x;\lambda) = (-1)^Z \frac{\sum_{j=0}^{Z} Q^*_{Z+1}(j+1-x)\lambda^j}{r(\lambda)}. \tag{13.36}$$

Proof Let us make the direct expansion

$$G(z) = \frac{e^{xz}}{e^z - \lambda} = \sum_{j=0}^{\infty} \lambda^j e^{-(j+1-x)z},$$

hence, by the definition of the function ϕ_Z it follows that:

$$\begin{aligned} A_Z(x;\lambda) &= [\lambda_1, \lambda_2, \ldots, \lambda_{Z+1}]_z G(z) \\ &= \sum_{j=0}^{\infty} \lambda^j \phi_Z(x-j-1) \\ &= \frac{\left(\sum_{j=0}^{\infty} \lambda^j \phi_Z(x-j-1)\right) \cdot \left(\sum_{j=0}^{Z+1} r_j \lambda^j\right)}{r(\lambda)} \\ &= (-1)^Z \cdot \frac{\sum_{j=0}^{Z} Q^*_{Z+1}(j+1-x)\lambda^j}{r(\lambda)} \end{aligned}$$

which completes the proof. ■

We now obtain an important representation of the *Euler–Frobenius* polynomials $\Pi_Z(\lambda;0)$.

Proposition 13.50 *We have the following symmetric representation of the polynomial* $\Pi_Z(\lambda;0)$:

$$\Pi_Z(\lambda;0) = (-1)^Z \cdot \lambda^{(Z-1)/2} \cdot e^{\lambda_1+\cdots+\lambda_{Z+1}} \cdot \sum_{l=-(Z-1)/2}^{(Z-1)/2} Q_{Z+1}\left(\frac{Z+1}{2}-l\right)\lambda^l. \tag{13.37}$$

Proof By formula (13.14), p. 235, we obtain

$$\Pi_Z(\lambda;x) := r(\lambda)A_Z(x;\lambda) = (-1)^Z \cdot \sum_{j=0}^{Z} Q^*_{Z+1}(j+1-x)\lambda^j.$$

Further we use formula (13.35), i.e., $Q^*_{Z+1}(Zh+h-x) = e^{(\lambda_1+\cdots+\lambda_{Z+1})h} \cdot Q_{Z+1}(x)$. After putting

$$\frac{Z-1}{2} - j = l$$

we obtain the following equalities:

$$\begin{aligned}
\Pi_Z(\lambda;0) &= (-1)^Z \cdot \sum_{j=0}^{Z} Q^*_{Z+1}(j+1)\lambda^j \\
&= (-1)^Z \cdot e^{\lambda_1+\cdots+\lambda_{Z+1}} \cdot \sum_{j=0}^{Z-1} Q_{Z+1}(Z-j)\lambda^j \\
&= (-1)^Z \cdot e^{\lambda_1+\cdots+\lambda_{Z+1}} \sum_{j=0}^{Z-1} Q_{Z+1}\left(\frac{Z-1}{2}-j+\frac{Z+1}{2}\right)\lambda^j \\
&= (-1)^Z \cdot \lambda^{Z-1/2} \cdot e^{\lambda_1+\cdots+\lambda_{Z+1}} \cdot \sum_{l=-(Z-1)/2}^{(Z-1)/2} Q_{Z+1}\left(\frac{Z+1}{2}+l\right)\lambda^{-l}.
\end{aligned} \tag{13.38}$$

Since the function $Q_{Z+1}(x+(Z+1)/2)$ is symmetrized around zero in the sense that its support is the interval $[-(Z+1)/2,(Z+1)/2]$ it follows that

$$\Pi_Z(\lambda;0) = (-1)^Z \cdot \lambda^{(Z-1)/2} \cdot e^{\lambda_1+\cdots+\lambda_{Z+1}} \cdot \sum_{l=-(Z-1)/2}^{(Z-1)/2} Q_{Z+1}((Z+1)/2-l)\lambda^l,$$

which completes the proof. ■

Note that in the polynomial case we have $Q_{Z+1}((Z+1)/2-\ell) = Q_{Z+1}((Z+1)/2+\ell)$ for every $\ell \in \mathbb{Z}$, which plays a key role for the symmetry of the zeros of the polynomial $\Pi_Z(\lambda)$. The above result will be used in Section 13.23, p. 261, to prove a remarkable symmetry property of the zeros of $\Pi_Z(\lambda;0)$ in the special case when the numbers λ_j arise through the spherical operators. We have the same symmetry so far in the case of a nonordered vector Λ which is symmetric.

Theorem 13.51 *Let the nonordered vector* Λ *be symmetric, i.e.* $\Lambda = -\Lambda$. *Then for every number* $x \in \mathbb{R}$ *we have*

$$Q_{Z+1}[\Lambda]\left(\frac{Z+1}{2} - x\right) = Q_{Z+1}[\Lambda]\left(\frac{Z+1}{2} + x\right), \tag{13.39}$$

or, equivalently,

$$Q_{Z+1}[\Lambda](Z+1-x) = Q_{Z+1}[\Lambda](x).$$

Proof Assuming for simplicity that all λ_js are different, on every interval $[\ell, \ell+1]$ we have the representation

$$Q_{Z+1}[\Lambda](x) = \sum_{j=1}^{Z+1} \alpha_j e^{\lambda_j x},$$

which implies that the function $Q_{Z+1}[\Lambda](Z+1-x)$ is again L-spline since

$$\begin{aligned} Q_{Z+1}[\Lambda](Z+1-x) &= \sum_{j=1}^{Z+1} \alpha_j e^{\lambda_j (Z+1-x)} \\ &= \sum_{j=1}^{Z+1} e^{\lambda_j (Z+1)} \alpha_j e^{-\lambda_j x} \end{aligned}$$

on every interval $[\ell, \ell+1]$. The function $Q_{Z+1}[\Lambda](Z+1-x)$ has the same support $[0, Z+1]$. Due to the uniqueness of the compactly supported L-spline $Q_{Z+1}[\Lambda](x)$ it follows that

$$Q_{Z+1}[\Lambda](Z+1-x) = C Q_{Z+1}[\Lambda](x)$$

for some constant C. After putting $x = (Z+1)/2$ we obtain $C = 1$, since $Q_{Z+1}[\Lambda]((Z+1)/2) \neq 0$. ■

