Potential Theory

Edited by
Josef Král
Czechoslovak Academy of Sciences
Prague, Czechoslovakia

and

Jaroslav Lukeš
Ivan Netuka and
Jiří Veselý
Charles University
Prague, Czechoslovakia

Plenum Press • New York and London

THE DISTRIBUTED MOMENT PROBLEM

Ognyan Kounchev

Institute of Mathematics
Bulgarian Academy of Sciences
Sofia, Bulgaria

INTRODUCTION

The present work continues the investigation of the socalled distributed moment problem (DMP) (see [13] and references there). We use the last name instead of its alternative one, inverse potential problem [19], to stress the analogy with the one-dimensional moment problem [1,8,15].

First we prove a result about the uniqueness of the solution of the extremal problems for the inverse problem of gravimetry [3, 19,22]. After that we describe some geometric properties of the mass distributions for the same inverse problem. Then we prove that the element of best approximation by harmonic functions in the uniform norm is unique.

We chose thes items since they use similar technics based on a classical density result of Brelot [4] and Deny [5].

1. THE INVERSE POTENTIAL PROBLEM

Let $\overline{G}$ be a bounded domain in R^n with a regular boundary $F = \partial G$ and the operator

$$L(x,D) = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha \tag{1.1}$$

be an elliptic differential operator of order 2m with sufficiently smooth coefficients in G [2,19], here we set as usually

$$D^\alpha = \partial^{|\alpha|}/(\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}), \quad |\alpha| = \alpha_1 + \ldots + \alpha_n .$$

Following [19] we introduce the following notations

$$N(L) = \{h(x) : L(x,D)h(x) = 0, \quad x \in G, \quad h(x) \in C^{m-1}(\overline{G}) \cap C^{2m}(G)\}; \tag{1.2}$$

$$N_+(L) = \{h(x) \in N(L) : h(x) \geq 0, \quad x \in \overline{G}\};$$

for the boundary operators

$$B_j(x,D) = \sum_{|\gamma| \leq m_j} b_{j\gamma}(x) D^\gamma \qquad (j=1,\dots,m) \tag{1.4}$$

We suppose that L, B_j, G satisfy the normality condition [2,19] which provides solubility of the boundary value problem

$$Lh(x) = 0 \text{ for } x \in G, \qquad B_j h(y) = g_j(y) \text{ for } y \in F, \quad j = 1,\dots,m, \tag{1.5}$$

for sufficiently smooth boundary data g_j on F.

Through $M(\overline{G})$ and $M(F)$ we denote the set of all Radon measures on G and F respectively [10,18,19].

Further we suppose that L, B_j and G are such that a priori estimates for problem (1.5) can be established and a corresponding sweeping-out (balayage) operator can be defined by means of the following

Proposition 1- ([19]) Let $\mu \in M(\overline{G})$. Then there exists a vector-measure $\nu = (\nu^j)$, $1 \leq j \leq m$, $\nu^j \in M(F)$ such that for every $u \in N(L)$ the equality (1.6) holds:

$$\int_{\overline{G}} u(x)\,d\mu(x) = \sum_{j=1}^{m} \int_F B_j u(y)\,d\nu^j(y) \tag{1.6}$$

Instead of (1.6) we will often use the following abbreviations

$$\mu(u) = \nu(Bu) \text{ or } P\mu = \nu. \tag{1.7}$$

Now let $\nu \in M^m(F)$ be a given vector-measure. Consider the set

$$B(\nu) = \{\mu \in M(G) : P\mu = \nu, \quad \mu \geq 0\}. \tag{1.8}$$

The inverse problem of potential theory consists in finding elements of the set $B(\nu)$ and more generally, in studying the structure of $B(\nu)$ [19].

Let $w(x)$ be a continuous function defined in $\overline{G}$. We are interested in the following extremal problem

$$\int_{\overline{G}} w(x)\,d\mu(x) \to \max(\min), \quad \mu \in B(\nu) \tag{1.9}$$

for a given $\nu \in M^m(F)$.

