СОФИЙСКИ ОКРЪЖЕН НАРОДЕН СЪВЕТ • СЪЮЗ НА МАТЕМАТИЦИТЕ [В Б]ЪЛГАРИЯ

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„100 ГОДИНИ ОТ РОЖДЕНИЕТО НА АКАДЕМИК ЛЮБОМИР НИКОЛОВ ЧАКАЛОВ"

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A NONLOCAL MAXIMUM PRINCIPLE FOR THE BIHARMONIC EQUATION, ALMANSI TYPE FORMULAE FOR OPERATORS WHICH ARE SQUARES OF ELLIPTIC OPERATORS OF SECOND ORDER AND APPROXIMATION BY THEIR SOLUTIONS

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A maximum principle of constructive type is proved using only the values of the biharmonic function. An analogue of the Almansi formula for some operators of a specific structure is proved. Some qualitative properties of the element of the best approximation by solutions of these equations are investigated.

A biharmonic function in Ω is a 4-differentiable solution of the equation $\Delta^2 u(x)=0,\ u(x)\in C^4(\Omega)$, where $\Delta=\sum_{i=1}^{n}\partial^2/\partial x_i^2$ is the Laplace operator. We denote the set of all such functions by $H_2(\Omega),\ \Omega\subset R^n$.

It was proved by Almansi[1] that under certain conditions on Ω (see[2]) a function $u(x)\in H_2(\Omega)$ can be represented by two functions $\varphi_j(x),\ j=\overline{0,1}$ harmonic in Ω in the following way

(1) $$u(x)=\varphi_0(x)+x_1\varphi_1(x),\quad x\in\bar\Omega .$$

In the case of a starlike domain Ω (see [3]) with respect to 0 there is a representation of the form

(2) $$u(x)=\varphi_0(x)+|x|^2\varphi_1(x),\quad x\in\bar\Omega ,$$

here $|x|^2=\sum_{i=1}^{n}x_i^2$ and $\varphi_0(x),\varphi_1(x)$ are harmonic in Ω.

Further by $B(x_0;R)$ we denote the ball $\{x\in R^n;\ |x-x_0|\le R\}$ with centre x_0 and radius R ; if G is a set in R^n then $\bar G$ and $\mathring{G}$ denote respectively its closure and interior.

Theorem 1. Let $\varphi(x)\in H_2(\Omega)$ be a biharmonic function in the convex domain Ω in R^n. If $B(x_0;R)\subset\Omega$ and $\bar G\subset\mathring{B}(x_0;R)$ for some $x_0\in\Omega$ and a regular domain G ([4]) then there exist two constants C_1, C_2 such that

(3) $$\|\varphi\|_{\bar G}\le C_1\|\varphi\|_{\partial B}\|_{\partial B}+C_1\|\varphi|\partial G\|_{\partial G} .$$

Here $\|\varphi\|_{\bar G}=\max_{x\in\bar G}|\varphi(x)|$ and similarly for the restrictions $\varphi|_{\partial B}$ and $\varphi|_{\partial G}$ of $\varphi(x)$, $\|\cdot\|_{\partial B}$ and $\|\cdot\|_{\partial G}$ denote the uniform norms.

Proof. Since Ω is starlike with respect to any point $x_0\in\Omega$ using (2) we obtain the following representation

(4) $$\varphi(x)=\varphi_0(x)+|x-x_0|^2\varphi_1(x),\quad x\in\Omega$$

for an arbitrary function $\varphi(x)\in H_2(\Omega)$.

Let us denote by K the operator which maps the Dirichlet boundary data on ∂B into the harmonic function having these data, i.e. if $f(x)\in C(\partial B)$ then $\Delta K[f](x)=0$ and $K[f]|_{\partial B}=f(x)$ (see[4]). Similar operator for the domain G exists [4] and we shall denote it by L.

For $x\in\partial B$ we have $\varphi(x)=\varphi_0(x)+R^2\varphi_1(x)$.

Applying the operator K to (4) we obtain

(5) $$\varphi_1|_{\partial G}=(K[\varphi|_{\partial B}]-\varphi|_{\partial G}).(R^2-|x-x_0|^2)^{-1},$$
$$\varphi_0|_{\partial G}=\varphi|_{\partial G}-(K[\varphi|_{\partial B}]-\varphi|_{\partial G}).(R^2-|x-x_0|^2)^{-1}.|x-x_0|^2 .$$

On the other hand, for $x\in G$ we have

$$\varphi(x)=L[\varphi_0|_{\partial G}]+|x-x_0|^2.L[\varphi_1|_{\partial G}],$$

and $|x-x_0|<R$ according to the inclusion $\bar G\in\mathring{B}$.

