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DISTRIBUTED MOMENT PROBLEM AND SOME RELATED QUESTIONS ON APPROXIMATION OF FUNCTIONS OF MANY VARIABLES

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The definition and some investigations of the distributed moment problem which arises in optimal control problems are presented. Statements which appear to be analogous to the results in the unidimensional moment problem, obtained by M.G. Krein, are proved.

Let Ω be a bounded region (connected open set) in $\mathbb{R}^n$ and the operator $L(x,D)=\sum_{|\alpha|\le 2m} a_\alpha(x)D^\alpha$ is an elliptic differential operator of order 2m with sufficiently smooth coefficients in the closed hull $\bar{\Omega}$ [1].

Following the notations in 2 , we introduce the sets of functions

$$N(L)=\{h(x);\ L(x,D)h(x)=0\ ,\ x\in\bar{\Omega},$$
$$h(x)\in C^{m-1}(\bar{\Omega})\cap C^{2m}(\Omega)\}\ ,$$
$$N_+(L)=\{h(x);\ h(x)\in N(L),\ h(x)\ge 0\}\ .$$

The following boundary operators are given [3] :

$$B_j(x,D)=\sum_{|\gamma|\le m_j} b_{j\gamma}(x)D^\gamma,\quad j=1,\dots,m\ .$$

The region Ω and the operators L, B_j satisfy the conditions considered in [2,3] which provide solvability of the following boundary value problem

$$L(x,D)h(x)=0\ ,\quad x\in\bar{\Omega}\ ,$$
(1)
$$B_j(x,D)h(x)=g_j(x)\ ,\quad x\in\partial\Omega\ ,\quad j=1,\dots,m\ ,$$

for smooth functions $g_j(x)$ defined on the boundary $\partial\Omega$.

Through $\mathfrak{M}(\bar{\Omega})$ and $\mathfrak{M}(\partial\Omega)$ we denote the set of all Radon measures in $\bar{\Omega}$, $\partial\Omega$ respectively [5] .

We use the following generalization of the sweeping-out method of Poincaré [7] , proved in [2] :

<u>Lemma</u>. For any measure $\mu\in\mathfrak{M}(\bar{\Omega})$ there exists a vector measure $\nu=(\nu_j)_{1\le j\le m}\in[\mathfrak{M}(\partial\Omega)]^m$ such that the following equality holds

(2) $$\int_{\bar{\Omega}} h(x)\,d\mu(x)=\sum_{j=1}^{m}\int_{\partial\Omega} B_j h(y)\,d\nu_j(y)\ ,\qquad \forall h(x)\in N(L)\ .$$

Further we often use the abbreviated form of (2):

(3) $$\mu(h)=\nu(Bh)\ ,\ \forall h(x)\in N(L)\quad \text{or}\quad \Pi\mu=\nu\ .$$

In what follows, some results are proved, which are analogs to the classical results of Krein for the unidimensional moment problem [6] .

For a given measure $\nu\in[\mathfrak{M}(\partial\Omega)]^m$ consider the set of masses (positive measures)

(4) $$B(\nu)=\{\mu\in\mathfrak{M}(\bar{\Omega});\ \Pi\mu=\nu\ ,\ \mu(1)=1\ ,\ \mu\ge 0\}\ .$$

<u>Definition</u>. The measure $\nu\in[\mathfrak{M}(\partial\Omega)]^m$ is called ϰ-positive if and only if the inequality holds

(5) $$\nu(Bh)\ge 0\quad \text{for}\quad \forall h(x)\in N(L)\ ,\ h(x)\ge 0\ .$$

<u>Theorem 1</u>. For a given measure $\nu\in[\mathfrak{M}(\partial\Omega)]^m$ the set $\mathbb{B}(\nu)$ is not empty if and only if ν is an ϰ-positive measure.

<u>Proof</u>. If there exists a mass $\mu\in B(\nu)$ then the relation $\mu(h)\ge 0, \forall h(x)\in N(L), h(x)\ge 0$ and (2) imply the inequality $\nu(Bh)=\mu(h)\ge 0$. This proves the necessity.

