Part III

Wavelet analysis

In the Introduction to Part II we explained how naturally the cardinal polysplines appear. This is especially non-trivial in the case of polysplines with break-surfaces on concentric spheres. Similarly the notion of *polyharmonic multiresolution analysis* appears in a natural manner. Let us explain how it happens in the case of polysplines on annuli.

First, recall that every cardinal polyspline on annuli has break-surfaces the spheres $S(0; e^{\nu})$ for all integers $\nu \in \mathbb{Z}$, and possesses an expansion in spherical harmonics of the form

$$h(x) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d_k} h^{k,\ell}(\log r)\, Y_{k,\ell}(\theta).$$

For every couple of indexes k and ℓ the function $h^{k,\ell}(v)$ is a cardinal L-spline for the constant coefficients operator $L = M_{k,p}$ with knots at $\nu \in \mathbb{Z}$.

What would be more natural than to make multiresolution analysis (MRA) of every one-dimensional component $h^{k,\ell}(v)$, i.e. of the L-splines for the operator $L = M_{k,p}$? And the refinement of the knot set $\mathbb{Z}$ will be $(1/2)\mathbb{Z}$, etc. Furthermore we are lucky that the basic elements of wavelet analysis for cardinal L-splines has been created by de Boor *et al.* [9].[1]

Having obtained MRA for every component $h^{k,\ell}$ it remains only to assemble the puzzle by means of the above formula (III).[2]

The above program is easy to describe in general terms but it takes a good deal of work to accomplish. First, let us recall that in the polynomial case the detailed cardinal spline wavelet analysis has been carried out by Chui [4], [5]. Accordingly, in Chapter 16 we provide a brief review of his results. This review will be our compass when studying the cardinal L-spline wavelet analysis. The transition from the polynomial spline case to the L-spline case is highly non-trivial and we were aware of that in the proof of the *Riesz inequalities* for the shifts of the TB-spline Q_{Z+1} in Chapter 14, p. 267.

The Cardinal L-spline wavelet analysis which we develop in Chapter 17 uses the whole machinery of cardinal L-splines which we have developed in Part II. It is amazing that all the main results of Chui's approach permit non-trivial generalizations for the cardinal L-spline wavelet analysis. The dependence on the vector Λ is essential and is emphasized in these results. The Chui's results are reduced to the special case of the vector $\Lambda = [0, ..., 0]$.

Finally in Chapter 18 we obtain the assembled "polyharmonic wavelet". It has some interesting properties. Needless to say, it does not satisfy the axioms of the MRA as established by Y. Meyer and S. Mallat, see [14]. So far it satisfies some of them and also some other properties which we provide in Theorem 18.9, p. 380. Put in a proper framework these properties may be considered as the axioms of what we call *Polyharmonic Multiresolution Analysis.*

[1] The Bibliography is at the end of the present Part.

[2] As we already mentioned in the Introduction to Part II, due to the lack of space we do not consider in detail the cardinal Polysplines on strips. For that reason we do not consider here the Wavelet Analysis generated by Polysplines on strips. We note only that its formulas are simpler than the annular case, in particular the refinement is by considering parallel hyperplanes having say first coordinate $t \in 2^{-j}\mathbb{Z}$, see Definition 9.1, p. 118.

It is remarkable that we may preserve the basic scheme of the usual MRA but if we introduce some proper substitutes to the basic notions:

1. There is no *refinement equation* but we have a *refinement operator*, see Theorem 18.9, p. 380. This refinement operator is generated by the *non-stationary scaling operator* for the L-splines defined in formula (17.6), p. 328.
2. In Section 18.5, p. 379, we see that there is a unique function which we call *father wavelet* and it generates the spaces PV_j of the *polyharmonic MRA* in a “non-stationary” way. In a similar way, in Section 18.6, p. 384, we see that there is a unique function which we call *mother wavelet* which generates the wavelet spaces PW_j in a “non-stationary” way.
3. To have the whole picture completed let us recall that we have also the *sampling operator* provided by formula (14.16), p. 285, of Part II.