For the case of the Laplace operator $L = \Delta = \sum_{i=1}^{n} \partial^2/\partial x_i^2$, in Theorem 1 below we prove that the minimization problem (1.9) has unique solution for subharmonic functions $w(x)$. In that case problem (1.9) may be interpreted as an extremal problem for the inverse problem of gravimetry [2,22]. For the proof we need the duality Proposition 3 [13]. In order to formulate it we introduce the sets

$$\underline{P}(w) = \{u \in N(L) : u(x) \leq w(x), \ x \in \overline{G}\},$$

$$\overline{P}(w) = \{u \in N(L) : u(x) \geq w(x), \ x \in \overline{G}\}.$$

They are not empty since all constants are in N(L).

The measure $\nu \in M^m(F)$ is called positive if for every $h \in N_+(L)$ we have $\nu(Bh) \geq 0$. It is called strictly positive if $\nu(Bh) > 0$ for every $h \in N_+(L)$, $h \not\equiv 0$, and singularly positive if it is positive but not strictly, i.e. there is an $h_0 \in N(L)$, $h_0 \not\equiv 0$ but $\nu(Bh) = 0$.

We have the following result:

Proposition 2- ([12]) For a given vector-measure $\nu \in M(F)$ the set $B(\nu)$ is not empty if and only if ν is a positive measure.

The following duality result is an analogue to a classical duality theorem [15, Ch. 4, Theorem 4.1] :

Proposition 3- Let the strictly positive vector-measure $\nu \in M^m(F)$ be given. Then

a) the equalities hold

$$\min \{\int_{\overline{G}} w(x)\,d\mu(x) : \mu \in B(\nu)\} = \max\{\nu(Bu) : u \in \underline{P}\}$$

$$\max \{\int_{\overline{G}} w(x)\,d\mu(x) : \mu \in B(\nu)\} = \min\{\nu(Bu) : u \in \overline{P}\};$$

b) a necessary and sufficient condition that the integral

$$\int_{\overline{G}} w(x)\,d\mu(x) \tag{1.10}$$

takes its maximal (minimal) value for $\mu = \mu_0 \in B(\nu)$ is the existence of some $u_0 \in \underline{P}(w)$ $(u_0 \in \overline{P}(w))$ which coincides with $w(x)$ for $x \in \overline{G}$ where $d\mu(x) \neq 0$.

The proof of this proposition follows closely the idea in [15].

In the following paragraphs we will consider the case of the Laplace operator $L = \Delta$ and its powers defined by $\Delta^{i+1} = \Delta\,\Delta^i$ and $\Delta^0 = 1$ for all positive integers i. We will impose no restrictions on the boundary operators B_j except the stated above. Other operators L for which the theorems may be proved are obtained through coordinate transformation of Δ and its powers. In the bidimensional case every strictly elliptic second order operator may be reduced to C.Δ [21] which extends our results to this generality.

2. UNIQUENESS FOR SOLUTIONS OF EXTREMAL PROBLEMS

For the sake of clarity, within the statement we formulate problem (1.9) for the Laplace operator $L = \Delta$ and the boundary operator $B_1 = 1$:

Theorem 1- Le ν be a given positive measure on F. Consider the set of positive measures

$$B(\nu) = \{\mu \in M(\overline{G}) : \mu \geq 0,\ \int_{\overline{G}} h(x)d\mu(x) = \int_F h(y)d\nu(y) \quad \text{for every function } h(x), \text{ harmonic in } G\}. \tag{2.1}$$

If $w(x)$ is a subharmonic function [7,16] then the extremal problem

$$\int_{\overline{G}} w(x)d\mu(x) \to \min \qquad (\mu \in B(\nu))$$

has unique solution.

Proof of Theorem 1- Since the set $B(\nu)$ is w*-compact in $M(\overline{G})$ [19] problem (2.2) has a solution. Let us call it μ_o. Applying Proposition 3 we obtain a harmonic in G function $h_o(x)$ satisfying

$$h_o(x) \leq w(x) \qquad (x \in \overline{G}) \tag{2.3}$$

and such that $\operatorname{supp}(\mu_o) \subset M = \{x : h_o(x) = w(x)\}$.

If for some measure $\mu \in B(\nu)$ $\operatorname{supp}(\mu) \not\subset M$ then we get

$$0 = \int_{\overline{G}} (w(x) - h_o(x))d\mu_o(x) < \int_{\overline{G}} (w(x) - h_o(x))d\mu(x). \tag{2.4}$$

This obviously implies

$$\int_{\overline{G}} w(x)d\mu_o(x) < \int_{\overline{G}} w(x)d\mu(x)$$

since both μ, μ_o satisfy (2.1).