In order to prove the sufficiency, consider the set of sweeped-out masses $S=\{\Pi\mu;\ \mu\in\mathfrak{M}(\bar{\Omega}),\ \mu(1)=1,\ \mu\ge 0\}$.

The set S is a closed and convex subset of $[\mathfrak{M}(\partial\Omega)]^m$

Let $\varphi(\cdot)$ be a linear continuous functional on the space $[\mathfrak{M}(\partial\Omega)]^m$, satisfying the condition $\varphi(s)\ge 0,\ s\in S$

According to the Riesz representation theorem [5] $\varphi(\cdot)=(\varphi_j(x))_{1\le j\le m}$ for some continuous functions $\varphi_j(x)\ ,\ x\in\partial\Omega\ ,\ 1\le j\le m$.

Following our initial proposition (1), there exists a solution to the boundary value problem with boundary data φ , i.e. a function $h(x)\in N(L)$ with boundary values $B_j h(x)=\varphi_j(x)$.

The positivity of $\varphi(\cdot)$ implies $h(x)\ge 0, x\in\Omega$ since $\Pi\mu(Bh)=\mu(h$. Consequently, the inequality $\varphi(\nu)=\nu(Bh)\ge 0$ holds for an ϰ-positive measure ν . Since $\varphi(\cdot)$ is an arbitrary continuous functional in $[\mathfrak{M}(\partial\Omega)]^m$, the theorem of Hahn-Banach [5] implies that $\nu\in S$. Q.E.D.

Let us remark that in (1) we proposed solvability for smooth boundary data (the Lipschitz classes [3]). The last form a dense set in the set of all continuous boundary data. On the other hand it is enough to consider only a dense set of functionals $\varphi(\cdot)$. This makes the reasoning correct.

Our second theorem extends the maximal mass method of Krein [6] to the case of distributed parameters.

Suppose a measure $\nu\in[\mathfrak{M}(\partial\Omega)]^m$ to be given.

For a fixed point $\xi\in\bar{\Omega}$ define the function

(6) $$\rho(\xi)=\inf_{h\in N_+(L)}\frac{\nu(Bh)}{h(\xi)}\ .$$

<u>Theorem 2</u>. The function $\rho(\xi)$ is the maximal mass, i.e.

(7) $$\rho(\xi)=\max_{\mu\in\mathbb{B}(\nu)}\tilde{\mu}(\xi)$$

where the function $\tilde{\mu}(\xi)$ is defined as the limit

$$\tilde{\mu}(\xi)=\lim_{\varepsilon\to 0}\int_{B_\varepsilon} d\mu(x)\ ;$$

the ball $B_\varepsilon(\xi)=\{x\in\mathbb{R}^n;\ |x-\xi|\le\varepsilon\}$.

<u>Proof</u>. Consider the measure $\nu_0\in[\mathfrak{M}(\partial\Omega)]^m$ defined through the equation

$$\nu_0(Bh)=\nu(Bh)-\rho(\xi)h(\xi)\ .$$

(The correctness of this definition follows from proposition (1)).

The measure ν_0 is ϰ-positive which follows from the definition (6) of $\rho(\xi)$.

Now, applying Theorem 1, a mass $\mu_0 \in \mathbb{B}(\nu_0)$ is obtained.

If the mass μ_ξ is defined through the equality $d\mu_\xi(x) = d\mu_0(x) + \rho(\xi)\delta(x-\xi)$ then the equation $\nu(Bh) = \rho(\xi)h(\xi) + \mu_0(h) = \mu_\xi(h)$ holds; here $\delta(x-\xi)$ is the Dirac function in the point ξ. As a result we obtain the inequality

$$(8) \qquad \rho(\xi) = \tilde{\mu}_\xi(\xi) \le \sup_{\mu \in \mathbb{B}(\nu)} \tilde{\mu}(\xi) .$$

On the other hand, the following inequalities hold

$$\mu(h) = \int_{\bar\Omega} h(x)\,d\mu(x) \ge h(\xi)\tilde{\mu}(\xi),\ \tilde{\mu}(\xi) \le \frac{\nu(Bh)}{h(\xi)},\ \forall \mu \in \mathbb{B}(\nu),\ \forall h(x) \in N(L)$$

which imply the estimate

$$(9) \qquad \sup_{\mu \in \mathbb{B}(\nu)} \tilde{\mu}(\xi) \le \rho(\xi) .$$

Both inequalities (8), (9) justify the equality (7). Q.E.D.