Hence, the main conclusion of the present Part is that the attempt to make a *reasonable Multiresolution Analysis* by means of a refining sequence of spaces of *cardinal polysplines (on annuli or strips)* leads to a considerable reconsideration of the whole store of basic notions of MRA.

The present Part provides a detailed study of only one example of “spherical polyharmonic wavelet analysis”. Similar wavelet analysis may be carried out for other elliptic differential operators of the form

$$A\,(r)\,\frac{\partial^2}{\partial r^2} + B\,(r)\,\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_\theta$$

which are possibly degenerate at the origin but split into infinitely many one-dimensional operators with constant coefficients.

What might be the area of application of such wavelets which have *by definition* singularities on whole $(n-1)$-dimensional surfaces? Let us point out to a possible application for analyzing, e.g. in $\mathbb{R}^2$, images having singularities on curves. The problem of efficient computational analysis of such images has been indicated by Meyer and Mallat – the standard wavelet paradigm is not efficient for analyzing images having $(n-1)$-dimensional singularities. This problem has been given a thorough consideration in a series of papers of D. Donoho with coauthors. In particular, the curvelets by D. Donoho and E. Candes [2] have been created with the main purpose to solve this problem. The polyharmonic wavelets may be considered as an alternative approach to this problem.

Finally, let us note that much as we do not like it many of the formulas in the present Part are overburdened with indexes and arguments which makes it somewhat heavy to read. On the other hand this detailed exposition would provide the reader with the opportunity to check the correctness of all formulas.

Chapter 16

Chui's cardinal spline wavelet analysis

As is usual in wavelet analysis, we will be working in the space $L_2(\mathbb{R})$ of square summable complex valued functions with the scalar product defined by

$$\langle f, g \rangle := \langle f, g \rangle_{L_2(\mathbb{R})} := \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx \tag{16.1}$$

for every two functions $f, g \in L_2(\mathbb{R})$. We have the norm

$$\| f \|^2 := \| f \|^2_{L_2(\mathbb{R})} := \langle f, f \rangle. \tag{16.2}$$

16.1 Cardinal splines and the sets V_j

Denote by V_j the closure in $L_2(\mathbb{R})$ of the space of all **cardinal polynomial splines** (which are in $L_2(\mathbb{R})$) of polynomial degree m having knots at the points

$$\mathbb{Z} \cdot 2^{-j} = \left\{ \frac{\ell}{2^j} : \text{ for all } \ell \in \mathbb{Z} \right\}. \tag{16.3}$$

By definition their smoothness is C^{m-1}. Evidently, since $\mathbb{Z} \cdot 2^{-j_1} \subset \mathbb{Z} \cdot 2^{-j_2}$ for every two integers j_1 and j_2 satisfying $j_1 < j_2$, we have the inclusions

$$\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots .$$

We use the term *cardinal* in the wider sense, understanding splines with knots at the set $\mathbb{Z} \cdot h = \{\ell h : \text{ for all } \ell \in \mathbb{Z}\}$ for some number $h > 0$.

The most important function in the theory of cardinal splines and also of spline wavelet analysis is the compactly supported B-spline $N_m(x) \in V_0$ with support coinciding with the interval $[0, m]$ and with knots at the integers. Following the tradition in

MRA we will denote it through $\phi(x)$, i.e.

$$\phi(x) := N_m(x), \tag{16.4}$$

and we will call it the **scaling function**. We know that N_m is symmetric around the point $m/2$ which is the center of the interval $[0, m]$, i.e.