Proposition 13.52 *Let* $A_Z^*(x;\lambda)$ *be the function corresponding by Definition 13.20, p. 233, to the polynomial* q_{Z+1}^*. *Then*

$$A_Z\left(1-x; \frac{1}{\lambda}\right) = (-1)^{Z-1} \lambda A_Z^*(x;\lambda) \quad \text{for } 0 \le x \le 1. \tag{13.40}$$

If the nonordered vector $\Lambda = [\lambda_1, \lambda_2, \ldots, \lambda_{Z+1}]$ *is symmetric with respect to zero, i.e.* $\Lambda = -\Lambda$, *then the unique zero of equation* $A_Z(x;-1) = 0$ *satisfying* $0 \le x < 1$ *is equal to* $1/2$ *for odd* Z, *and is equal to* 0 *for even* Z.[14]

[14] In Micchelli's paper [13, p. 216, Remark 2.3], the last statement is obviously wrong since the formula is wrong; it is correct on p. 213 and on p. 224 of the paper. It is correct in Schoenberg's paper on L-splines [19, p. 268].

Proof (1) Assuming for simplicity that all λ_js are different we apply formula (13.10), p. 234, and obtain

$$A_Z^*(x;\lambda) = \sum_{j=1}^{Z+1} \frac{1}{q_{Z+1}^{*\prime}(-\lambda_j)} \frac{e^{-\lambda_j x}}{e^{-\lambda_j} - \lambda} = \frac{(-1)^Z}{\lambda} \cdot \sum_{j=1}^{Z+1} \frac{1}{q_{Z+1}'(\lambda_j)} \frac{e^{(1-x)\lambda_j}}{1/\lambda - e^{\lambda_j x}}$$
$$= \frac{(-1)^{Z-1}}{\lambda} A_Z\left(1 - x; \frac{1}{\lambda}\right).$$

(2) Now if $\Lambda = -\Lambda$ it follows that $A_Z^*(x;\lambda) = A_Z(x;\lambda)$, hence

$$A_Z(1-x;-1) = (-1)^Z A_Z(x;-1).$$

If Z is even it follows that $A_Z(1-x;-1) = A_Z(x;-1)$. For $x = 0$ this gives $A_Z(1;-1) = A_Z(0;-1)$. On the other hand by the very definition (Definition 13.20, p. 233) of $A_Z(x;\lambda)$ we have $A_Z(1;-1) = -A_Z(0;-1)$, which implies $A_Z(0;-1) = 0$.

(3) Let Z be odd. It follows that $A_Z(1-x;-1) = -A_Z(x;-1)$. For $x = 1/2$ we obtain $A_Z(1/2;-1) = -A_Z(1/2;-1)$ which implies that $A_Z(1/2;-1) = 0$. ■

We immediately obtain the following useful corollary.

Corollary 13.53 *If the nonordered vector Λ is symmetric, i. e. $\Lambda = -\Lambda$, then the zeros of the equation $\Pi_Z(\lambda) = 0$, which have been defined in Theorem 13.31, p. 237, are all different from -1, i.e. $\Pi_Z(-1) \neq 0$.*

Proof By Proposition 13.52, p. 252, we find that $\xi = 1/2$ is the only zero of the equation $A_Z(x;-1) = 0$. Let some τ_i be a solution to $\Pi_Z(\tau_i) = 0$, and $\tau_i = -1$. Then by $\Pi_Z(\lambda) = r(\lambda)A_Z(0;\lambda)$ it follows that $A_Z(0;-1) = 0$. This contradiction proves the corollary. ■

Remark 13.54 *Since $r(-1) = \prod(e^{\lambda_j} + 1) > 0$ it follows that the zeros of A_Z and Π_Z are the same.*

13.18 The leading coefficient of the Euler–Frobenius polynomial $\Pi_Z(\lambda)$

We will now compute the leading coefficient of the Euler–Frobenius polynomial $\Pi_Z(\lambda) = \Pi_Z(\lambda;0)$.

Since $\Pi_Z(\lambda;0) = r(\lambda)A_Z(0;\lambda)$ and $Q_{Z+1}^*(Z+1) = 0$ we find from formula (13.36), p. 250, that the leading coefficient of $\Pi_Z(\lambda;0)$ is

$$(-1)^Z Q_{Z+1}^*(Z)\lambda^{Z-1}.$$

By formula (13.35), p. 249, we obtain the equalities:

$$
\begin{aligned}
Q^*_{Z+1}(Z) &= e^{\lambda_1+\cdots+\lambda_{Z+1}} Q_{Z+1}(1),\\
Q_{Z+1}(1) &= \phi^+_Z(1)\cdot s_0,\\
s_0 &= e^{-\lambda_1-\cdots-\lambda_{Z+1}},\\
\phi^+_Z(1) &= \sum_{j=1}^{Z+1} \frac{e^{\lambda_j}}{q'_{Z+1}(\lambda_j)},
\end{aligned}
$$

which imply

$$Q^*_{Z+1}(Z) = \sum_{j=1}^{Z+1} \frac{e^{\lambda_j}}{q'_{Z+1}(\lambda_j)}.$$

Hence,

$$\Pi_Z(\lambda;0) = (-1)^Z \sum_{j=1}^{Z+1} \frac{e^{\lambda_j}}{q'_{Z+1}(\lambda_j)} \cdot \prod_{j=1}^{Z-1} (\lambda - v_j), \tag{13.41}$$

where v_j are the zeros of the polynomial $\Pi_Z(\lambda;0)$ which, we will see, are all real and negative.