Since $\varphi_0|_{\partial G}$ and $\varphi_1|_{\partial G}$ are given by formulae (5) and the operators K and L satisfy the usual maximum principle [4] : $\|Kf_1\|_B\le\|f_1\|_{\partial B}$, $\|Lf_2\|_{\bar G}\le\|f_2\|_{\partial G}$, we easily obtain the estimate (3) with $C_1=2M\delta^{-1}$ and $C_2=1+2M\delta^{-1}$; here $M=\max_{x\in\bar G}|x-x_0|$, $\delta=\min_{x\in\partial G}(R^2-|x-x_0|^2)$ Q.E.D.

Let us note that (3) implies uniqueness of the solution of the nonlocal problem

(6) $$\varphi|_{\partial B}=f_1(x),\ x\in\partial B;\quad \varphi|_{\partial G}=f_2(x),\quad x\in\partial G .$$

Using formulae (1), an estimate similar to (3) can be proved when B is replaced by a half space.

It is very probable that estimates of type (3) could be proved when B is replaced by more general domains for which problem (6) has a unique solution.

We saw that the Almansi formulae are useful for obtaining qualitative information concerning the biharmonic functions. Similar formulae for operators of the form $Q^2(D_x)$ where $Q(D_x)=\sum_{i+j\le 2}a_{ij}\,\partial^2/\partial x_i\partial x_j$ is a second order elliptic operator with constant coefficients are established in the following:

Theorem 2. Let $\varphi(x)\in C^4(R^n_+)$ be a solution of the equation $Q^2(D_x)\varphi(x)=0$ satisfying the condition $U(x)=O(|x|)$ for large $|x|$ (see [5]). Then $\varphi(x)$ can be represented in the following way:

(7) $$\varphi(x)=\varphi_0(x)+x_1\varphi_1(x),\quad x\in R^n_+ ,$$

where $\varphi_0(x),\varphi_1(x)$ are solutions of the equation $Q\varphi_0(x)=Q\varphi_1(x)=0,\ x\in R^n_+=\{x\in R^n;\ x_1\ge 0\}$.

Proof. To prove this we shall use the Poisson kernels in the half-space [5], denoting x_1 by t and working in R^{n+1}_+ where we shall prove formula (7).

Let $\varphi_0(x)$ be the solution of the following boundary value problem

(8) $$Q\varphi_0(x,t)=0,\ (x,t)\in R^{n+1}_+(t\ge 0);\quad \varphi(x,t)=f(x),\ t=0 .$$

It is given by formula (2.3) in [5].

According to [5] for the operator $Q^2(D_x)$ we have:

$$m=2,\ L=(Q(\xi,\tau))^2,\ Q=(\tau-\tau^+(\xi))(\tau-\tau^-(\xi));$$

$$M^+=(\tau-\tau^+(\xi))^2=\tau^2-2\tau\tau^+ + (\tau^+)^2;$$

$$B_1=1,\ m_1=0;\ B_2=\tau,\ m_2=1,\ b_{ij}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$ (see [5]);

and

$$M_1^+=\tau-2\tau^+,\ M_0^+=1,$$

which gives $N_1=\tau-2\tau^+$, $N_2=1$.

Solution of the boundary value problem

(9)
$$Q^2(Dx,Dt)\varphi(x,t)=0,\ (x,t)\in \mathbb{R}_+^{n+1}$$
$$\varphi(x,0)=0,$$
$$\varphi_t(x,0)=f(x),\ x\in\mathbb{R}^n$$

is given by formula (2.3) in [5] :

(10)
$$\varphi(x,t)=\int K_2(x-y,t)f(y)dy,$$

where simple calculations show that

$$K_2(x,t)=C_1\int_{|\xi|=1} d\omega_\xi\left(\int_\gamma \frac{d\tau}{(\tau-\tau^+)^2(x\xi+t\tau)^{n-1}}\right).$$

Here $d\omega_\xi$ denotes the surface element of the unit sphere and γ is a Jordan curve in the half-plane $Im\tau>0$ which contains all zeros of the polynomial $M^+(\xi,\tau)$ for $\xi\in\mathbb{R}^n$ with $|\xi|=1$.