$$N_m(x) = N_m(m - x) \quad \text{for } x \in \mathbb{R}.$$

An important formula is the one providing the **Fourier transform** of N_m

$$\widehat{N_m}(\xi) = \left(\frac{1 - e^{-i\xi}}{i\xi} \right)^m, \tag{16.5}$$

[5, p. 53]. It is remarkable that the cardinal spline $N_m(x)$ which has knots at the integer points $\mathbb{Z}$ generates through shifts not only the space V_0, i.e.[3]

$$V_0 = \operatorname*{clos}_{L_2(\mathbb{R})} \{\phi(x - \ell) : \text{ for all } \ell \in \mathbb{Z}\}, \tag{16.6}$$

but also the spaces V_j. For that purpose one forms the 2^j-dilates of the function $N_m(x)$, namely $N_m(2^j x)$, and considers its shifts, thus obtaining for all $j \in \mathbb{Z}$, the equality

$$V_j = \operatorname*{clos}_{L_2(\mathbb{R})} \{\phi(2^j x - \ell) : \text{ for all } \ell \in \mathbb{Z}\}. \tag{16.7}$$

Actually, this is due to the fact that the function $\phi(2^j x) = N_m(2^j x)$ is again piecewise polynomial of degree $\leq m$ but with knots on the mesh $\mathbb{Z}2^{-j} = \{\ell/2^j : \text{ for all } \ell \in \mathbb{Z}\}$.

Since $V_0 \subset V_1$ we have, [4, p. 91, formula (4.3.2)] the central relation in MRA called the **two-scale relation** or **refinement equation**[4] for $\phi(x)$, namely

$$\phi(x) = \sum_{j=-\infty}^{\infty} p_j \phi(2x - j), \tag{16.8}$$

where in fact the sequence $\{p_j\}$ is finite and is given by the coefficients of the polynomial

$$P(z) := \frac{1}{2} \sum_{j=0}^{m} p_j z^j := \left(\frac{1 + z}{2} \right)^m. \tag{16.9}$$

In order to make clear this *most fundamental relation* in wavelet theory, let us show that it is easy to prove. After taking the Fourier transform of (16.8), we obtain

$$\widehat{\phi}(\xi) = \frac{1}{2} \sum_{j=0}^{m} p_j e^{-ij(\xi/2)} \cdot \widehat{\phi}\left(\frac{\xi}{2} \right), \tag{16.10}$$

[3] The notation $\text{clos}_{L_2(\mathbb{R})}$ means the linear and topological hull in the norm of $L_2(\mathbb{R})$.

[4] In the setting of some authors these are *prewavelets* (de Boor *et al.* [9]), in the setting of Chui these are *semi-orthogonal wavelets.*.

and by substituting the explicit formula for the Fourier transform (16.5) above we obtain

$$\left(\frac{1-e^{-i\xi}}{i\xi}\right)^m = P(z)\left(\frac{1-e^{-i\xi/2}}{i\xi/2}\right)^m .$$

Now the identity

$$(1-e^{-i\xi})^m = \left(1-e^{-i\xi/2}\right)^m \left(1+e^{-i\xi/2}\right)^m$$

implies (16.9).

The principal property of the scaling function ϕ is that the set of shifts $\{\phi(x-\ell):\ \text{for all } \ell \in \mathbb{Z}\}$ is a **Riesz basis** of V_0. The notion of Riesz basis has become a replacement condition for orthogonal basis since every Riesz basis may be orthonormalized and many properties of the orthogonal basis are preserved. By the definition of a Riesz basis there exist two constants A, B with $0 < A \leq B < \infty$, and for every sequence $\{c_j\} \in \ell_2$ holds

$$A \sum_{j=-\infty}^{\infty} |c_j|^2 \leq \left\| \sum_{j=-\infty}^{\infty} c_j \phi(x-j) \right\|_{L_2(\mathbb{R})} \leq B \sum_{j=-\infty}^{\infty} |c_j|^2. \tag{16.11}$$

The constants A, B are called **Riesz bounds**. An *equivalent condition* for a basis to be Riesz is that

$$\lfloor \phi \rfloor := \sum_{j=-\infty}^{\infty} \left|\widehat{\phi}(\xi + 2\pi j)\right|^2 < \infty \quad \text{a.e. in } \mathbb{R}. \tag{16.12}$$

Below we will find an explicit expression for this infinite sum.