13.19 Schoenberg's "exponential" Euler L-spline $\Phi_Z(x;\lambda)$ and $A_Z(x;\lambda)$

The word *exponential* is used in a different sense by Schoenberg [20, 21, p. 256] where he introduces "exponential L-splines of basis λ". This sense has nothing to do with the exponential splines used by other authors [17]. For that reason we have put it in quotation marks and used the expression "exponential" Euler.

Now we will obtain an expression for the *"exponential" Euler L-spline*, through the basic function $A_Z(x;\lambda)$.

The *"exponential" Euler L-spline* is defined in a natural way, generalizing the polynomial case of Schoenberg by putting

$$\Phi_Z(x;\lambda) = \sum_{j=-\infty}^{\infty} \lambda^j Q_{Z+1}(x-j). \tag{13.42}$$

It is always a convergent series since only a finite number of terms are nonzero. It evidently has the remarkable "*exponential property*" (and for that reason Schoenberg has called it exponential):

$$\Phi_Z(x+1;\lambda) = \sum_{j=-\infty}^{\infty} \lambda^j Q_{Z+1}(x+1-j) \tag{13.43}$$
$$= \lambda \sum_{j=-\infty}^{\infty} \lambda^{j-1} Q_{Z+1}(x-(j-1))$$
$$= \lambda \Phi_Z(x;\lambda);$$

it reminds us of the property of the exponential function λ^x which satisfies the same equation

$$\lambda^{x+1} = \lambda \cdot \lambda^x.$$

Since $Q_{Z+1}(x)$ is differentiable $Z-1$ times, if we differentiate the above equality (13.43) l times where $l \leq Z-1$, and put $x=0$, it follows that:

$$\Phi_Z^{(l)}(1;\lambda) = \lambda \Phi_Z^{(l)}(0;\lambda) \quad \text{for } l = 0, 1, \ldots, Z-1.$$

Hence, by Definition 13.20, p. 233, and Lemma 13.19, p. 233, for $0 \leq x \leq 1$ the function $\Phi_Z(x;\lambda)$ is proportional to $A_Z(x;\lambda)$.

Now we establish a link between these two fundamental functions, $\Phi_Z(x;\lambda)$ and $A_Z(x;\lambda)$.

Proposition 13.55 *The following relation holds for* $0 \leq x \leq 1$:

$$\Phi_Z(x;\lambda) = \frac{(-1)^Z}{\lambda^Z} e^{-\lambda_1 - \ldots - \lambda_{Z+1}} \cdot \Pi_Z(x;\lambda) \tag{13.44}$$
$$= \frac{(-1)^Z}{\lambda^Z} e^{-\lambda_1 - \ldots - \lambda_{Z+1}} r(\lambda) A_Z(x;\lambda)$$
$$= -\lambda s\left(\lambda^{-1}\right) A_Z(x;\lambda).$$

Proof Now let us use equality (13.35), p. 249, namely $Q_{Z+1}^*(Zh+h-x) = e^{(\lambda_1+\cdots+\lambda_{Z+1})h} \cdot Q_{Z+1}(x)$ for $h=1$. In (13.36), p. 250, we have obtained the equality

$$A_Z(x;\lambda) = (-1)^Z \cdot [r(\lambda)]^{-1} \cdot \sum_{j=0}^{Z} Q_{Z+1}^*(j+1-x)\lambda^j.$$

Hence for every x satisfying $0 \leq x \leq 1$, we obtain

$$A_Z(x;\lambda) = \frac{(-1)^Z e^{\lambda_1+\cdots+\lambda_{Z+1}}}{r(\lambda)} \cdot \lambda^Z \cdot \sum_{j=0}^{Z} Q_{Z+1}(x+Z-j)\lambda^{j-Z}$$
$$= \frac{(-1)^Z e^{\lambda_1+\cdots+\lambda_{Z+1}}}{r(\lambda)} \cdot \lambda^Z \cdot \sum_{j=-\infty}^{\infty} \lambda^j Q_{Z+1}(x-j)$$
$$= \frac{(-1)^Z e^{\lambda_1+\cdots+\lambda_{Z+1}}}{r(\lambda)} \cdot \lambda^Z \cdot \Phi_Z(x;\lambda). \tag{13.45}$$

By the definition of the polynomial Π_Z in formula (13.51), p. 261, we obtain

$$\Phi_Z(x;\lambda) = \frac{(-1)^Z}{\lambda^Z} e^{-\lambda_1-\cdots-\lambda_{Z+1}} \cdot \Pi_Z(x;\lambda).$$

■

We have the following symmetry property.

Theorem 13.56 *If the nonordered vector* Λ *is symmetric, i.e.* $\Lambda = -\Lambda$, *then*

$$\Phi_Z\left(\frac{Z+1}{2}; \frac{1}{z}\right) = \Phi_Z\left(\frac{Z+1}{2}; z\right) \quad \text{for all } z \text{ in } \mathbb{C}. \tag{13.46}$$

Proof In Theorem 13.51, p. 252, we have proved

$$Q_{Z+1}(Z+1-x) = Q_{Z+1}(x) \quad \text{for all } x \text{ in } \mathbb{R},$$

and in equality (13.42), p. 254, we have

$$\begin{aligned}\Phi_Z(0;z) &= \sum_{j=-\infty}^{\infty} z^j Q_{Z+1}(-j) \\ &= \sum_{j=0}^{Z+1} z^{-j} Q_{Z+1}(j).\end{aligned}$$

These imply by the exponential property of Φ_Z the following:

$$\begin{aligned}\Phi_Z\left(\frac{Z+1}{2}; z\right) &= z^{(Z+1)/2} \sum_{j=0}^{Z+1} Q_{Z+1}(j) z^{-j} \\ &= z^{-(Z+1)/2} \sum_{j=0}^{Z+1} Q_{Z+1}(j) z^{j},\end{aligned}$$

which completes the proof. ■

We immediately obtain the following useful corollary.