It follows that if $\mu_1 \in B(\nu)$ is another solution of problem (2.2) then $\operatorname{supp}(\mu_1) \subset M$.

On the other hand the set M is a set of zeros of the positive subharmonic function $w(x) - h_o(x)$ in $\overline{G}$. This implies that every point of M is regular for its complement CM [7,16]. We shall prove that the set CM is connected.

Let us suppose the opposite: the set CM has a compact component G_o. Then $w(x) - h_o(x) = 0$ on ∂G_o and a basic property of subharmonic functions [7] implies that $w(x) - h_o(x) < 0$ in G_o which contradicts (2.3).

Now a classical theorem [4,5] may be applied giving that the set of functions harmonic in G is dense in the set of continuous functions C(M) (in the uniform norm). Since we have $\mu_0(h) = \mu_1(h)$ for every $h \in N(\Delta)$ in G, the above implies $\mu_1 = \mu_0$. Q.E.D.

It is quite plausible that Theorem 1 holds for the case $L = \Delta^m$ with $w(x)$ - subharmonic function of order m [17], i.e. a function satisfying $(-1)^m \Delta^m w(x) \leq 0$, $x \in \overline{G}$.

3. GEOMETRIC STRUCTURE OF THE MASS DISTRIBUTIONS

Here we prove a geometric property of the measures in $B(\nu)$ which have only regular points [7,16] in their support.

Theorem 2- Let $L = \Delta^m$ and the set $B(\nu)$, for a given vector-measure $\nu \in M^m(F)$, is not empty. If μ_1, μ_2 are two different measures in $B(\nu)$ which have supports regular sets and the distance between the two supports is not zero, i.e. $\operatorname{dist}(\operatorname{supp}(\mu_1), \operatorname{supp}(\mu_2)) > 0$, then one of the sets $\operatorname{supp}(\mu_1)$, $\operatorname{supp}(\mu_2)$ contains a closed surface Γ (the boundary of a simply connected domain in R^n) and the other one contains a point x_o which is inside Γ.

This result is especially interesting for the case of $L = \Delta$ since it justifies, in some sense, results of D. Zidarov [22] who studied methods for obtaining one mass distribution in $B(\nu)$ from another through a method of "concentration of masses" (see also [3]).

Proof of Theorem 2- We proceed in a way quite similar to the proof of the characterization theorem [6,11].

Consider the measure $d\mu = d(\mu_1 - \mu_2)$ which is not zero. We have that $\mu(h) = 0$ for every function $h(x)$, harmonic in $\overline{G}$.

Now, let us suppose that $\operatorname{supp}(\mu)$ has a connected complement in R^n. Since $\operatorname{supp}(\mu)$ is a regular set we may apply a classical theorem [4,5] which states that the set of harmonic functions is dense in the set of continuous functions $C(\operatorname{supp}(\mu))$. According to a standard argument this contradicts $\mu(h) = 0$ for every $h \in N(\Delta)$ with a nonzero measure $d\mu$. It follows that $\operatorname{supp}(\mu)$ has nonconnected complement and we may find a compact simply connected component of $G \setminus \operatorname{supp}(\mu)$, let us call it G_o. Since the boundary of G_o is a connected set and $\operatorname{supp}(\mu_1)$, $\operatorname{supp}(\mu_2)$ are distant sets it is clear that $\Gamma = \partial G_o$ is contained either in $\operatorname{supp}(\mu_1)$ or in $\operatorname{supp}(\mu_2)$.

In order to prove the existence of a point x_o inside Γ, let us suppose that no such exists, i.e. every compact simply connected component of $G \setminus \operatorname{supp}(\mu)$ which has a boundary in $\operatorname{supp}(\mu_1)$ does not meet $\operatorname{supp}(\mu_2)$ and the same for $\operatorname{supp}(\mu_2)$. So we find a maximal open set G_1 which is simply connected (but in general not connected) with boundary in $supp(\mu_1)$ and a similar G_2 for $supp(\mu_2)$ such that G_1 and G_2 are distant and the set $G_1 \cup G_2 \cup supp(\mu_1) \cup supp(\mu_2)$ has a connected complement. The same theorem quoted above implies that there is a function $h_1(x)$, harmonic in $\bar{G}$ which is close to 1 on $G_1 \cup supp(\mu_1)$ and close to -1 on $G_2 \cup supp(\mu_2)$. The last contradicts $\mu(h_1) = 0$ or $\mu_1(h_1) = \mu_2(h_1)$, since $\mu_1(h_1)$ is a positive but $\mu_2(h_1)$ is a negative number. The proof is finished. Q.E.D.