16.2 The wavelet spaces W_j

The **wavelet spaces** W_j are defined uniquely through the properties holding for all j in $\mathbb{Z}$, namely:

$$V_{j+1} = V_j \oplus W_j,$$
$$W_j \subset V_{j+1},$$

or, equivalently,

$$W_j := V_{j+1} \ominus V_j \quad \text{for all } j \text{ in } \mathbb{Z}. \tag{16.13}$$

Here $\oplus$ means the orthogonal sum of two linear spaces, and in the context above we have two mutually orthogonal subspaces, V_j and W_j; their usual sum gives V_{j+1}. The last means that in $V_{j+1} = V_j \oplus W_j$ we have a sum but not simply isomorphism!

Since

$$\operatorname*{clos}_{L_2(\mathbb{R})} \left\{ \bigcup_{j=-\infty}^{\infty} V_j \right\} = L_2(\mathbb{R}) \tag{16.14}$$

we obtain the expansion

$$L_2(\mathbb{R}) = \bigoplus_{j=-\infty}^{\infty} W_j. \tag{16.15}$$

Any function may be expanded in a **generalized Fourier series**

$$f(x) = \sum_{j=-\infty}^{\infty} w_j \quad \text{with } w_j \in W_j \text{ for all } j \in \mathbb{Z}. \tag{16.16}$$

The major factor contributing to the charm of the *classical cardinal spline wavelets*[5] is that, roughly speaking, one has only one generating function ψ for all spaces W_j. More precisely, ψ is called the mother **wavelet** if its shifts generate W_0, i.e.

$$W_0 = \operatorname*{clos}_{L_2(\mathbb{R})} \{\psi(x-\ell) \colon \text{ for all } \ell \in \mathbb{Z}\} \tag{16.17}$$

and the 2^j-dilates

$$\psi_{j,\ell}(x) := 2^{j/2}\psi(2^j x - \ell)$$

generate the spaces W_j, i.e.

$$W_j = \operatorname*{clos}_{L_2(\mathbb{R})} \{\psi_{j,\ell}(x) \colon \text{ for all } \ell \in \mathbb{Z}\}. \tag{16.18}$$

The norming of the functions $\psi_{j,\ell}(x)$ is important since it preserves the L_2-norm of linear combinations at every level j for $j \in \mathbb{Z}$, namely

$$\int_{-\infty}^{\infty} \left| \sum_{\ell=-\infty}^{\infty} c_\ell 2^{j/2} \psi(2^j x - \ell) \right|^2 dx = \int_{-\infty}^{\infty} \left| \sum_{\ell=-\infty}^{\infty} c_\ell \psi(x-\ell) \right|^2 dx,$$

hence the set of shifts $\{\psi_{j,\ell}(x)\}_{\ell\in\mathbb{Z}}$ has the same Riesz constants A, B as the shifts of $\psi(x)$.

Let us note that the polynomial spline wavelets owe their algorithmic effectiveness to the fact that the polynomial splines are scale-invariant. That is: if $f(x)$ is a polynomial spline, then for every number h the function $f(hx)$ is also a polynomial spline. In some sense, dilations do not change the physical nature of the basic space of functions used to construct MRA. It is important to note that in a similar way all classical wavelet constructions and the axioms of MRA rely upon this principle [8, 14], namely that the *physical nature* of the dilated function $f(hx)$ is the same as that of the original function $f(x)$. We will see that this is not the case for the general L-spline wavelets.