Corollary 13.57 *If the nonordered vector* Λ *is symmetric, i.e.* $\Lambda = -\Lambda$, *then* $\Phi_Z(0;z) \neq 0$ *for all complex numbers* z, *with* $|z| = 1$.

The proof follows directly from Corollary 13.53, p. 253, and the relation between Π_Z and Φ_Z given by formula (13.44), p. 255, above.

Let us apply formula (13.44). We use the relation for $\lambda = e^{\lambda_s}$, $1 \le s \le Z+1$. This gives the equality

$$\Phi_Z(x; e^{\lambda_s}) = (-1)^Z e^{-(\lambda_1+\cdots+\lambda_{Z+1})} e^{-\lambda_s Z} \cdot \Pi_Z(x; e^{\lambda_s}).$$

By formula (13.15), p. 235, we see that in the case of pairwise different λ_j we obtain:

$$\Pi_Z(x; e^{\lambda_s}) = r(e^{\lambda_s}) \cdot A_Z(x; e^{\lambda_s}) = \frac{-r'(e^{\lambda_s})}{q'_{Z+1}(\lambda_s)} \cdot e^{\lambda_s x}.$$

Hence,

$$\Phi_Z(x; e^{\lambda_s}) = (-1)^Z e^{-(\lambda_1+\cdots+\lambda_{Z+1})} e^{-\lambda_s Z} \cdot \frac{-r'(e^{\lambda_s})}{q'_{Z+1}(\lambda_s)} \cdot e^{\lambda_s x}. \tag{13.47}$$

13.20 Marsden's identity for cardinal L-splines

There is an important *normalization* property which is analogous to the classical *Marsden identity* for polynomial splines.

Proposition 13.58 *Assume that all λ_j are pairwise different. Then for every λ_s in Λ and for every x in $\mathbb{R}$ we have the following identity:*

$$\sum_{j=-\infty}^{\infty} e^{j\lambda_s} \cdot Q_{Z+1}(x-j) = \Phi_Z(x; e^{\lambda_s}) \tag{13.48}$$

$$= (-1)^{Z+1} e^{-(\lambda_1+\cdots+\lambda_{Z+1})} e^{-\lambda_s Z} \cdot \frac{r'(e^{\lambda_s})}{q'_{Z+1}(\lambda_s)} \cdot e^{\lambda_s x}.$$

It is clear that the sum on the left-hand side is finite over j satisfying $0 < x - j < Z + 1$. The proof is obtained by applying the above formula for $\Phi_Z(x; \lambda)$ in (13.47), p. 257. This result is useful for estimating the norm of Q_{Z+1}.

13.21 Peano kernel and the divided difference operator in the cardinal case

Here we provide a direct proof that the *TB*-spline $Q_{Z+1}(x)$ is indeed the Peano kernel for the divided difference operator defined in formula (13.18), p. 236, through the polynomial $s(\lambda)$.

We compute the divided difference in the case of different λ_js. First, we recall the adjoint operator of formula (13.31), p. 248,

$$\mathcal{L}^*_{Z+1}\left(\frac{d}{dx}\right) := (-1)^{Z+1} \prod_{j=1}^{Z+1} \left(-\frac{d}{dx} - \lambda_j\right) = (-1)^{Z+1} \prod_{j=1}^{Z+1} \mathcal{D}^*_j$$

where $\mathcal{D}^*_j = -(d/dx) - \lambda_j$ is the operator formally adjoint to the operator $\mathcal{D}_j = d/dx - \lambda_j$ defined in formula (13.6), p. 226.[15]

[15] These operators differ from those of Dyn and Ron [7, p. 5]. However, the difference between the operators $\mathcal{L}^*_{P+1}$ is not large.

Recalling the properties of the functions $\phi_Z(x)$ in Proposition 13.12, p. 229, and the definition of the TB-spline Q_{Z+1} in (13.19), p. 239, now we have the following *Peano identity* for the *generalized divided difference operator* given by formula (13.18), p. 236:

Theorem 13.59 *We assume that the function f is C^∞. Then the following Peano-type identity holds:*

$$\int_{-\infty}^{\infty} Q_{Z+1}(x)\mathcal{L}_{Z+1}^{*} f(x)dx = (-1)^{Z+1}\sum_{j=0}^{Z+1} s_j \cdot f(j). \tag{13.49}$$

Proof First, recall the properties of the function $\phi_Z(x)$ which are stated in Proposition 13.12, p. 229. We assume without restricting the generality that f has a compact support. By the definition of Q_{Z+1} in (13.19), p. 239, we obtain

$$\begin{aligned}
I &:= \int_{-\infty}^{\infty} Q_{Z+1}(x)\mathcal{L}_{Z+1}^{*} f(x)\,dx \\
&= (-1)^{Z+1}\int_{-\infty}^{\infty} Q_{Z+1}(x)\mathcal{D}_1^{*}\cdots\mathcal{D}_{Z+1}^{*} f(x)\,dx \\
&= (-1)^{Z+1}\sum_{j=0}^{Z+1} s_j \int_j^{\infty} \mathcal{D}_Z\cdots\mathcal{D}_1\phi_Z(x-j)D_{Z+1}^{*} f(x)\,dx \\
&= (-1)^{Z}\sum_{j=0}^{Z+1} s_j \int_j^{\infty} \mathcal{D}_Z\cdots\mathcal{D}_1\phi_Z(x-j)\left(\frac{d}{dx}+\lambda_{Z+1}\right) f(x)\,dx.
\end{aligned}$$