Calculation of the integral over γ gives

$$K_2(x,t)=C_2\int_{|\xi|=1}\frac{t}{(x\xi+t\tau^+(\xi))^n}d\omega_\xi$$

In the same way we find the Poisson kernel for Q :

$$m=1,\ L=Q(\xi,\tau)=(\tau-\tau^+(\xi))(\tau-\tau^-(\xi)),$$

$$M^+=\tau-\tau^+(\xi),\ B_1=1,\ m_1=0,\ (b_{ij})=1$$ (see [5]),

$$M_0^+=1,\ N_1=1,$$

which gives (for Q) the kernel :

$$K_1(x,t)=C_3\int_{|\xi|=1}d\omega_\xi\int_\gamma\frac{d\tau}{(\tau-\tau^+(\xi))(x\xi+t\tau)^n}=C_3'\int_{|\xi|=1}\frac{d\omega_\xi}{(x\xi+t\tau^+)^n}.$$

Evidently $K_2=C.t.K_1$.

The representation (7) follows through formula (2.3) in [5] since (10) becomes

$$\varphi(x,t)=C.t.\int K_1(x-y,t)f(y)dy,$$

where the integral solves a boundary value problem of the form (8), Q.E.D.

Here we restricted ourselves to the case of a half-space so that we could freely use the formulae of [5] but similar formulae of the Almansi type can be proved for a wider class of domains [2]. For the solutions of the equation in (9) it is possible to prove a nonlocal maximum principle like theorem 1 for the half-space applying formula (7).

Next we apply some arguments used in the proof of Theorem 1 to demonstrate a partial case of the criterion for the element of the best uniform approximation by polyharmonic functions ([6]).

Consider the following approximation problem

(11)
$$\|\varphi-f\|=\max_{x\in\bar\Omega}|\varphi(x)-f(x)|\to\inf_{\varphi\in N}\ (=\delta),$$

where the set of biharmonic functions N is such that the problem has a solution (see e.g. 7) $f(x)\in C(\bar\Omega)$.

Denote by T the set of "critical points":

$$T=T(\varphi)=\{x\in\bar\Omega;\ \|\varphi-f\|=|\varphi(x)-f(x)|\}.$$

Theorem 3. Let the set $T(\varphi)$ contains the sphere $\partial B(x_0;R)$, the boundary ∂G of a regular domain $\bar G\subset \mathring{B}(x_0;R)$ and also $x_0\in T$, so that the following "alternance" relations hold:

(12)
$$\varphi(x)-f(x)=\varepsilon\delta,\ x\in\partial B(x_0;R);$$

(13)
$$\varphi(x)-f(x)=-\varepsilon\delta,\ x\in\partial G;$$

(14)
$$\varphi(x_0)-f(x_0)=\varepsilon\delta,\ \text{where}\ \varepsilon=\pm1.$$

Then $\varphi(x)$ is the element of best approximation in problem (11).

Proof. Let us suppose, for the sake of simplicity, that $x_0=0, R=1$ and $\varepsilon=+1$.

If the function $\tilde\varphi(x)\in N$ is such that $\|\tilde\varphi-f\|<\|\varphi-f\|$ then for the function $\psi(x)=\tilde\varphi(x)-\varphi(x)$ relations (12-14) imply:

(15)
$$\psi(x)>0,\ x\in\partial B(0;1);$$

(16)
$$\psi(x)<0,\ x\in\partial G;$$

(17)
$$\psi(0)>0.$$

For $\psi(x)$ we have the Almansi representation in $B(0;1)$ $\psi=\psi_0(x)+|x|^2\psi_1(x)$ with harmonic functions $\psi_0(x),\psi_1(x)$.

Applying the positive operator K to (15), (16) gives the inequality $\psi_0(x)+|x|^2\psi_1(x)<0<\psi_0(x)+\psi_1(x),\ x\in\partial G$. Since $|x|<1, x\in\partial G$, this implies

(18)
$$\psi_1(x)>0,\ x\in\partial G,$$

and (16) implies:

(19)
$$\psi_0(x)<0,\ x\in\partial G.$$

Since the operator L is positive, (19) gives $\psi_0(0)<0$ which contradicts(17).

This contradiction proves that $\varphi(x)$ is the best uniform approximation of $f(x)$, Q.E.D.

Similar theorem can be proved for operators of the type Q^m considered in Theorem 2. In that case the set T contains $m-1$ concentric spheres, ∂G and the centre of the spheres x_0.