[5] In the setting of some authors these are *prewavelets* (de Boor *et al.* [9]), in the setting of Chui these are *semi-orthogonal wavelets.*

16.3 The mother wavelet ψ

There is an **explicit formula** for the function ψ in *cardinal spline wavelet analysis*. Since $W_0 \subset V_1$, we have the representation

$$\psi(x) = \sum_{j=-\infty}^{\infty} r_j \phi(2x - j), \tag{16.19}$$

where r_j are the coefficients of a Laurent polynomial $R(z)$. A Laurent polynomial also has negative exponents. We have

$$R(z) = \frac{1}{2} \cdot \sum_{j=-\infty}^{\infty} r_j z^j.$$

An important property is that only a finite number of the coefficients r_j are non-zero, hence ψ has a **compact support**. Indeed, if we pass to the Fourier images (in the frequency variable ξ), we obtain

$$\widehat{\psi}(\xi) = R\left(e^{-i\xi/2}\right)\widehat{\phi}\left(\frac{\xi}{2}\right) \quad \text{for all } \xi \text{ in } \mathbb{R}. \tag{16.20}$$

On the other hand, $\psi \perp V_0$. These two conditions and the fact that ψ has shifts $\{\psi\,(x-\ell)\colon \text{ for all } \ell \in \mathbb{Z}\}$ which are the basis of W_0 determine ψ uniquely up to a constant factor

$$R(z) = -z^{2m-1} P\left(-\frac{1}{z}\right) E_\phi\left(-\frac{1}{z}\right).$$

Here we have denoted by $E_\phi(z)$ the so-called **Euler–Frobenius polynomial** (which is a Laurent polynomial) which was introduced by Schoenberg. It is given by

$$E_\phi(z) = \sum_{\ell=-m+1}^{m-1} \phi_{2m}(m+\ell) z^\ell; \tag{16.21}$$

where again

$$\phi_{2m}(x) := N_{2m}(x).$$

We see that $R(z)$ is a classical polynomial (it does not contain negative exponents of z) and its degree is evidently $3m-2$. Hence, we easily prove by equality (16.19), p. 317, that the wavelet function $\psi(x)$ has a compact support coinciding with the interval $[0, 2m-1]$.

The polynomial $E_\phi(z)$ is the *key function* to the whole approach. Let us note the important equality

$$\lfloor \phi \rfloor = \sum_{\ell=-\infty}^{\infty} \left|\widehat{\phi}(\xi + 2\pi\ell)\right|^2 = E_\phi(e^{-i\xi}), \tag{16.22}$$

where we have used the notation for this norm introduced above in (16.12), p. 315. Since the function $E_\phi(z)$ has no zeros on the unit circle $|z| = 1$, owing to the Riesz inequalities

(16.11) and the norm in (16.12), p. 315, this might be considered as a proof that the shifts of the function $\phi(x)$ provide a Riesz basis in V_0. Due to the symmetry of the cardinal spline $N_{2m}(x)$ around the point m we see that

$$E_\phi(z) = E_\phi\left(\frac{1}{z}\right).$$

16.4 The dual mother wavelet $\widetilde{\psi}$

Another important property of the function $\psi(x)$ which *makes it a wavelet* is the existence of the **dual wavelet** $\widetilde{\psi}(x)$. The dual wavelet function gives rise to the functions

$$\widetilde{\psi}_{j,\ell}(x) := 2^{j/2}\widetilde{\psi}(2^j x - \ell), \tag{16.23}$$

which satisfy the basic **bi-orthogonality** property, for all indexes $j, j_1, \ell, \ell_1 \in \mathbb{Z}$, namely[6]

$$\langle\widetilde{\psi}_{j,\ell}, \psi_{j_1,\ell_1}\rangle := \int_{-\infty}^{\infty} \widetilde{\psi}_{j,\ell}(x)\overline{\psi_{j_1,\ell_1}(x)}\, dx = \delta_{j,j_1} \cdot \delta_{\ell,\ell_1}.$$

Now if the generalized Fourier series representation (16.16), p. 316, is written as follows, see Chui [5, p. 89]:

$$f(x) = \sum_{j,\ell=-\infty}^{\infty} d_{j,\ell}\psi_{j,\ell}(x),$$

we see that after taking the scalar product with $\widetilde{\psi}_{j,\ell}(x)$ for every two indexes j and ℓ in $\mathbb{Z}$ we obtain

$$d_{j,\ell} = \langle f, \widetilde{\psi}_{j,\ell}\rangle \quad \text{for all } j, \ell \text{ in } \mathbb{Z}.$$