Further we integrate by parts and apply the properties of the function ϕ_Z in Proposition 13.12, p. 229,

$$\begin{aligned}
I &= (-1)^{Z}\sum_{j=0}^{Z+1} s_j\mathcal{D}_Z\cdots\mathcal{D}_1\phi_Z(x-j)\cdot f(x)\Big|_{x=j+}^{x=\infty} \\
&\quad + (-1)^{Z}\sum_{j=0}^{Z+1} s_j\left(-\int_j^{\infty}\frac{d}{dx}\mathcal{D}_Z\cdots\mathcal{D}_1\phi_Z(x-j)f(x)\,dx\right. \\
&\qquad\qquad \left. + \int_j^{\infty}\mathcal{D}_Z\cdots\mathcal{D}_1\phi_Z(x-j)\lambda_{Z+1} f(x)\,dx\right)
\end{aligned}$$

$$= (-1)^{Z+1} \sum_{j=0}^{Z+1} s_j \cdot f(j) + (-1)^{Z+1} \sum_{j=0}^{Z+1} s_j \int_j^{\infty} \mathcal{D}_{Z+1}\mathcal{D}_Z \cdots \mathcal{D}_1 \phi_Z(x-j) f(x)\, dx$$

$$= (-1)^{Z+1} \sum_{j=0}^{Z+1} s_j \cdot f(j),$$

which completes the proof. ■

13.22 Two-scale relation (refinement equation) for the *TB*-splines $Q_{Z+1}[\Lambda; h]$

Assuming the nonordered vector Λ given, we denote by $Q_{Z+1}[\Lambda](x)$ the *TB*-spline defined according to formula (13.19), p. 239, for the mesh $\mathbb{Z}$. As before we denote, by $Q_{Z+1}[t\Lambda; h](x)$ the *TB*-spline for the mesh $h\mathbb{Z}$ and for the nonordered vector $t\Lambda = [t\lambda_1, \ldots, t\lambda_{Z+1}]$. Up to now we have mainly used the notation

$$Q_{Z+1}(x) = Q_{Z+1}[\Lambda](x)$$

without indicating the dependence on h.[16] We note again that the index $Z+1$ is redundant but useful to have.

It is important for *wavelet analysis* to consider the relation between the *TB*-spline Q_{Z+1} for the cardinal L-splines on the mesh $h\mathbb{Z} := \{jh:\ \text{for } j \text{ in } \mathbb{Z}\}$ and the *TB*-spline on the mesh $2h\mathbb{Z} := \{2jh:\ \text{for } j \text{ in } \mathbb{Z}\}$, where as above h is a fixed positive number. One says that $h\mathbb{Z}$ is a *refinement* of $2h\mathbb{Z}$. We have seen in Section 13.10, p. 239, that the *TB*-spline $Q_{Z+1}[\Lambda; h]$ has support on the interval $[0, Zh + h]$ and break-points jh for $j = 0, 1, \ldots, Z+1$. In a similar way on the mesh $2h\mathbb{Z}$ the compactly supported TB-spline $Q_{Z+1}[\Lambda; 2h]$ has a support $[0, (Z+1)2h]$ with break-points $j2h$ for all $j = 0, 1, \ldots, Z+1$. On the other hand, obviously $Q_{Z+1}[\Lambda; 2h](x)$ is also an L-spline on the mesh $h\mathbb{Z}$. According to Theorem 13.38, p. 241, the integer shifts $Q_{Z+1}[\Lambda; h](x-\ell h)$ form a basis for all compactly supported splines on $\mathbb{R}$, hence it is possible to express $Q_{Z+1}[\Lambda; 2h]$ as a linear combination of the shifts $Q_{Z+1}[\Lambda; h](x - \ell h)$. Theorem 13.60 provides the exact linear combination.

Theorem 13.60 *We have the representation, called the* two-scale relation *or* refinement equation

$$Q_{Z+1}[\Lambda; 2h](x) = \sum_{\ell=0}^{Z+1} \gamma_\ell Q_{Z+1}[\Lambda; h](x - \ell h), \tag{13.50}$$

where the two-scale sequence *is*

$$\gamma_\ell = (-1)^\ell s_\ell \quad \text{for } \ell = 0, 1, \ldots, Z+1,$$

and the two-scale symbol *is* $Z(e^{-i\xi h}) = s_h(-e^{-i\xi h})$.[17]

[16] In the notation of de Boor *et al*. [5], we have $Q_{Z+1}[\Lambda](x) = N_\Lambda(x)$.

[17] See Part III for this terminology.

Proof Let us take the Fourier transform on both sides of the equality (13.50). Due to

$$\widehat{Q_{Z+1}[\Lambda; h]}(x-\ell h)(\xi) = \int_{-\infty}^{\infty} Q_{Z+1}[\Lambda; h](x-\ell h) e^{-i\xi x}\, dx$$
$$= e^{-i\xi\ell h} \widehat{Q_{Z+1}[\Lambda; h]}(\xi),$$

we obtain

$$\widehat{Q_{Z+1}[\Lambda; 2h]}(\xi) = \sum_{\ell=0}^{Z+1} \gamma_\ell e^{-i\xi\ell h} \widehat{Q_{Z+1}[\Lambda; h]}(\xi).$$

We obtain from formula (13.25), p. 242,

$$\widehat{Q_{Z+1}[\Lambda; 2h]}(\xi) = \prod_{j=1}^{Z+1} (e^{-\lambda_j h} + e^{-i\xi h}) \widehat{Q_{Z+1}[\Lambda; h]}(\xi)$$
$$= s_h(-e^{-i\xi h}) \widehat{Q_{Z+1}[\Lambda; h]}(\xi).$$

Since

$$s_h(-e^{-i\xi h}) = \sum_{\ell=0}^{Z+1} s_\ell (-e^{-i\xi h})^\ell = \sum_{\ell=0}^{Z+1} s_\ell (-1)^\ell e^{-i\xi\ell h},$$

the proof will be completed by taking the inverse Fourier transform. ■

Theorem 13.60 is another interpretation of Proposition 13.40, p. 243, where we have established a relation between the Fourier transforms of $Q_{Z+1}[\Lambda; 2h]$ and of $Q_{Z+1}[\Lambda; h]$.