Let us note that an interesting interpretation, from the point of view of elasticity theory, for the elements of the set $B(\nu)$ when $L = \Delta^2$ was recently considered in [9].

4. ON THE UNIFORM APPROXIMATION BY SOLUTIONS OF PDEs

For a given function $f(x) \in C(\bar{G})$ consider the approximation problem in the Chebyshev norm:

$$\|h-f\| = \max_{x \in \bar{G}} |h(x) - f(x)| \to \inf_{h \in N(L)} \quad (= \delta). \tag{4.1}$$

This problem has a solution for an appropriate extension of $N(L)$. For example, when $L = \Delta^m$, we consider the bounded in $\bar{G}$ solutions of $Lh(x) = 0$, $x \in G$ [17].

We define the important set where the norm is attained:

$$T(h) = \{x \in \bar{G}: |h(x)-f(x)| = \|h-f\|\} \tag{4.2}$$

For $L = \Delta$, when the element of best approximation in problem (4.1) is a continuous function, there is a characterization for it in the terms of the set $T(h)$ [6,11]. In the next theorem we show that this solution is unique:

Theorem 3. Let $L = \Delta$. Then the corresponding problem (4.1) has unique solution if its solution $\tilde{h}$ is a continuous function and the corresponding set $T(\tilde{h})$ in (4.2) has only regular points.

Proof of Theorem 3. According to [20] the function $\tilde{h} \in N(L)$ is an element solving (4.1) if and only if there exist Borel nonnegative measures μ_+, μ_- on $\bar{G}$ such that the measure $\mu = \mu_+ - \mu_-$ satisfies the following conditions:

$$Var(\mu) = 1, \tag{4.3}$$

$$\int_{\bar{G}} h(x)\, d\mu(x) = 0 \text{ for every } h \in N(\Delta), \tag{4.4}$$

$$supp(\mu_+) \subseteq T_+(\tilde{h}), \quad supp(\mu_-) \subseteq T_-(\tilde{h}), \tag{4.5}$$

where the sets T_+ and T_- are defined as follows

$$T_\pm(\tilde{h}) = \{x \in \bar{G}: \tilde{h}(x)-f(x) = \pm\|\tilde{h}-f\| = \pm\delta\}.$$

Evidently, $T = T_+ \cup T_-$.

Let us suppose that problem (4.1) has two solutions h_1, h_2 with the properties stated in the theorem, and $\|h_1-f\| = \|h_1-f\| = \delta$. Then the function $\tilde{h} = (h_1+h_2)/2$ is a solution of (4.1) too. This may be seen from the relations

$$\|\tilde{h}-f\| = \|(h_1-f)/2 + (h_2-f)/2\| \le \delta/2 + \delta/2 = \delta.$$

The inclusions

$$T_\pm(\tilde{h}) \subseteq T_\pm(h_1) \cap T_\pm(h_2) \tag{4.6}$$

follow from the relations

$$\delta = |\tilde{h}(x)-f(x)| = |(h_1(x)-f(x))/2 + (h_2(x)-f(x))/2| \le$$
$$\le \delta/2 + \delta/2 = \delta.$$

This implies that the set $T(\tilde{h})$ has only regular points. Suppose further that the complement of the set $T(\tilde{h})$ is connected. According to the result in [4,5] we get that the set $N(L)$ is dense in the set of continuous functions $C(T(\tilde{h}))$ which contradicts (4.4) when applied to our function $\tilde{h}$, since (4.3) says that $d\mu$ is a nonzero measure. This contradiction shows that the complement of the set $T(\tilde{h})$ is not connected and we may find a compact component of $G \setminus T(\tilde{h})$ which is a simply connected set. Let us call it G_o. Since the sets $T_+(\tilde{h})$, $T_-(\tilde{h})$ are distant, and the boundary ∂G_o is a connected set, it follows that either $\partial G_o \subseteq T_+(\tilde{h})$ or $\partial G_o \subseteq T_-(\tilde{h})$. In both cases the inclusion (4.6) gives that $h_1(x) = h_2(x)$ for $x \in \partial G_o$. According to the maximum principle $h_1(x) = h_2(x)$ for $x \in G_o$ and the real analyticity of harmonic functions implies that $h_1(x) = h_2(x)$ everywhere in G. Q.E.D.