Let us denote

$$\widetilde{W}_j := \operatorname*{clos}_{L_2(\mathbb{R})} \left\{\widetilde{\psi}_{j,\ell} : \text{ for all } j, \ell \text{ in } \mathbb{Z}\right\}.$$

Then the following orthogonality properties are satisfied:

$$\begin{cases} \widetilde{W}_j \perp V_j & \text{for all } j \in \mathbb{Z}, \\ W_j \perp V_j & \text{for all } j \in \mathbb{Z}. \end{cases} \tag{16.24}$$

There is an explicit expression for the *unique dual wavelet* in terms of its Fourier transform

$$\widehat{\widetilde{\psi}}(\xi) = \frac{\widehat{\psi}(\xi)}{\sum_{\ell=-\infty}^{\infty} |\widehat{\psi}(\xi + 2\pi\ell)|^2}.$$

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[6] Here δ is the Kronecker symbol defined as $\delta_{\alpha\beta} = 1$ for $\alpha = \beta$ and $\delta_{\alpha\alpha} = 0$.

The last expression makes sense since the sum is convergent, and we have the following elegant explicit expression [5, p. 106, formula (5.3.11)]:

$$\lfloor \psi \rfloor = \sum_{\ell=-\infty}^{\infty} |\widehat{\psi}(\xi + 2\pi\ell)|^2 = |R(z)|^2 E_\phi(z) + |R(-z)|^2 E_\phi(-z) \tag{16.25}$$

$$= E_\phi(z) E_\phi(-z) E_\phi(z^2) \quad \text{for all } z = e^{-i\xi/2}.$$

The last is nonzero since the Euler–Frobenius polynomial $E_\phi(z)$ has no zeros on the unit circle $|z| = 1$. This remark provides a proof that the set of shifts $\{\psi(x-\ell)\colon \text{ for all } \ell \in \mathbb{Z}\}$ is a Riesz basis of W_0 by applying the criterion for a Riesz basis using the Fourier transform of ψ. It is similar to that for the scaling function $\phi(x)$ which we have seen in (16.11) and (16.12), p. 315.

Another important identity is

$$E_\phi(z^2) = |P(z)|^2 E_\phi(z) + |P(-z)|^2 E_\phi(-z). \tag{16.26}$$

16.5 The dual scaling function $\widetilde{\phi}$

Now we consider the **dual scaling function** $\widetilde{\phi}(x) \in V_0$. It satisfies the orthogonality property

$$\langle \phi(x-j), \widetilde{\phi}(x-\ell) \rangle = \delta_{j,\ell} \quad \text{for all } j, \ell \text{ in } \mathbb{Z}.$$

There is an explicit expression for $\widetilde{\phi}(x)$ in terms of its Fourier transform

$$\widehat{\widetilde{\phi}}(\xi) = \frac{\widehat{\phi}(\xi)}{\sum_{\ell=-\infty}^{\infty} |\widehat{\phi}(\xi + 2\pi\ell)|^2} = \frac{\widehat{\phi}(\xi)}{E_\phi(e^{-i\xi})}. \tag{16.27}$$

If we denote

$$\widetilde{V_j} := \operatorname*{clos}_{L_2(\mathbb{R})} \{\widetilde{\phi}(x-\ell)\colon \text{ for all } \ell \in \mathbb{Z}\},$$

then we obtain

$$\widetilde{V_j} = V_j$$

and also the following orthogonality relations which together with the relations in (16.24) read as follows:

$$W_j \perp \widetilde{V_j}, \quad \widetilde{W_j} \perp V_j, \quad W_j \perp V_j \quad \text{for all } j \text{ in } \mathbb{Z}.$$