For the approximation by n-harmonic functions φ (i.e. $\Delta^n\varphi(x)=0, x\in\bar\Omega$) in the problem (11) we have the following .

Corollary. Let a continuous in the unit ball $B(0;1)$ function $f(x)$ be rotationally symmetric, i.e. $f(x)=f(|x|)$. Then the n-harmonic function $\varphi(x)$ which is the element of best uniform approximation of $f(x)$ by n-harmonic functions (solving the problem (11)) has the form $\varphi(x)=P_{n-1}(|x|^2)$, where the polynomial $P_{n-1}(t)$ is the element of the best uniform approximation of function $f(\sqrt{t})$ in the interval $[0,1]$ by polynomials of d[illegible]ee less than or equal to $n-1$.

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НЕЛОКАЛЕН ПРИНЦИП ЗА МАКСИМУМА ЗА БИХАРМОНИЧНИ ФУНКЦИИ, ФОРМУЛА НА АЛМАНЗИ ЗА ОПЕРАТОРИ, КОИТО СА КВАДРАТИ НА ЕЛИПТИЧНИ ОПЕРАТОРИ ОТ ВТОРИ РЕД И АПРОКСИМАЦИЯ С РЕШЕНИЯТА ИМ

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Обоснован е един критерий за елемента на най-доброто приближение с бихармонични функции в равномерната метрика. Технически това е свързано с формулата на Алманзи и с един нелокален принцип за максимума за бихармонични функции. Резултатите се обобщават и в случая на диференциални оператори, които са степени на елиптични оператори от втори ред с постоянни коефициенти, в частност полихармонични.

СОФИЙСКИ ОКР[illegible] НАРОДЕН СЪВЕТ • СЪЮЗ НА МАТЕМАТИЦИТЕ В БЪЛГАРИЯ

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„100 ГОДИНИ ОТ РОЖДЕНИЕТО НА АКАДЕМИК ЛЮБОМИР НИКОЛОВ ЧАКАЛОВ“
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ДОЛНА ГРАНИЦА ЗА РАЗМЕРА НА НОСИТЕЛЯ НА БАЛАНСИРАНИ НЕПЪЛНИ БЛОК-ДИЗАЙНИ

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В статията се дава подобрение на оценката на Фуди и Хедаят за размера на носителя на балансиран непълен блок-дизайн с повтарящи се блокове.

1. Въведение. Комбинаторните понятия и означения, използувани в настоящата статия, са съгласувани с тези от книгата на Тончев [1]. Дефиницията на t-дизайн допуска появата на повтарящи се блокове. Множеството от различните блокове в даден дизайн $\mathcal{D}$ ще бележим с $\mathcal{D}^*$ и ще наричаме носител на дизайна. Мощността на носителя ще бележим с b^* и ще наричаме размер на носителя. За $t-(v,k,\lambda)$-дизайн с размер на носителя b^* ще използуваме означението $t-(v,k,\lambda|b^*)$. Дизайн $\mathcal{D}$, за който $b^*<b$, ще наричаме дизайн с повтарящи се блокове. Ако даден блок се повтаря s пъти в $\mathcal{D}$, ще го наричаме s-блок. За всеки блок-дизайн, съдържащ s-блок, е в сила неравенството на Ман [4]:

(1.1) $$b \geqq s.v$$.

По-нататък ще разглеждаме само блок-дизайни. За произволни v, k и λ с b^*_{min} ще бележим най-малкото b^*, за което съществува $2-(v,k,\lambda|b^*)$-дизайн. В [2] Фуди и Хедаят дават следната оценка за b^*_{min} :

(1.2) $$b^*_{min} \geqq \left\lceil \frac{v}{k} \left\lceil \frac{v-1}{k-1} \right\rceil \right\rceil ,$$

където $\lceil a \rceil$ е горната цяла част на a .

Цел на настоящата статия е подобряването на този резултат. Нека с m_i^* означаваме максималното число, за което съществува $2-(v,k,\lambda|b^*)$-дизайн с $b^*=i$, съдържащ m_i^* различни λ-блока. Ако за $b^*=i$ не съществува $2-(v,k,\lambda|b^*)$-дизайн, $m_i^*=0$. Сега нека

$$m^* = \max_{1\leqq i<b} m_i^* .$$

В настоящата статия е показано, че

(1.3) $$b^*_{min} \geqq 2\frac{\binom{v}{2}}{\binom{k}{2}} - m^* .$$

Дадена е горна оценка за m^* , която след заместване в (1.3) води до по-сил-