The duality for the so-called L-problem for distributed parameters is the content of our final theorem. First we shall formulate the distributed moment problem and its dual.

Consider the Lebesgue space $L_1(\bar\Omega)$ of real functions $\phi(x)$ with the norm $\|\phi(x)\| = \int_{\bar\Omega} |\phi(x)|\,dx$. It is well known that any linear continuous functional $\mu(\cdot)$ in this space has the following analytic expression $\mu(\phi) = \int_{\bar\Omega} \phi(x) f(x)\,dx$, where $f(x)$ is some measurable essentially bounded function; the norm of the functional $\mu(\cdot)$ is calculated through the formulas [5]:

$$\|\mu\| = \sup_{x \in \bar\Omega} \operatorname{ess} |f(x)| .$$

Suppose now that the measure $\nu \in [\mathfrak{M}(\partial\Omega)]^m$ is given and also k+1 measurable functions $u_0(x), \dots, u_k(x)$ on $\bar\Omega$ and k+1 constants $C_0, \dots, C_k$.

Consider the extremal problem

$$(10) \qquad \|\mu\| \longrightarrow \min = \Lambda ;$$

here $\mu = f(x)dx$ runs the space of all linear continuous functionals on $L_1(\Omega)$ satisfying the equalities $\Pi\mu = \nu$, $\int_{\Omega} u_j(x) f(x)\,dx = c_j$, $j = 0, \dots, k$, and the extremal problem

$$(11) \qquad \|h(x) + \sum_{j=0}^{k} \lambda_j u_j(x)\|_{L_1} \longrightarrow \min_P = \frac{1}{M}$$

here the minimization is on the set P of pairs, satisfying some condition:

$$P = \{(h,\lambda);\ h(x) \in N(L),\ \lambda = (\lambda_0, \dots, \lambda_k) \in \mathbb{R}^{k+1},\ \nu(Bh) + \sum_{j=0}^{k} \lambda_j c_j = 1\} .$$

Theorem 3. The minimum in problem (10) coincides with the reciprocal to the minimum in problem (11), i.e. $\Lambda = M$.

Proof. Let μ be a measure satisfying the conditions of problem (10). For any function $h(x) \in N(L)$ we have $|\nu(Bh) + \sum \lambda_j c_j| = |\mu(h) + \sum \lambda_j c_j| \le \|h(x) + \sum \lambda_j u_j(x)\| \cdot \|\mu\|$.

Consequently, the following estimate is true:

$$\frac{1}{\|\mu\|} \le \min_{(h,\lambda)} \frac{\|h(x) + \sum \lambda_j u_j(x)\|}{|\nu(Bh) + \sum \lambda_j c_j|} = \min_P \|h(x) + \sum \lambda_j u_j(x)\| .$$

This implies the relation $\Lambda \ge M$.

To prove the opposite inequality, consider the space of functions

$$H = \{h(x) + \sum \lambda_j u_j(x);\ h(x) \in N(L),\ \lambda = (\lambda_j) \in \mathbb{R}^{k+1}\} .$$

which is contained in the space $C(\bar\Omega)$.

Define the linear functional $F(\cdot)$ in H as follows: $F(h + \sum \lambda_j u_j) = \nu(Bh) + \sum \lambda_j c_j$. The norm of $F(\cdot)$ in H is

$$\|F\|_H = \sup_H \frac{|F(h + \sum \lambda_j u_j)|}{\|h + \sum \lambda_j u_j\|} = \sup_P \frac{1}{\|h + \sum \lambda_j u_j\|} = M .$$

The Hahn-Banach theorem [5] confirms that $F(\cdot)$ can be extended to a linear continuous functional $F(\cdot)$ (we use the same letter for the extension) in the space of continuous functions $C(\bar\Omega)$ with norm, satisfying the inequality $\|F\|_{C(\bar\Omega)} \le \|F\|_H\ (\le M)$. This implies the relation Q.E.D.