16.6 Decomposition relations

Since the **Euler–Frobenius** polynomial $E_\phi(z)$ does not take on zero values on the unit circle $|z| = 1$, there always exists the inverse Laurent polynomial which is convergent near the unit circle, and we may put

$$E_\phi(z) = \sum_{j=-\infty}^{\infty} \beta_j z^j,$$

$$\frac{1}{E_\phi(z)} = \sum_{j=-\infty}^{\infty} \alpha_j z^j.$$

The coefficients α_j decay exponentially. By taking the inverse Fourier transform in equality (16.27) we see that we may express the basis of V_0 in both directions

$$\widetilde{\phi}(x) = \sum_{j=-\infty}^{\infty} \alpha_j \phi(x-j),$$

$$\phi(x) = \sum_{j=-\infty}^{\infty} \beta_j \widetilde{\phi}(x-j).$$

Since

$$V_1 = V_0 \oplus W_0,$$

it is obvious that the functions $\phi(2x)$ and $\phi\,(2x-1)$ may be represented as

$$\phi(2x) = \sum_{s=-\infty}^{\infty} \{a_{-2s}\phi(x-s) + b_{-2s}\psi(x-s)\},$$

$$\phi(2x-1) = \sum_{s=-\infty}^{\infty} \{a_{1-2s}\phi(x-s) + b_{1-2s}\psi(x-s)\},$$

by means of the sequences a_{2j}, b_{2j} and a_{2j+1}, b_{2j+1}, respectively. We may combine these two representations in one called the **decomposition relation**, distinguishing only the case of odd and even index ℓ, namely

$$\phi(2x-\ell) = \sum_{s=-\infty}^{\infty} \{a_{\ell-2s}\phi(x-s) + b_{\ell-2s}\psi(x-s)\}. \tag{16.28}$$

For the sequences a_j and b_j we define the corresponding symbols by putting

$$A(z) = \frac{1}{2} \cdot \sum_{j=-\infty}^{\infty} a_j z^j, \tag{16.29}$$

$$B(z) = \frac{1}{2} \cdot \sum_{j=-\infty}^{\infty} b_j z^j. \tag{16.30}$$

The functions $A(z)$ and $B(z)$, which are Laurent polynomials, may be found as a solution of the following algebraic system:

$$P(z)A(\overline{z}) + R(z)B(\overline{z}) = 1 \quad \text{for all } |z| = 1,$$

$$P(-z)A(\overline{z}) + R(-z)B(\overline{z}) = 0 \quad \text{for all } |z| = 1.$$

They may be found explicitly by applying identities (16.25) and (16.26), p. 319.

$$A(z) = \frac{E_\phi(z)}{E_\phi(z^2)} P(z),$$
$$B(z) = -\frac{z^{2m-1}}{E_\phi(z^2)} P\left(-\frac{1}{z}\right).$$

Evidently, $A(z)$ and $B(z)$ are Laurent polynomials.

It is interesting that the symbols $A(z)$ and $B(z)$ also provide the **two-scale relations** for $\widetilde{\phi}$ and $\widetilde{\psi}$ (which are the "dual" relations to the two-scale relations for ϕ and ψ in (16.8), p. 314, and (16.19), p. 317, respectively, by

$$\begin{cases} \widehat{\widetilde{\phi}}(\xi) = A(z)\widehat{\widetilde{\phi}}(\xi/2), \\ \widehat{\widetilde{\psi}}(\xi) = B(z)\widehat{\widetilde{\phi}}(\xi/2). \end{cases} \tag{16.31}$$

16.7 Decomposition and reconstruction algorithms

Let us return to the main point of MRA. Assume that we are given an arbitrary function $f \in L_2(\mathbb{R})$. Then for every $\varepsilon > 0$ we find an approximation $f_N \in V_N$ for a sufficiently large N such that

$$\|f - f_N\| < \varepsilon.$$

We consider the expansion of the function $f_N(x) \in V_N$ given by

$$f_N(x) = \sum_{j=-\infty}^{\infty} c_{N,\ell}\phi(2^N x - \ell).$$

Due to

$$V_N = V_{N-1} \oplus W_{N-1}, \tag{16.32}$$

we have the representation

$$f_N(x) = f_{N-1}(x) + g_{N-1}(x)$$

with $f_{N-1} \in V_{N-1}$ and $g_{N-1} \in W_{N-1}$. Then the coefficients in the representations