It is quite natural to conjecture that Theorem 3 holds for the polyharmonic operator $L = \Delta^m$.

REFERENCES

[1] Achieser, N.I.: The classical moment problem, Fizmatgiz, Moscow, 1961 (in Russ., there is Engl. transl.).

[2] Agmon, S., Douglis, A., Nirenberg, L.: Estimates near the boundary for the solutions of elliptic partial differential equations satisfying general boundary conditions. 1., Comm. Pure Appl. Math., 12 (1959), pp. 623-727.

[3] Anger, G.: Lectures on potential theory and inverse problems, in: Geodaetische und Geophysikalische Veroffentlichungen, Reihe 3, No. 45 (1980), pp. 15-95, published by the Nation. Committee for Geodesy and Geophysics, Acad. Sci. GDR, Berlin.

[4] Brelot, M.: Sur l'approximation et la convergence dans la theorie des functions harmoniques ou holomorphes, Bull. Soc. Math. France, 73 (1945), pp. 55-70.

[5] Deny, J.: Systemes totaux de functions harmoniques, Ann. Inst. Fourier (Grenoble), 1 (1949), pp. 103-113.

[6] Hayman, W.K., Kershaw, D., Lyons, T.J.: The best harmonic approximant to a continuous function, in: International Series of Numerical Mathematics, Vol. 65, Birkhauser Verlag, Basel, 1984.

[7] Helms, L.L.: Einfuehrung in die Potentialtheorie, De Gruyter, Berlin, 1973.

[8] Karlin, S., Studden, W.J.: Tchebycheff systems: with applications in analysis and statistics, Interscience Publishers, New York, 1970.

[9] Kleine, E.: An inverse problem for biharmonic potentials, Mathematische Nachrichten, 128 (1986), pp. 7-27.

[10] Kolmogorov, A.N., Fomin, S.V.: Elements of theory of functions and functional analysis, Nauka, Moscow, 1968 (in Russ., there is Engl. transl.).

[11] Kounchev, O.I.: On the harmonic function which deviates least from a given continuous one in the circle, DAN BSSR, Ser. fiz.-mat. nav., No. 4 (1985) (in Russ.).

[12] Kounchev, O.I.: Distributed moment problem and some related questions on approximation of functions of many variables, in: Mathemattics and education in mathematics, Publishing House of the Bulgarian Academy of Sciences, Sofia, 1985, pp. 454-458.

[13] Kounchev, O.I.: The distributed moment problem, in: Proc. Confer. Contr. and Identif. Distrib. Param. Sys., Vorau-Austria, 1986, Lect. Notes Inf. Contr. (to appear).

[14] Kounchev, O.I.: The Chebyshev alternance set and the theory of canonical representations of Markov, (in preparation).

[15] Krein, M.G., Nudel'man, A.A.: The problem of Markov and extremal problems, Nauka, Moscow, 1973 (in Russ., there is Engl. transl.).

[16] Landkof, N.S.: Foundations of modern potential theory, Nauka, Moskow, 1966 (in Russ., there is Engl. transl.).

[17] Nicolescu, M.: Opèra matematica: Functii poliarmonice, Editura academiei Republicii Socialiste Romania, Bucuresti, 1980.

[18] Rudin, W.: Real and complex analysis, Mc Graw Hill Publ. Co., New York, 1976.

[19] Schulze, B.-W., Wildenhain, H.: Methoden der Potentialtheorie fuer elliptische Differentialgleichungen beliebiger Ordnung, Akademie Verlag, Berlin, 1977.

[20] Singer, I.: Best approximation in normed linear spaces by elements of linear subspaces, Springer, Berlin, 1970.

[21] Vekua, I.N.: Generalized analytic functions, GIFML, Moscow, 1959 (in Russ., there is Engl. transl.).

[22] Zidarov, D.: New approach to the solution of the integral equation of the first kind, in: Inverse and improperly posed problems in differential equations, Akademie Verlag, Berlin, 1979, pp. 271-288.