For the case of elliptic operators, satisfying the so-called $(U)_S$ property [4] we announce two approximation theorems.

Theorem A. For any function $f(x) \in C(\bar\Omega)$ the minimization problem $\int_{\bar\Omega} |h(x) - f(x)|\,dx \longrightarrow \min_{h \in N(L)}$ has unique solution.

This theorem is an analogue to the theorem of Jackson which confirms uniqueness for the L_1-approximation with polynomials. The proof is based on Theorem 3.

The next theorem is an analogue to the Chebyshev alternance theorem. Under a closed surface in $\mathbb{R}^n$ we understand the boundary of a simply connected domain in $\mathbb{R}^n$.

Theorem B. For a given function $f(x) \in C(\bar\Omega)$ consider the approximation problem

$$(12) \qquad \|h(x) - f(x)\|_C \longrightarrow \min_{h \in N(L)} \quad (=\delta)$$

where $\|\cdot\|_C$ denotes the uniform norm in $C(\bar\Omega)$.

A necessity and sufficient condition for the function $h(x) \in N(L)$ to be solution of problem (12) is:

i) The set $T = \{x \in \bar\Omega;\ |h(x) - f(x)| = \|h - f\|_C\}$ should contain a sequence of m closed surfaces $\Gamma_1, \dots, \Gamma_m$ such that the surface Γ_{j+1} is contained in the domain surrounded by the surface Γ_j;

ii) The set T contains a point x_0 which lies inside Γ_m;

iii) The following alternance condition is true:

$$h(x) - f(x) = \varepsilon(-1)^j\delta,\ \forall x \in \Gamma_j,\ j = 1, \dots, m;\ h(x_0) - f(x_0) = \varepsilon(-1)^{m+1}\delta$$

for $\varepsilon = 1$ or $\varepsilon = -1$.

The correctness of the approximation problems (existence of solution) considered in Theorem A and B follows from the compactness of the space $N(L)$ [8].

For the case of second order elliptic operators Theorem A will appear.

For elliptic operators of high order the proofs of Theorem A and B need deep results, concerning the sets of zeroes of the elements of the space $N(L)$.

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РАЗПРЕДЕЛЕН ПРОБЛЕМ НА МОМЕНТИТЕ И НЯКОИ СВЪРЗАНИ С НЕГО ВЪПРОСИ ОТ АПРОКСИМАЦИЯ НА ФУНКЦИИ НА МНОГО ПРОМЕНЛИВИ

Огнян Ив. Кунчев

В работата е определен и изследван разпределителният проблем на моментите, който възниква в някои задачи на оптималното управление на частни диференциални уравнения. Получени са някои твърдения, аналогични на резултатите на М.Г. Крейн за едномерния проблем на моментите.

отсъствуват "удари". Крайните резултати са представени в компактна матрична форма, удобна за реализация на ЕИМ.

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APPLICATION OF A SLIDING MODE IN PLANNING THE MOTION OF A ROBOT ARM WITH REDUNDANCY

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One of the most significant problems comprising the end effector's motion planning is to satisfy the constraints on the joint variables. In the method, here proposed, a smooth approximation with accuracy given in advance is applied to the domain defined by the joint variables constraints. Thus, the joint rates obtained are continuous on the closure of the approximated domain. In most of the methods known so far, arbitrarily chosen joints are fixed or the constraints are being considered implicitly. In this paper the constraints are treated explicitly in such a way that when the approximated boundary has been reached, a motion is defined on it (sliding mode). This method introduces a unified approach to the definition of the equation for continuous path control on the closure of the approximated domain. The final results are represented in a form suitable for application in computer programs.