$$f_{N-1}(x) = \sum_{j=-\infty}^{\infty} c_{N-1,\ell}\phi(2^{N-1}x - \ell),$$
$$g_{N-1}(x) = \sum_{j=-\infty}^{\infty} d_{N-1,\ell}\psi(2^{N-1}x - \ell),$$

may be computed by applying the above decomposition relations. We obtain the **decomposition algorithm**, holding for all s in $\mathbb{Z}$, namely

$$c_{N-1,s} = \sum_{\ell=-\infty}^{\infty} a_{\ell-2s} c_{N,\ell},$$

$$d_{N-1,s} = \sum_{\ell=-\infty}^{\infty} b_{\ell-2s} d_{N,\ell}.$$

Conversely, we have the representation for the coefficients of f_N by means of the coefficients of f_{N-1} and g_{N-1}, which is known as the **reconstruction algorithm**, and for every $s \in \mathbb{Z}$ given by

$$c_{N,s} = \sum_{\ell=-\infty}^{\infty} \{p_{s-2\ell} c_{N-1,\ell} + r_{s-2\ell} d_{N-1,\ell}\}. \tag{16.33}$$

The "reconstruction algorithm" is practically reasonable since the sequences p_j and r_j are finite. However, from this point of view the "decomposition algorithm" is not very practical since the sequences a_j and b_j are infinite. For that reason, for the decomposition it is better to apply the dual representation of f_N, namely

$$\begin{aligned} f_N(x) &= \sum_{j=-\infty}^{\infty} \widetilde{c}_{N,\ell} \widetilde{\phi}(2^N x - \ell) \\ &= \sum_{j=-\infty}^{\infty} \widetilde{c}_{N-1,\ell} \widetilde{\phi}(2^{N-1} x - \ell) + \sum_{j=-\infty}^{\infty} \widetilde{d}_{N-1,\ell} \widetilde{\psi}(2^{N-1} x - \ell). \end{aligned}$$

Now thanks to the dual two-scale relations (16.31), p. 321, we obtain the *dual decomposition algorithm*, holding for all s in $\mathbb{Z}$, namely

$$\widetilde{c}_{N-1,s} = \sum_{\ell=-\infty}^{\infty} p_{2s-\ell} \widetilde{c}_{N,\ell},$$

$$\widetilde{d}_{N-1,s} = \sum_{\ell=-\infty}^{\infty} r_{2s-\ell} \widetilde{d}_{N,\ell}.$$

Note that the order of the indices of the coefficients p_j and r_j has changed.

16.8 Zero moments

An important property of the spline wavelets is that they have **zero moments** up to order m, i.e.

$$\int_{-\infty}^{\infty} x^{\ell} \psi(x)\, dx = 0 \quad \text{for } \ell = 0, \ldots, m, \tag{16.34}$$

the last being equivalent to the fact that $W_0 \perp V_0$ [5, p. 59, p. 61]. This is due to the fact that every compactly supported spline is a finite linear combination of the shifts of the function $\phi(x)$. An equivalent statement is that for every integer $\ell \leq m$ and for every compact interval $[a, b]$ we have the representation

$$x^{\ell} = \sum_{i=-\infty}^{\infty} a_{\ell,i} \phi(x - i),$$

where $a_{\ell,i}$ is a finite sequence.

16.9 Symmetry and asymmetry

The **symmetry** and the **antisymmetry** properties distinguish the cardinal spline wavelets from other wavelets. Namely, if we put $\psi_1(x) = \psi\left(x + \frac{2m-1}{2}\right)$ then we have

$$\psi_1(x) = \begin{cases} \psi_1(-x) & \text{for even } m, \\ -\psi_1(-x) & \text{for odd } m. \end{cases} \tag{16.35}$$