This relation is quite close to being understood as a *generalized two-scale relation*. Anyway, we have a simple transition from one level to the other in the wavelet spaces, which will be much exploited in Part III.

Remark 13.61 *Due to the translation invariance we have the same coefficients for all shifts* $Q_{Z+1}(x - 2lh)$.

Remark 13.62 *If* $\Lambda = [0, \ldots, 0]$, *which corresponds to the usual polynomial case, we see that due to* $h\Lambda = \Lambda$ *it follows that:*

$$Q_{Z+1}[\Lambda; h](x) = h^Z \cdot Q_{Z+1}[\Lambda] \left(\frac{x}{h}\right),$$

which provides us with a scale invariant set of compactly supported functions. Chui [3] uses this in his cardinal spline wavelet analysis. For the nonzero vector Λ *we have the nonstationary wavelet analysis of de Boor* et al. *[5].*

13.23 Symmetry of the zeros of the Euler–Frobenius polynomial $\Pi_Z(\lambda)$

We now consider the special case of the nonordered vector Λ which is generating the spherical operator $M_{k,p}$, see formula (10.26), p. 169. We will prove a remarkable symmetry property of the compactly supported spline Q_{Z+1} and of the *Euler–Frobenius* polynomial $\Pi_Z(\lambda) = \Pi_Z(\lambda; 0)$ which are available due to the "almost" symmetry properties of the corresponding vector $\Lambda = [\lambda_1, \lambda_2, \ldots, \lambda_{2p}]$.

We consider the operator $L = M_{k,p}$. We have

$$Z = 2p - 1$$

and the nonordered vector $\Lambda = [\lambda_1, \lambda_2, \ldots, \lambda_{2p}]$ is given by

$$\begin{aligned} &\lambda_1 = -n - k + 2, \quad \lambda_2 = -n - k + 4, \quad \ldots, \quad \lambda_p = -n - k + 2p, \\ &\lambda_{p+1} = k, \qquad\qquad \lambda_{p+2} = k + 2, \qquad\quad \ldots, \quad \lambda_{2p} = k + 2p - 2. \end{aligned} \tag{13.51}$$

By the definition of the Euler–Frobenius polynomial and by the proof of Proposition 13.50, p. 250, namely equality (13.38) we have

$$\begin{aligned} \Pi_Z(\lambda) &= e^{\lambda_1 + \cdots + \lambda_{Z+1}} \cdot (-1)^Z \cdot \sum_{j=0}^{Z} Q_{Z+1}(j)\lambda^{Z-j} \qquad (13.52) \\ &= e^{\lambda_1 + \cdots + \lambda_{Z+1}} \cdot (-1)^Z \cdot \sum_{j=0}^{Z-1} Q_{Z+1}(Z-j)\lambda^{j}. \end{aligned}$$

Let us note that in the case of arbitrary symmetric set $\Lambda = -\Lambda$ we will have $Q_{Z+1}(j) = Q_{Z+1}(Z+1-j)$. Indeed, in such a case the function

$$Q_{Z+1}(Z + 1 - x)$$

is a piecewise linear combination of

$$\{e^{-\lambda_1 x}, \ldots, e^{-\lambda_{Z+1} x}\} = \{e^{\lambda_1 x}, \ldots, e^{\lambda_{Z+1} x}\},$$

hence, due to the uniqueness of the compactly supported TB-spline Q_{Z+1} with support $[0, Z+1]$ it follows that:

$$Q_{Z+1}(Z + 1 - x) = C \cdot Q_{Z+1}(x)$$

for some constant $C > 0$. But for $x = (Z+1)/2$ we obtain $Q_{Z+1}((Z+1)/2) = C \cdot Q_{Z+1}((Z+1)/2)$, hence $C = 1$. Thus by Proposition 13.50, p. 250, we obtain

$$\Pi_Z(\lambda) = \lambda^{Z-1}\Pi_Z\left(\frac{1}{\lambda}\right).$$

Hence

$$\Pi_Z(\lambda) = 0$$

implies

$$\Pi_Z\left(\frac{1}{\lambda}\right) = 0.$$

We will see that for the above special choice of the vector Λ in (13.51) we have a rather similar picture since the set Λ "symmetrizes" for $k \to \infty$. We know that the function $Q_{Z+1}(Z+1-x)$ is a piecewise linear combination of the functions

$$\{e^{-\lambda_1 x}, e^{-\lambda_2 x}, \ldots, e^{-\lambda_{2p} x}\}.$$

Due to the "almost" symmetry of the vector Λ we see that after multiplying with $e^{(\lambda_1+\lambda_{2p})x}$ the basis for $-\Lambda$ changes into the basis for Λ, namely

$$e^{(\lambda_1+\lambda_{2p})x} \cdot \{e^{-\lambda_1 x}, e^{-\lambda_2 x}, \ldots, e^{-\lambda_{2p} x}\} = \{e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_{2p} x}\}.$$

We have used the equalities

$$\lambda_1 + \lambda_{2p} = -n - k + 2 + k + 2p - 2 = -n + 2p,$$

$$\begin{aligned} -\lambda_j + \lambda_1 + \lambda_{2p} &= k + 2(p - j) \\ &= \lambda_{p+p-j} \quad \text{for } j = 1, \ldots, p, \end{aligned}$$

$$\begin{aligned} -\lambda_{p+j} + \lambda_1 + \lambda_{2p} &= -n - k + 2(p + 1 - j) \\ &= \lambda_{p+1-j} \quad \text{for } j = 1, \ldots, p. \end{aligned}$$

Thus by the uniqueness of the compactly supported spline we obtain

$$e^{(\lambda_1+\lambda_{2p})x} Q_{Z+1}(Z+1-x) = C \cdot Q_{Z+1}(x).$$

By putting $x = (Z+1)/2$ it follows that

$$C = e^{(\lambda_1+\lambda_{2p})(Z+1)/2} = e^{(\lambda_1+\lambda_{2p})p} = e^{(-n+2p)p}.$$

Thus we have proved the following result about the symmetry of the compactly supported TB-spline.

Theorem 13.63 *For the special choice of the set Λ given by (13.51), p. 261, we have*

$$\begin{aligned} Q_{Z+1}(Z+1-x) &= e^{(-n+2p)p} \cdot e^{-(-n+2p)x} \cdot Q_{Z+1}(x) \\ &= e^{(-n+2p)(p-x)} \cdot Q_{Z+1}(x). \end{aligned} \tag{13.53}$$

It should be noted that this result is independent of k.

Now we will draw some consequences about the symmetry of the polynomial Π_Z and its zeros. We obtain from (13.52) the equalities

$$\begin{aligned} \Pi_Z(\lambda) &= e^{\lambda_1+\cdots+\lambda_{Z+1}} \cdot (-1)^Z \cdot \sum_{j=0}^{Z} Q_{Z+1}(j)\lambda^{Z-j} \\ &= e^{\lambda_1+\cdots+\lambda_{Z+1}} \cdot (-1)^Z \cdot e^{-(-n+2p)p} \cdot \sum_{j=0}^{Z} e^{(-n+2p)j} Q_{Z+1}(Z+1-j)\lambda^{Z-j}. \end{aligned}$$

Let us recall that since $Q_{Z+1}(Z+1) = 0$ the term with $j = 0$ is zero. If we put $i = Z - j$ or $j = Z - i$ we see that

$$\begin{aligned}
\Pi_Z(\lambda; 0) &= e^{\lambda_1+\cdots+\lambda_{Z+1}} \cdot (-1)^Z \cdot e^{-(-n+2p)p} \cdot \sum_{i=0}^{Z} e^{(-n+2p)(Z-i)} Q_{Z+1}(i+1)\lambda^i \\
&= \left(\frac{\lambda}{e^{-n+2p}}\right)^{-1} e^{\lambda_1+\cdots+\lambda_{Z+1}} \cdot (-1)^Z \cdot e^{-(-n+2p)p} \cdot e^{(-n+2p)Z} \\
&\quad \times \sum_{i=0}^{Z} Q_{Z+1}(i+1) \left(\frac{\lambda}{e^{-n+2p}}\right)^{i+1} \\
&= \left(\frac{\lambda}{e^{-n+2p}}\right)^{-1+Z} e^{\lambda_1+\cdots+\lambda_{Z+1}} \cdot (-1)^Z \cdot e^{-(-n+2p)p} \cdot e^{(-n+2p)Z} \\
&\quad \times \sum_{i=0}^{Z} Q_{Z+1}(i+1) \left(\frac{e^{-n+2p}}{\lambda}\right)^{Z-(i+1)} \\
&= \lambda^{Z-1} \cdot C \cdot \Pi_Z\left(\frac{e^{-n+2p}}{\lambda}; 0\right)
\end{aligned}$$

for a constant C which may be defined by the above and it is clear that $C \neq 0$. We find this constant by putting $\widetilde{\lambda} = \sqrt{e^{-n+2p}}$. This gives

$$\Pi_Z(\widetilde{\lambda}; 0) = \widetilde{\lambda}^{Z-1} \cdot C \cdot \Pi_Z(\widetilde{\lambda}; 0),$$

hence since $\Pi_Z(\lambda; 0)$ has only negative zeros we obtain

$$C = e^{-p(-n+2p)/2}.$$

By the general theory, see Theorem 13.31, p. 237, we know that all $Z - 1 = 2p - 2$ zeros of $\Pi_Z(\lambda; 0)$ satisfy

$$\mu_{Z-1} < \cdots < \mu_1 < 0,$$

hence we see that all zeros separate into two groups. Thus we have proved Theorem 13.64.

Theorem 13.64 *For the special choice of Λ given by (13.51), p. 261, we have the symmetry*

$$\Pi_Z(\lambda; 0) = \lambda^{Z-1} \cdot e^{-p(-n+2p)/2} \cdot \Pi_Z\left(\frac{e^{-n+2p}}{\lambda}; 0\right).$$

If for some $\lambda \neq 0$ we have

$$\Pi_Z(\lambda; 0) = 0$$

then also

$$\Pi_Z\left(\frac{e^{-n+2p}}{\lambda}; 0\right) = 0.$$

Hence, the $Z-1=2p-2$ *zeros of the equation* $\Pi_Z(\lambda;0)=0$ *satisfy*

$$\mu_j\mu_{2p-2-j+1}=e^{-n+2p}\quad \textit{for } j=1,\ldots,p-1,$$

and

$$\mu_{2p-2}<\cdots<\mu_p<-\sqrt[2]{e^{(-n+2p)}}<\mu_{p-1}<\cdots<\mu_1<0.$$

We see again the *remarkable fact that this symmetry is completely independent of* k, in particular the constant $-\sqrt[2]{e^{(-n+2p)}}$ is independent of k.

These results will be used in the cardinal interpolation with polysplines in Section 15.7.

13.24 Estimates of the functions $A_Z(x;\lambda)$ and $Q_{Z+1}(x)$

We will provide some important estimates of the function $A(x;\lambda)$ for the special choice of the set Λ above in (13.51), p. 261, and the somewhat more general cases considered in [9].

Using the residuum representation (13.11), p. 235, of the function $A(x;\lambda)$ we prove the following.

Theorem 13.65 *Let the vector* Λ *be the one given by (13.51), p. 261. Let* K *be a compact subset of the complex plane,* $0\notin K$ *and hence* $e^{\lambda_j}\notin K$ *for large* k*. Then for every* $\varepsilon>0$ *there exist a constant* $C>0$ *and an integer* k_0 *such that for all* $k\geq k_0$*, for all* $\lambda\in K$*, and for all* x *satisfying* $0\leq x\leq 1$*, the following estimate holds:*

$$|A_Z(x;\lambda)|\leq \frac{C}{k^Z}. \tag{13.54}$$

Proof We will prove the estimate first for all x satisfying $0\leq x\leq 1-\delta$ for every small $\delta>0$. Then it will follow for all $0\leq x\leq 1$ by the symmetry property (13.40), p. 252, i.e.

$$A_Z\left(1-x;\frac{1}{\lambda}\right)=(-1)^{Z-1}\lambda A_Z^*(x;\lambda)\quad \text{for } 0\leq x\leq 1.$$

For simplicity we consider the case $p=2$, $Z=2p-1=3$, and $K=\{|\lambda|=1\}$. By formula (13.11), p. 235, we have

$$A_3(0;\lambda)=\frac{1}{2\pi i}\int_\Gamma \frac{dz}{q_4(z)(e^z-\lambda)},$$

where Γ is a contour (or a sum of contours with the same orientation) in the complex plane which surrounds all points $\{t_1,\ldots,t_8\}$ and does not surround the points $i\varphi$ for real φ such that $e^{i\varphi}=\lambda$.

We will choose $\Gamma=\Gamma_1\cup\Gamma_2$, where Γ_1 is a circle which surrounds the points λ_1,λ_2, and Γ_2 is a circle which surrounds the points λ_3,λ_4, namely we put

$$\Gamma_j:=\{z\in C:|z-z_j|=R_j\}\quad \text{for } j=1,2,$$

where

$$z_1 := \frac{\lambda_1 + \lambda_2}{2} = -n - k + 3,$$
$$R_1 := |z_1| - 2 = +n + k + 1,$$

and

$$z_2 := \frac{\lambda_3 + \lambda_4}{2} = k + 1,$$
$$R_2 := |z_2 - \lambda_3| + 1 = 2.$$

As will become clear, we have chosen these circles in order to obtain the best possible estimate.

Indeed, for large k and some constant $C_1 > 0$ we have the inequality

$$|q_4(z)| \geq C_1 |2k - 1|^2 \quad \text{for } z \in \Gamma_2.$$

On the other hand, $|e^z - \lambda| \geq |e^z| - |\lambda|$ implies for large k the inequality

$$|e^z - \lambda| \geq e^{k-1} - 2 \quad \text{for } z \in \Gamma_2 \text{ and } \lambda \in K.$$

The above inequalities imply

$$I_2 := \left| \int_{\Gamma_2} \frac{dz}{q_4(z)(e^z - \lambda)} \right| \leq \frac{R_2}{C_1 |2k - 1|^2 (e^{k-1} - 2)}.$$

This estimate provides exponential decay for the integral over Γ_2 for $k \to \infty$.

On the other hand for the integral over the circle Γ_1 for an appropriate constant $C_2 > 0$ we have

$$|q_4(z)| \geq C_2 R_1^2 \lambda_3^2 \quad \text{for } z \in \Gamma_1,$$
$$|e^z - \lambda| \geq 1 - e^{-2} \quad \text{for } z \in \Gamma_1 \text{ and } \lambda \in K,$$

and obtain for an appropriate $C_2' > 0$ the estimate

$$I_1 := \left| \int_{\Gamma_1} \frac{dz}{q_4(z)(e^z - \lambda)} \right| \leq C_2' \frac{R_1}{k^4 |e^{-2} - 1|} = C_2' \frac{n + k - 1}{k^4 |e^{-2} - 1|}.$$

Since $A_3(x; \lambda)$ is the sum of the two integrals the statement of the theorem follows. ■

We can now provide an optimal estimate for the compactly supported spline.

According to formulas (13.42) and (13.44) we obtain for $0 \leq x \leq 1$ the representation

$$\sum_{j=-\infty}^{\infty} Q_{Z+1}(x - j) = (-1)^Z e^{-\lambda_1 - \lambda_2 - \cdots - \lambda_{Z+1}} r_{Z+1}(1) A_Z(x; 1).$$

Taking all terms in the sum we see that

$$\max_{x\in\mathbb{R}} Q_{Z+1}(x) \le e^{-\lambda_1-\lambda_2-\cdots-\lambda_{Z+1}} |r_{Z+1}(1)| \max_{x\in[0,1]} |A_Z(x;1)|.$$

Theorem 13.66 *Let the compactly supported spline $Q_{Z+1}(x)$ correspond to the vector Λ of (13.51), p. 261. Then for $k \to \infty$ it satisfies the asymptotic order*

$$\max_{x\in\mathbb{R}} Q_{Z+1}(x) \approx \frac{e^{pk}}{k^Z}. \tag{13.55}$$

Proof The estimate of $\max_{x\in[0,1]} |A_Z(x;1)|$ comes from the above theorem. Since $\lambda_j \to 0$ for $j = 1, 2, \ldots, p$, and $\lambda_j \to \infty$ for $j = p+1, \ldots, 2p$, the estimate of the asymptotic order of $r_{Z+1}(1)$ is

$$|r_{Z+1}(1)| \le \prod_{j=1}^{Z+1} |e^{\lambda_j} - 1| \le Ce^{pk}.$$

This completes the proof. ∎